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SOLUTIONS FOR SECOND ORDER NONLOCAL BVPS VIA THE GENERALIZED MIRANDA THEOREM

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ABSTRACT. In this paper, the generalized Miranda theorem is applied for second-order systems of differential equations with one boundary condition given by Riemann-Stieltjes integral

$$x'' = f(t, x, x'), \quad x(0) = 0,\ x'(1) = \int_0^1 x(s)\, dg(s),$$

where $f : [0,1] \times \mathbb{R}^k \times \mathbb{R}^k \to \mathbb{R}^k$ is continuous and $g : [0,1] \to \mathbb{R}^k$ has bounded variation. Under suitable assumptions upon f and g we prove the existence of solutions to such posed problem.

1. Introduction. We aim to prove the existence of solutions to the following problem

$$\begin{cases} x'' = f(t, x, x') \\ x(0) = 0 \\ x'(1) = \int_0^1 x(s)\, dg(s), \end{cases} \tag{1.1}$$

where $f : [0,1] \times \mathbb{R}^k \times \mathbb{R}^k \to \mathbb{R}^k$ is continuous and $g : [0,1] \to \mathbb{R}^k$, with $g = \operatorname{diag}(g_1, \dots, g_k)$ a function of bounded variation. Moreover, we consider the problem (1.1) in the case where

$$\int_0^1 s\, g_i(s)\, ds = 1$$

for every $i = 1, \dots, k$. Then, the problem is resonant since, when $f \equiv 0$, the linear problem

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$$\begin{cases} x'' = 0 \\ x(0) = 0 \\ x'(1) = \int_0^1 x(s)\,dg(s), \end{cases}$$

has nontrivial solutions $x(t) = a\,t$ with $a \in \mathbb{R}^k$.

The non-resonant and scalar cases, i.e., when

$$\int_0^1 s\,g(s)\,ds \neq 1,$$

were considered by Webb and Infante **[11]** and Webb and Zima **[12]**. In **[11]**, the authors investigated the existence of positive solutions to the following problem:

$$\begin{cases} -x''(t) = q(t)f(t,x(t)) \\ x(0) = 0 \\ x'(1) = \int_0^1 x(s)\,dg(s), \end{cases}$$

where

$$f : [0,1] \times \mathbb{R}_+ \longrightarrow \mathbb{R}_+, \qquad q : [0,1] \longrightarrow \mathbb{R}_+,$$

and the integral is meant in the sense of Riemann-Stieltjes. The authors wrote the problem as a Hammerstein integral equation and used the fixed point index theory of compact mappings.

In **[12]**, the authors studied the existence of positive solutions for nonlinear nonlocal boundary value problem of the form:

$$\begin{cases} -x''(t) = f(t,x(t)) \\ x(0) = 0 \\ x'(1) = \int_0^1 x(s)\,dg(s). \end{cases}$$

The case where $f(t,x)$ is not positive for all positive x was considered such that $f(t,x) + \omega^2 x \geq 0$ for $x \geq 0$ for some constant $\omega > 0$. The authors also investigated the perturbed equation $-x''(t) + \omega^2 x(t) = h(t,x(t))$, with $h(t,x(t)) \geq 0$. They established the existence and multiplicity of positive solutions for the non-perturbed boundary value problem at resonance by considering equivalent non-resonant perturbed problems with the same boundary condition.

The corresponding problem for systems has been much less studied. In this case, imposing an a priori bound condition on f and g and applying the Leray-Schauder fixed point theorem, we proved the existence of at least one solution to the non-resonant problem (1.1) **[9]**.

As far as we know, there are no existence results for the resonant problem (1.1) in the case where the function f is a vector function and when f depends on x'.

The method considered in this paper may be found, for example, in **[8, 10]**, where it was applied to two-point boundary value problems on finite and infinite intervals. The idea is based on the consideration of an auxiliary problem. For such a problem, we repeat some of the results from **[10]** in Lemma 3.1. Using the initial value problem, it suffices to define appropriate functions h and H and apply the generalized Miranda theorem to show that problem (1.1) has at least one solution (see Theorem 2.5).

In order to obtain the theorem of the existence of solutions to problem (1.1), we impose some additional conditions on the functions f and g. We assume that, for every $i = 1, \dots, k$, the functions g_i are nondecreasing. Moreover, function f has linear growth and satisfies some sign condition (Theorem 4.1, (A3)). The sign condition is standard and has been considered in many papers, see, for instance, **[2, 5, 6]**.

2. Preliminaries. First, we recall the necessary topological concepts used throughout the paper. A detailed discussion of the following definitions may be found in **[1, 3]**.

Definition 2.1. A topological space (X, τ) is said to be *contractible* if it is null-homotopic, i.e., there exist a homotopy $\mathcal{H} : [0, 1] \times X \to X$ and a point $x \in X$ such that $\mathcal{H}(0, \cdot) = \mathrm{id}_X$ and $\mathcal{H}(1, \cdot) = x$.

Definition 2.2. Let (X, τ_X), (Y, τ_Y) be topological spaces. A set-valued map $H : X \multimap Y$ is said to be *upper semicontinuous* if, for any $V \in \tau_Y$, the set $\{x \in X : H(x) \subset V\}$ is τ_X-open.

Definition 2.3. A compact space (X, τ) is called an R_δ*-set* (which we denote $X \in R_\delta$) if there is a decreasing sequence X_n of compact, contractible spaces such that

$$X = \bigcap_{n=1}^{\infty} X_n.$$

Definition 2.4. Let (X, τ_X) and (Y, τ_Y) be topological spaces. We say that $H : X \multimap Y$ is an R_δ *map* if it is upper semicontinuous and for every $x \in X$ we have $H(x) \in R_\delta$.

In order to show that the problem (1.1) has at least one solution, we shall use the following generalization of the Miranda theorem [**10,** Theorem 5]:

Theorem 2.5 (Generalized Miranda theorem). *Let $M_i > 0$, $i = 1, \ldots, k$, and F be an admissible map from $\prod_{i=1}^k [-M_i, M_i]$ to $\mathbb{R}^k$, i.e., there exist a Banach space $(E, \|\cdot\|_E)$ with $\dim(E) \geq k$, a linear, bounded and surjective map*

$$h : E \longrightarrow \mathbb{R}^k$$

and an R_δ-map H from $\prod_{i=1}^k [-M_i, M_i]$ to E such that $F = h \circ H$. If, for any $i = 1, \ldots, k$ and every $y \in F(r)$, where $|r_i| = M_i$, we have $r_i \cdot y_i \geq 0$, then there exists an r_* such that $0 \in F(r_*)$.

3. An initial value problem.

Denote by $C([0,1], \mathbb{R}^k)$ the Banach space of all continuous functions $x : [0,1] \to \mathbb{R}^k$, by $C^1([0,1], \mathbb{R}^k)$ the Banach space of all continuous functions which have continuous first derivatives and by $C^2([0,1], \mathbb{R}^k)$ the Banach space of twice continuously differentiable functions, with the usual norms.

We consider an initial value problem

$$\begin{cases} x'' = f(t, x, x') \\ x(0) = 0 \\ x'(0) = r, \end{cases} \tag{3.1}$$

where $r \in \mathbb{R}^k$ is fixed.

Problem (3.1) is an auxiliary problem. In Section 4, we will show that the set of solutions to problem (3.1) contains a solution to problem (1.1).

We shall now show that problem (3.1) has global solutions.

Lemma 3.1. *Assume that the following conditions hold*:

(A1) $f : [0,1] \times \mathbb{R}^k \times \mathbb{R}^k \to \mathbb{R}^k$ *is continuous.*

(A2) *There are constants* $a_1, a_2, a_3 \geq 0$ *such that*

$$|f(t,x,y)| \leq a_1|x| + a_2|y| + a_3,$$

for all $(t,x,y) \in [0,1] \times \mathbb{R}^k \times \mathbb{R}^k$.

Then, for every $r \in \mathbb{R}^k$, *the problem* (3.1) *has at least one global solution, i.e., any possible solution can be extended to the interval* $[0,1]$.

Proof. Let $r \in \mathbb{R}^k$ be fixed. The existence of at least one local solution to problem (3.1) follows from Assumption (A1). We will show that every such solution is a global one, using the theorem on a priori bounds, **[7]**.

Let x be a local solution to (3.1), and define $\omega(t) := \sup_{u\in[0,t]} |x'(u)|$, $t \in [0,1]$. Since

$$x(t) = \int_0^t x'(s)\,ds, \tag{3.2}$$

by (A2), we have

$$\begin{aligned} |x'(t)| &\leq |r| + \int_0^t |f(t,x(t),x'(t))|\,ds \\ &\leq (|r| + a_3) + \int_0^t (a_1 s + a_2)\omega(s)\,ds. \end{aligned}$$

Consequently,

$$\omega(t) \leq (|r| + a_3) + \int_0^t (a_1 s + a_2)\omega(s)\,ds$$

and, by Gronwall's lemma,

$$\omega(t) \leq (|r| + a_3) \exp\left(\int_0^t a_1 s + a_2\,ds\right). \tag{3.3}$$

Hence, $|x'(t)|$ is bounded on $[0,1]$ and, by (3.2), $|x(t)|$ is also bounded on $[0,1]$. Consequently, using the theorem on a priori bounds, **[7,** page 146], we can extend the solution x to the entire interval $[0,1]$. $\square$

We consider a completely continuous nonlinear operator T_r,

$$T_r : \mathbb{R}^k \times C^1([0,1],\mathbb{R}^k) \longrightarrow C^1([0,1],\mathbb{R}^k),$$

associated with (3.1), given by

$$T_r x(t) := rt + \int_0^t (t-s) f(s, x(s), x'(s))\, ds. \tag{3.4}$$

It is easy to observe that x is a fixed point of the operator T_r if and only if x is a solution to the problem (3.1).

4. Boundary value problem. Let us consider the family of initial value problems (3.1) with $r \in \mathbb{R}^k$. Using Theorem 2.5, we shall show that there is an $r_* \in \mathbb{R}^k$ for which the solution x to problem (3.1) is a solution to problem (1.1), i.e., x also satisfies the second boundary condition of problem (1.1)

$$x'(1) = \int_0^1 x(s)\, dg(s).$$

Theorem 4.1. *Assume that, in addition to* (A1) *and* (A2)*, the following conditions hold*:

(A3) *for every $i = 1, \ldots, k$, there exists an $M_i > 0$ such that, for every $t \in [0,1]$, $x \in \mathbb{R}^k$ and $y \in \mathbb{R}^k$, $|y_i| \geq M_i$ implies that $y_i f_i(t, x, y) > 0$.*

(A4) *For every $i = 1, \ldots, k$, the function g_i is nondecreasing and*

$$\int_0^1 s\, dg_i(s) = 1.$$

Then, problem (1.1) *has at least one solution.*

Proof. Define a map

$$h : C^1([0,1], \mathbb{R}^k) \longrightarrow \mathbb{R}^k$$

by

$$h(x) := x'(1) - \int_0^1 x(s)\, dg(s). \tag{4.1}$$

It can easily be seen that h is linear and continuous. Moreover, h is surjective: for any $c \in \mathbb{R}^k$, it suffices to consider a function $x \in C^1([0,1], \mathbb{R}^k)$ such that

$$x_i(t) := -\frac{c_i}{\int_0^1 dg_i(s)}, \quad i = 1, \ldots, k. \tag{4.2}$$

Note that, by Assumption (A4), (4.2) is well defined, meaning

$$\int_0^1 dg_i(s) \neq 0 \quad \text{for } i = 1, \ldots, k.$$

Indeed, if g_i were nondecreasing and satisfied $g_i(1) = g_i(0)$, then it would be constant. However, for such functions, the equality $\int_0^1 s\, dg_i(s) = 1$ does not hold.

Now, let us consider a set-valued map

$$H : \mathbb{R}^k \multimap C^1([0,1], \mathbb{R}^k)$$

given by

$$H(r) := \{x \in C^1([0,1], \mathbb{R}^k) : T_r(x) = x\}.$$

Since the operator T_r is completely continuous, it may be shown that H is USC with compact values [**10,** Lemma 2]. The fact that H is an R_δ-map follows from Assumption (A2). Indeed, it is well known that, if f has a linear growth, then the set of all solutions of problem (3.1) is an R_δ-set, [**4,** page 162].

Observe that we have just proved that the maps h and H satisfy the assumptions of Theorem 2.5.

Let M_i be as in Assumption (A3). Now, it remains to show that, for any $i = 1, \ldots, k$ and $y \in h \circ H(r)$, we have $r_i \cdot y_i \geq 0$ with $|r_i| = M_i$.

We consider the case when $r_i = M_i$. Let x be a solution to problem (3.1) with $r_i = M_i$, $i = 1, \ldots, k$, and observe that $x \in C^2([0,1], \mathbb{R}^k)$. We shall prove that then x_i' is increasing.

Indeed, since $x_i'(0) = M_i$, by Assumption (A3), we get

$$x_i'(0)x_i''(0) = x_i'(0)f_i(0, x(0), x'(0)) > 0.$$

Consequently, $x_i''(0) > 0$, and there is an $\varepsilon > 0$ such that $x_i''(t) > 0$ for $t \in [0, \varepsilon)$. Moreover, x_i' is increasing on $[0, \varepsilon)$. Suppose that

$$t_1 := \inf\{t \in (\varepsilon, 1] : x_i''(t) < 0\}$$

exists. Then, we have $x_i'(t_1) \geq M_i$ and

$$0 = x_i'(t_1)x_i''(t_1) = x_i'(t_1)f_i(t_1, x(t_1), x'(t_1)) > 0,$$

a contradiction. Consequently, x_i' is increasing on $[0,1]$.

Now, since x_i' is increasing, by (3.2) and Assumption (A4), we obtain

$$\int_0^1 x_i(s)\,dg_i(s) = \int_0^1 \int_0^s x_i'(u)\,du\,dg_i(s) < x_i'(1)\int_0^1 s\,dg_i(s) = x_i'(1).$$

Thus,

$$r_i\left(x_i'(1) - \int_0^1 x_i(s)\,dg_i(s)\right) > 0.$$

In the case where $r_i = -M_i$ we proceed in an analogous manner.

Consequently, by Theorem 2.5, there exists an $r_* \in \prod_{i=1}^k [-M_i, M_i]$ such that $0 \in h \circ H(r_*)$. Hence, $H(r_*) \subset C^1([0,1],\mathbb{R}^k)$ contains a solution to problem (1.1). □

Remark 4.2. Observe that, by Assumption (A4), the measures dg_i are positive, $i = 1, 2$.

5. Example. Now, we shall present an example to illustrate the application of Theorem 4.1.

Let $k = 2$, and let $g_i(s) = (i+2)s^i$, $i = 1, 2$. Consider the boundary value problem (1.1) with the function $f = (f_1, f_2)$ given by

$$\begin{aligned} f_1(t,x,y) &= \alpha_1(t,x,y_2) + \beta_1(t,x,y_2)y_1 \\ f_2(t,x,y) &= \alpha_2(t,x,y_1) + \beta_2(t,x,y_1)y_2, \end{aligned} \tag{5.1}$$

where

$$\alpha_i, \beta_i : [0,1] \times \mathbb{R}^2 \times \mathbb{R} \longrightarrow \mathbb{R}$$

are positive, bounded and continuous functions.

Define

$$\beta_1^{\inf} := \inf_{(t,x,y_2)\in[0,1]\times\mathbb{R}^2\times\mathbb{R}} \beta_1(t,x,y_2),$$

$$\beta_2^{\inf} := \inf_{(t,x,y_1)\in[0,1]\times\mathbb{R}^2\times\mathbb{R}} \beta_2(t,x,y_1),$$

and

$$\alpha_1^{\sup} := \sup_{(t,x,y_2)\in[0,1]\times\mathbb{R}^2\times\mathbb{R}} |\alpha_1(t,x,y_2)|,$$
$$\alpha_2^{\sup} := \sup_{(t,x,y_1)\in[0,1]\times\mathbb{R}^2\times\mathbb{R}} |\alpha_2(t,x,y_1)|.$$

We assume that $\beta_1^{\inf},\ \beta_2^{\inf} > 0$.

Obviously, Assumptions (A1), (A2) and (A4) are satisfied. Moreover, we have

$$\begin{aligned} y_1 f_1(t,x,y) &= y_1(\alpha_1(t,x,y_2) + \beta_1(t,x,y_2)y_1) \\ &\geq -\alpha_1(t,x,y_2)|y_1| + \beta_1(t,x,y_2)|y_1|^2 \\ &\geq -\alpha_1^{\sup}|y_1| + \beta_1^{\inf}|y_1|^2. \end{aligned}$$

Setting

$$M_1 := 1 + \frac{\alpha_1^{\sup}}{\beta_1^{\inf}},$$

we obtain that $y_1 f_1(t,x,y) > 0$ for $|y_1| \geq M_1$. Analogous reasoning applies for $y_2 f_2(t,x,y)$ with

$$M_2 := 1 + \frac{\alpha_2^{\sup}}{\beta_2^{\inf}}.$$

Consequently, Assumption (A3) is satisfied.

Finally, applying Theorem 4.1, we obtain the existence of at least one solution to problem (1.1) with f given by (5.1).

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