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SYMMETRY AND MONOTONICITY OF SOLUTIONS FOR EQUATIONS INVOLVING THE FRACTIONAL LAPLACIAN OF HIGHER ORDER

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ABSTRACT. The aim of this paper is to establish symmetry and monotonicity of solutions to the equation involving fractional Laplacians of higher order. For this purpose, we first reduce the equation into a system via the composition of lower fractional Laplacians and then obtain symmetry and monotonicity of solutions to the system by applying the method of moving planes.

1. Introduction. Let $\mathcal{S}$ be Schwartz space of rapidly decreasing smooth functions on $\mathbf{R}^n$, $n \geq 2$. For $0 < \alpha < 1$, the fractional Laplacian $(-\Delta)^\alpha$ is a non-local operator defined by

$$\begin{aligned}(-\Delta)^\alpha u(x) &= c(n,\alpha) \text{ P.V.} \int_{\mathbf{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2\alpha}}\,dy \\ &= c(n,\alpha) \lim_{\epsilon\to 0} \int_{\mathbf{R}^n\setminus B_\epsilon} \frac{u(x)-u(y)}{|x-y|^{n+2\alpha}}\,dy,\end{aligned}$$

where $u \in \mathcal{S}$, P.V. denotes the principal value of the integral and $c(n,\alpha)$ is a positive normalization constant. The fractional Laplacian $(-\Delta)^\alpha$ can also be equivalently viewed as a pseudo-differential operator $\widehat{(-\Delta)^\alpha} u(\xi) = |\xi|^{2\alpha}\widehat{u}(\xi)$, $u \in S$, where $\widehat{u}$ is the Fourier transform of u. Let

$$L_\alpha = \left\{ u : \mathbf{R}^n \to \mathbf{R} \mid \int_{\mathbf{R}^n} \frac{|u(x)|}{1+|x|^{n+2\alpha}} < \infty \right\}.$$

Then, $(-\Delta)^\alpha$ can be extended to an operator on the $L_\alpha \cap C^{1,1}_{\mathrm{loc}}(\mathbb{R}^n)$.

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Positive solutions of semi-linear elliptic equations involving the fractional Laplacian $(-\Delta)^{\alpha}$ have recently been investigated by many authors. Brandle, et al., [**1**] reduced the non-local equation

$$(-\Delta)^{\alpha} u(x) = u^{p}(x), \quad x \in \mathbf{R}^{n}, \tag{1.1}$$

into a local one in higher dimensions by using the extension method of Caffarelli and Silvestre [**3**] and then proved the non-existence of positive solutions to (1.1) in the subcritical exponent by using the method of moving planes for local problems. Zhuo, et al., established the equivalence between (1.1) and the integral equation

$$u(x) = c \int_{\mathbf{R}^{n}} \frac{u^{p}(y)}{|x-y|^{n-2\alpha}}\, dy \tag{1.2}$$

by employing a Liouville theorem for α-harmonic functions and then obtained radial symmetry in the critical case and non-existence in the subcritical case for positive solutions to (1.1) via the method of moving planes in integral forms. Chen, Li and Li [**5**] proved the strong maximum principle for antisymmetric functions, narrow region principle and decay at infinity, developed a direct method of moving planes to fractional Laplacian and employed it to some semi-linear elliptic equations involving fractional Laplacian to obtain symmetry and non-existence of positive solutions. Felmer and Wang [**10**] established a version of the maximum principle for small domains by applying the Aleksandrov-Bakelman-Pucci (ABP) estimate, which was proven by Guillen and Schwab [**12**], and used it with the method of moving planes to prove symmetry and monotonicity of positive solutions to some problems involving fractional Laplacians in the unit ball. Quaas and Aliang [**14**] extended the idea in [**10**] to unbounded domains and obtained nonexistence of positive solutions for a class of fractional Laplacian equations and systems on the half space.

This paper extends the results of Felmer and Wang [**10**] to equations involving fractional Laplacians of higher order.

For $0 < \alpha < 1, u \in S$, the higher fractional Laplacian $(-\Delta)^{\alpha+1}$ is defined by

$$\begin{aligned} &(-\Delta)^{\alpha+1} u(x) \\ &\quad = c(n, \alpha+1)\ \mathrm{P.V.} \int_{R^{n}} \frac{u(x) + (1/2n)\Delta u(x)|x-y|^{2} - u(y)}{|x-y|^{n+2\alpha+2}}\, dy, \end{aligned} \tag{1.3}$$

where $c(n, \alpha+1)$ is a positive constant. Denote

$$L_{2,\alpha} = \left\{ u : \mathbf{R}^n \longrightarrow \mathbf{R} \mid \int_{\mathbf{R}^n} \frac{|u(x)|}{1+|x|^{n+2\alpha+2}}\, dx + \int_{\mathbf{R}^n} \frac{|D^2 u(x)|}{1+|x|^{n+2\alpha+2}}\, dx < \infty \right\}.$$

Then, $(-\Delta)^{\alpha+1}$ can be extended to wider spaces $L_{2,\alpha} \cap C^{3,1}_{\mathrm{loc}}(\mathbb{R}^n)$. Note that, for $u \in L_{2,\alpha} \cap C^{3,1}_{\mathrm{loc}}(\mathbb{R}^n)$, it follows that (see [**13**])

$$(-\Delta)^{\alpha+1} u(x) = (-\Delta)^{\alpha} \circ (-\Delta) u(x). \tag{1.4}$$

The aim of this paper is to study symmetry and monotonicity of solutions to the following problem

$$\begin{cases} (-\Delta)^{\alpha+1} u(x) = f(u(x)) + g(x) & x \in B_1, \\ (-\Delta) u(x) > 0 & x \in B_1, \\ u(x) = 0 & x \in \mathbf{R}^n \setminus B_1, \end{cases} \tag{1.5}$$

where B_1 is the open unit ball centered at origin. The function

$$f : \mathbf{R} \longrightarrow \mathbf{R}$$

is assumed to be locally Lipschitz continuous, increasing, and

$$g : B_1 \longrightarrow \mathbf{R}$$

is radially symmetric and decreasing in $|x|$. We say that a continuous function

$$u : \mathbf{R}^n \longrightarrow \mathbf{R}$$

is a classical solution to problem (1.5), if $u \in C^{3,1}_{\mathrm{loc}}(B_1)$ satisfies the equation and the conditions of (1.5) in the point-wise sense.

Our main result is:

Theorem 1.1. *Suppose that f is locally Lipschitz continuous, increasing on $\mathbf{R}$, and g is radially symmetric and decreasing. If u is a classical solution to* (1.5)*, then u is positive, radially symmetric and strictly decreasing in $r = |x|$ with $r \in (0, 1)$.*

Unlike elliptic equations involving the fractional Laplacian $(-\Delta)^{\alpha}$, there are few results (due to the lack of maximum principle) to equations involving $(-\Delta)^{\alpha+1}$. In order to overcome this difficulty, we change equation (1.5) into a system and obtain symmetry and monotonicity of positive solutions for the system by applying the method of moving planes.

This paper is organized as follows. In Section 2, we describe the fractional Laplacian $(-\Delta)^{\alpha+1}$ in pseudo-differential form. In Section 3, we present the ABP estimates involving $(-\Delta)^{\alpha}$ and $(-\Delta)$ in a bounded domain, and, finally, we prove the main result by applying the method of moving planes.

2. Higher fractional Laplacians in pseudo-differential forms.

In this section, we show that the higher fractional Laplacian $(-\Delta)^{\alpha+1}$ can also be defined through the Fourier transform. Toward this aim, we need the following lemma.

Lemma 2.1. *For $0 < \alpha < 1$, let $(-\Delta)^{\alpha+1}$ be the fractional Laplacian defined by* (1.3). *Then, for any $x \in \mathbf{R}^n$, $u \in \mathcal{S}$,*

(2.1)
$$\begin{aligned}(-\Delta)^{\alpha+1}u(x) &= \frac{c(n,\alpha+1)}{2} \\ &\quad\cdot \int_{\mathbf{R}^n} \frac{2u(x)+(1/n)\Delta u(x)|z|^2-u(x-z)-u(x+z)}{|z|^{n+2\alpha+2}}\,dz \\ &= \frac{c(n,\alpha+1)}{2} \\ &\quad\cdot \int_{\mathbf{R}^n} \frac{2u(x)+zD^2u(x)z^T-u(x-z)-u(x+z)}{|z|^{n+2\alpha+2}}\,dz.\end{aligned}$$

Proof. By choosing $y = x - z$ and $y = x + z$ in (1.3), respectively, we have

(2.2)
$$(-\Delta)^{\alpha+1}u(x) = c(n,\alpha+1) \int_{\mathbf{R}^n} \frac{u(x)+(1/2n)\Delta u(x)|z|^2-u(x-z)}{|z|^{n+2\alpha+2}}\,dz,$$

and

(2.3)
$$(-\Delta)^{\alpha+1}u(x) = c(n,\alpha+1) \int_{\mathbf{R}^n} \frac{u(x)+(1/2n)\Delta u(x)|z|^2-u(x+z)}{|z|^{n+2\alpha+2}}\,dz.$$

Equalities (2.2) and (2.3) infer that

(2.4)

$$(-\Delta)^{\alpha+1}u(x) = \frac{c(n,\alpha+1)}{2} \cdot \int_{\mathbf{R}^n} \frac{2u(x)+(1/n)\Delta u(x)|z|^2 - u(x-z) - u(x+z)}{|z|^{n+2\alpha+2}}\, dz.$$

Since the integral in (2.4) is, in principle, value sense, a fourth order Taylor expansion yields

$$(-\Delta)^{\alpha+1}u(x) = \frac{c(n,\alpha+1)}{2} \cdot \int_{\mathbf{R}^n} \frac{2u(x) + zD^2u(x)z^T - u(x-z) - u(x+z)}{|z|^{n+2\alpha+2}}\, dz.$$

The proof is complete. $\square$

We point out that, by Lemma 2.1, $(-\Delta)^{\alpha+1}$ can be viewed as a pseudo-differential operator of symbol $|\xi|^{2(\alpha+1)}$.

Proposition 2.2. *For $0 < \alpha < 1$, let $(-\Delta)^{\alpha+1}$ be the fractional Laplacian defined by* (1.3)*. Then, for any $u \in \mathcal{S}$,*

$$(-\Delta)^{\alpha+1}u(x) = \mathcal{F}^{-1}[C|\xi|^{2(\alpha+1)}\widehat{u}(\xi)],$$

where C is a positive constant depending only upon α and n.

Proof. For $|z| < 1$, the Taylor expansion yields

$$2u(x) + zD^2u(x)z^T - u(x-z) - u(x+z) = O(|z|^4) \tag{2.5}$$

and

$$\frac{|2u(x) + zD^2u(x)z^T - u(x-z) - u(x+z)|}{|z|^{n+2\alpha+2}} \le \frac{c}{|z|^{n+2\alpha-2}}. \tag{2.6}$$

For $|z| \ge 1$, it follows from $u \in \mathcal{S}$ that

$$\begin{aligned} &\frac{|2u(x) + zD^2u(x)z^T - u(x-z) - u(x+z)|}{|z|^{n+2\alpha+2}} \\ &\quad \le c\frac{4\max_{x\in R^n}|u(x)| + |z|^2 \max_{x\in R^n}|D^2u(x)|}{|z|^{n+2\alpha+2}} \\ &\quad \le \frac{c}{|z|^{n+2\alpha}}. \end{aligned} \tag{2.7}$$

As a consequence of (2.6) and (2.7), we obtain

$$\begin{aligned}
&\frac{|u(x)+(1/2n)\Delta u(x)|z|^2-(x-z)|}{|z|^{n+2\alpha+2}}\\
&\qquad\le c\left(\frac{1}{|z|^{n+2\alpha-2}}\chi_{B_1(0)}+\frac{1}{|z|^{n+2\alpha}}\chi_{\mathbf{R}^n\setminus B_1(0)}\right)\frac{1}{1+|x|^{n+1}},
\end{aligned}$$

and thus, apply the Fourier transform with respect to x in (2.1) to obtain

(2.8)

$$\begin{aligned}
\mathcal{F}[(-\Delta)^{\alpha+1}u](\xi)&=\frac{c(n,\alpha+1)}{2}\int_{\mathbf{R}^n}\frac{2-(1/n)|\xi|^2|z|^2-e^{i\xi\cdot z}-e^{-i\xi\cdot z}}{|z|^{n+2\alpha+2}}\,dz\widehat{u}(\xi)\\
&=\frac{c(n,\alpha+1)}{2}\int_{\mathbf{R}^n}\frac{2-(1/n)|\xi|^2|z|^2-2\cos(\xi\cdot z)}{|z|^{n+2\alpha+2}}\,dz\widehat{u}(\xi).
\end{aligned}$$

By making a change of variable $z=y/|\xi|$ in (2.8), it yields

$$\begin{aligned}
&\mathcal{F}[(-\Delta)^{\alpha+1}u](\xi)\\
&\quad=\frac{c(n,\alpha+1)}{2}\int_{\mathbf{R}^n}\frac{2-(1/n)|y|^2-2\cos((\xi/|\xi|)\cdot y)}{|y|^{n+2\alpha+2}}\,dy|\xi|^{2(\alpha+1)}\widehat{u}(\xi).
\end{aligned}$$

Choosing a rotation A such that $\xi/|\xi|=Ae_1$, where $e_1=(1,0,\dots,0)$, and substituting $z=A^Ty$, we see

$$\begin{aligned}
&\mathcal{F}[(-\Delta)^{\alpha+1}u](\xi)\\
&\quad=\frac{c(n,\alpha+1)}{2}\int_{\mathbf{R}^n}\frac{2-(1/n)|y|^2-2\cos(Ae_1\cdot y)}{|y|^{n+2\alpha+2}}\,dy|\xi|^{2(\alpha+1)}\widehat{u}(\xi)\\
&\quad=\frac{c(n,\alpha+1)}{2}\int_{\mathbf{R}^n}\frac{2-(1/n)|y|^2-2\cos(e_1\cdot A^Ty)}{|y|^{n+2\alpha+2}}\,dy|\xi|^{2(\alpha+1)}\widehat{u}(\xi)\\
&\quad=\frac{c(n,\alpha+1)}{2}\int_{\mathbf{R}^n}\frac{2-(1/n)|z|^2-2\cos(e_1\cdot z)}{|z|^{n+2\alpha+2}}\,dz|\xi|^{2(\alpha+1)}\widehat{u}(\xi)\\
&\quad=\frac{c(n,\alpha+1)}{2}\int_{\mathbf{R}^n}\frac{2-|z_1|^2-2\cos(z_1)}{|z|^{n+2\alpha+2}}\,dz|\xi|^{2(\alpha+1)}\widehat{u}(\xi)\\
&\quad=\frac{c(n,\alpha+1)}{2}\int_{\{|z|<1\}\cup\{|z|\ge1\}}\frac{2-|z_1|^2-2\cos(z_1)}{|z|^{n+2\alpha+2}}\,dz|\xi|^{2(\alpha+1)}\widehat{u}(\xi),
\end{aligned}$$

where $z=(z_1,z_2,\dots,z_n)$, A^T the transpose of A.

If $|z| < 1$, it follows from the Taylor expansion that

$$\left| \frac{2 - |z_1|^2 - 2\cos(z_1)}{|z|^{n+2\alpha+2}} \right| \le \frac{c}{|z|^{n+2\alpha-2}}. \tag{2.9}$$

If $|z| \ge 1$, we have

$$\left| \frac{2 - |z_1|^2 - 2\cos(z_1)}{|z|^{n+2\alpha+2}} \right| \le \frac{c}{|z|^{n+2\alpha}}. \tag{2.10}$$

Inequalities (2.9) and (2.10) imply the integral

$$\int_{\{|z|<1\}\cup\{|z|\ge 1\}} \frac{2 - |z_1|^2 - 2\cos(z_1)}{|z|^{n+2\alpha+2}}\, dz$$

is finite. Hence,

$$\mathcal{F}[(-\Delta)^{\alpha+1} u](\xi) = C|\xi|^{2(\alpha+1)}\widehat{u}(\xi),$$

which completes the proof. □

3. ABP estimates and the proof of the main result. The method of moving planes is based upon the ABP estimate. We first state the ABP estimate for fractional Laplacian $(-\Delta)^\alpha$, which was proven by Guillen and Schwab **[12]** for general integro-differential operators.

Proposition 3.1 (**[10, 12]**). *Let Ω be a bounded, open subset of $\mathbf{R}^n$. For $0 < \alpha < 1$, suppose that $h : \Omega \to \mathbf{R}$ is in $L^\infty(\Omega)$, and $w(x) \in L^\infty(\mathbf{R}^n)$ is a classical solution to the problem*

$$\begin{cases} (-\Delta)^\alpha w(x) \ge -h(x) & x \in \Omega, \\ w(x) \ge 0 & x \in \mathbf{R}^n \setminus \Omega. \end{cases}$$

Then, there exists a positive constant C depending upon n and α, such that

$$-\inf w(x) \le C d^\alpha \|h^+\|^{1-\alpha}_{L^\infty(\Omega)} \|h^+\|^{\alpha}_{L^n(\Omega)},$$

where $d = \mathrm{diam}(\Omega)$ is the diameter of Ω and $h^+ = \max\{h(x), 0\}$.

The ABP estimate for Laplace $(-\Delta)$ is also useful for our purposes, which is stated as follows:

Proposition 3.2 (**[10]**). *Let Ω, h, d and h^+ be described as in Proposition 3.1. Suppose that $w(x) \in L^\infty(R^n)$ is a classical solution to the* *problem*

$$\begin{cases} (-\Delta)w(x) \geq -h(x) & x \in \Omega, \\ w(x) \geq 0 & x \in \partial\Omega. \end{cases}$$

Then, there exists a positive constant C *depending upon* n*, such that*

$$-\inf w(x) \leq Cd\|h^+\|_{L^n(\Omega)}.$$

In the sequel, we apply the method of moving planes with Propositions 3.1 and 3.2 to prove Theorem 1.1.

Proof of Theorem 1.1*.* The proof is divided into two steps.

Step 1. The positivity of u can be derived from $(-\Delta)u > 0$ by the maximum principle. Using (1.4), we rewrite problem (1.5) as:

$$\begin{cases} (-\Delta)u(x) = v(x) & x \in B_1, \\ (-\Delta)^\alpha v(x) = f(u(x)) + g(x) & x \in B_1, \\ v(x) > 0 & x \in B_1, \\ u(x) = 0 & x \in \mathbf{R}^n \setminus B_1, \\ v(x) = 0 & x \in \mathbf{R}^n \setminus B_1. \end{cases} \tag{3.1}$$

We first consider the x_1 direction. For $x = (x_1, x') \in \mathbf{R}^n$, $0 < \lambda < 1$, denote

$$T_\lambda = \{x = (x_1, x') \in \mathbf{R}^n \mid x_1 = \lambda\},$$

$$\Sigma_\lambda = \{x = (x_1, x') \in B_1 \mid x_1 > \lambda\},$$

$$u_\lambda(x) = u(x^\lambda), \qquad w_{\lambda,u}(x) = u_\lambda(x) - u(x),$$

$$v_\lambda(x) = v(x^\lambda), \qquad w_{\lambda,v}(x) = v_\lambda(x) - v(x),$$

$$\Sigma^-_{\lambda,u} = \{x \in \Sigma_\lambda \mid w_{\lambda,u} < 0\},$$

$$\Sigma^-_{\lambda,v} = \{x \in \Sigma_\lambda \mid w_{\lambda,v} < 0\},$$

and let $x^\lambda = (2\lambda - x_1, x')$ be the reflection point of x about the plane T_λ. For $A \subset \mathbf{R}^n$, write $A_\lambda = \{x^\lambda \mid x \in A\}$. Hence, (3.1) becomes

$$
(3.2) \qquad \begin{cases} (-\Delta) w_{\lambda,u}(x) = w_{\lambda,v}(x) & x \in \Sigma_\lambda, \\ (-\Delta)^\alpha w_{\lambda,v}(x) = f(u_\lambda(x)) & \\ \quad -f(u(x)) + g(x^\lambda) - g(x) & x \in \Sigma_\lambda, \\ w_{\lambda,u}(x),\ w_{\lambda,v}(x) \geq 0 & x \in \mathbf{R}^n \setminus B_1. \end{cases}
$$

We will show that $w_{\lambda,u}, w_{\lambda,v} > 0$ in Σ_λ, if $\lambda \in (0,1)$ is close to 1. For this purpose, we use the truncation technique in **[10, 14]**. Assume that

$$
(3.3) \qquad \Sigma^-_{\lambda,v} \neq \emptyset,
$$

and define

$$
w^+_{\lambda,v}(x) = \begin{cases} w_{\lambda,v} & x \in \Sigma^-_{\lambda,v}, \\ 0 & x \in \mathbf{R}^n \setminus \Sigma^-_{\lambda,v}, \end{cases}
$$

$$
w^-_{\lambda,v}(x) = \begin{cases} 0 & x \in \Sigma^-_{\lambda,v}, \\ w_{\lambda,v} & x \in \mathbf{R}^n \setminus \Sigma^-_{\lambda,v}. \end{cases}
$$

It is obvious that $w_{\lambda,v}(x) = w^+_{\lambda,v}(x) + w^-_{\lambda,v}(x)$, for all $x \in \mathbf{R}^n$. By the definitions of fractional Laplacian operator and $w^-_{\lambda,v}$ for $x \in \Sigma^-_{\lambda,v}$, we obtain

(3.4)

$$
\begin{aligned}
(-\Delta)^\alpha w^-_{\lambda,v}(x) &= c(n,\alpha) \int_{R^n} \frac{w^-_{\lambda,v}(x) - w^-_{\lambda,v}(y)}{|x-y|^{n+2\alpha}}\, dy \\
&= -c(n,\alpha) \int_{R^n \setminus \Sigma^-_{\lambda,v}} \frac{w_{\lambda,v}(y)}{|x-y|^{n+2\alpha}}\, dy \\
&= -c(n,\alpha) \int_{(\Sigma_\lambda \setminus \Sigma^-_{\lambda,v}) \cup (\Sigma_\lambda \setminus \Sigma^-_{\lambda,v})_\lambda} \frac{w_{\lambda,v}(y)}{|x-y|^{n+2\alpha}}\, dy \\
&\quad - c(n,\alpha) \int_{(\Sigma^-_{\lambda,v})_\lambda} \frac{w_{\lambda,v}(y)}{|x-y|^{n+2\alpha}}\, dy \\
&\quad - c(n,\alpha) \int_{(B_1 \setminus (B_1)_\lambda) \cup ((B_1)_\lambda \setminus B_1)} \frac{w_{\lambda,v}(y)}{|x-y|^{n+2\alpha}}\, dy \\
&= -I_1 - I_2 - I_3.
\end{aligned}
$$

We estimate integrals I_i, $i = 1, 2, 3$, respectively, which yields

$$
\begin{aligned}
(3.5)\quad I_1 &= c(n,\alpha)\int_{\Sigma_\lambda\setminus\Sigma^-_{\lambda,v}} \frac{w_{\lambda,v}(y)}{|x-y|^{n+2\alpha}}\,dy \\
&\quad + c(n,\alpha)\int_{\Sigma_\lambda\setminus\Sigma^-_{\lambda,v}} \frac{w_{\lambda,v}(y^\lambda)}{|x-y^\lambda|^{n+2\alpha}}\,dy \\
&= c(n,\alpha)\int_{\Sigma_\lambda\setminus\Sigma^-_{\lambda,v}} w_{\lambda,v}(y)\left(\frac{1}{|x-y|^{n+2\alpha}} - \frac{1}{|x-y^\lambda|^{n+2\alpha}}\right)dy \\
&\geq 0,
\end{aligned}
$$

since $w_{\lambda,v}(y) \geq 0$ in $\Sigma_\lambda \setminus \Sigma^-_{\lambda,v}$ and $|x - y^\lambda| > |x - y|$ for $x \in \Sigma^-_{\lambda,v}$, $y \in \Sigma_\lambda \setminus \Sigma^-_{\lambda,v}$.

Considering $w_{\lambda,v}(y) \leq 0$ for $y \in \Sigma^-_{\lambda,v}$, we have

$$
\begin{aligned}
(3.6)\qquad I_2 &= c(n,\alpha)\int_{\Sigma^-_{\lambda,v}} \frac{w_{\lambda,v}(y^\lambda)}{|x-y^\lambda|^{n+2\alpha}}\,dy \\
&= -c(n,\alpha)\int_{\Sigma^-_{\lambda,v}} \frac{w_{\lambda,v}(y)}{|x-y^\lambda|^{n+2\alpha}}\,dy \geq 0.
\end{aligned}
$$

Observing $v = 0$ in $(B_1)_\lambda \setminus B_1$ and $v_\lambda = 0$ in $B_1 \setminus (B_1)_\lambda$, we obtain

(3.7)

$$
\begin{aligned}
I_3 &= c(n,\alpha)\int_{(B_1)_\lambda\setminus B_1} \frac{v_\lambda(y)}{|x-y|^{n+2\alpha}}\,dy \\
&\quad - c(n,\alpha)\int_{B_1\setminus(B_1)_\lambda} \frac{v(y)}{|x-y|^{n+2\alpha}}\,dy \\
&= c(n,\alpha)\int_{(B_1)_\lambda\setminus B_1} v_\lambda(y)\left(\frac{1}{|x-y|^{n+2\alpha}} - \frac{1}{|x-y^\lambda|^{n+2\alpha}}\right)dy \geq 0,
\end{aligned}
$$

since $v_\lambda(y) \geq 0$ in $(B_1)_\lambda \setminus B_1$ and $|x - y^\lambda| > |x - y|$ for $x \in \Sigma^-_{\lambda,v}$, $y \in (B_1)_\lambda \setminus B_1$.

Putting (3.5), (3.6) and (3.7) into (3.4), we have that, for $0 < \lambda < 1$, $x \in \Sigma^-_{\lambda,v}$,

$$(-\Delta)^\alpha w^-_{\lambda,v}(x) \leq 0.$$

Hence, for $x \in \Sigma^-_{\lambda,v}$,

$$\begin{aligned}(-\Delta)^\alpha w^+_{\lambda,v}(x) &= (-\Delta)^\alpha w_{\lambda,v}(x) - (-\Delta)^\alpha w^-_{\lambda,v}(x)\\ &\geq (-\Delta)^\alpha w_{\lambda,v}(x)\\ &= f(u_\lambda(x)) - f(u(x)) + g(x^\lambda) - g(x)\\ &\geq \frac{f(u_\lambda(x)) - f(u(x))}{u_\lambda(x) - u(x)} w_{\lambda,u}(x)\\ &= \varphi_u(x) w_{\lambda,u}(x),\end{aligned}$$

where $\varphi_u(x) = (f(u_\lambda(x)) - f(u(x)))/(u_\lambda(x) - u(x))$. It follows from Proposition 3.1 that

(3.8)
$$\|w^+_{\lambda,v}\|_{L^\infty(\Sigma^-_{\lambda,v})} \leq C\|(-\varphi_u w_{\lambda,u})^+\|^{1-\alpha}_{L^\infty(\Sigma^-_{\lambda,v})}\|(-\varphi_u w_{\lambda,u})^+\|^{\alpha}_{L^n(\Sigma^-_{\lambda,v})}.$$

Denote $\Sigma^-_\lambda = \Sigma^-_{\lambda,u} \cap \Sigma^-_{\lambda,v}$. Noting

$$-\varphi_u w_{\lambda,u}(x) = f(u(x)) - f(u_\lambda(x)) \begin{cases} \leq 0 & x \in R^n \setminus \Sigma^-_{\lambda,u},\\ > 0 & x \in \Sigma^-_{\lambda,u},\end{cases}$$

we have

$$\|w^+_{\lambda,v}\|_{L^\infty(\Sigma^-_{\lambda,v})} \leq C\|(-\varphi_u w_{\lambda,u})^+\|_{L^\infty(\Sigma^-_\lambda)}|\Sigma^-_\lambda|^{\alpha/n}.$$

Furthermore, we have

$$\|w_{\lambda,v}\|_{L^\infty(\Sigma^-_{\lambda,v})} \leq C\|w_{\lambda,u}\|_{L^\infty(\Sigma^-_\lambda)}|\Sigma^-_\lambda|^{\alpha/n}. \tag{3.9}$$

If $\Sigma^-_{\lambda,u} = \emptyset$, then (3.9) infers that $\Sigma^-_{\lambda,v}$ is empty, which is a contradiction with (3.3). If $\Sigma^-_{\lambda,u} \neq \emptyset$, then we have that, for $x \in \Sigma^-_{\lambda,u}$,

$$(-\Delta)w_{\lambda,u}(x) = w_{\lambda,v}(x).$$

Using Proposition 3.2, we obtain

$$\|w_{\lambda,u}\|_{L^\infty(\Sigma^-_{\lambda,u})} \leq C\|w_{\lambda,v}\|_{L^\infty(\Sigma^-_\lambda)}|\Sigma^-_\lambda|^{(1/n)}. \tag{3.10}$$

Inequalities (3.9) and (3.10) show that

$$\begin{aligned}\|w_{\lambda,v}\|_{L^\infty(\Sigma^-_{\lambda,v})} &\leq C\|w_{\lambda,v}\|_{L^\infty(\Sigma^-_\lambda)}|\Sigma^-_\lambda|^{(1+\alpha)/n}\\ &\leq C\|w_{\lambda,v}\|_{L^\infty(\Sigma^-_{\lambda,v})}|\Sigma^-_\lambda|^{(1+\alpha)/n}\end{aligned} \tag{3.11}$$

and

$$\|w_{\lambda,u}\|_{L^\infty(\Sigma^-_{\lambda,u})} \le C\|w_{\lambda,u}\|_{L^\infty(\Sigma^-_\lambda)}|\Sigma^-_\lambda|^{(1+\alpha)/n} \le C\|w_{\lambda,u}\|_{L^\infty(\Sigma^-_{\lambda,u})}|\Sigma^-_\lambda|^{(1+\alpha)/n}. \tag{3.12}$$

Now, we choose λ close to 1 such that $C|\Sigma^-_\lambda|^{(1+\alpha)/n} < 1$ and conclude that $\Sigma^-_{\lambda,v}$ is empty from (3.11). This is a contradiction to (3.3) and proves that $w_{\lambda,v} \ge 0$ in Σ_λ if λ is close to 1.

It follows from (3.10) that

$$\Sigma^-_{\lambda,u} = \emptyset. \tag{3.13}$$

Thus, $w_{\lambda,u} \ge 0$ in Σ_λ if λ is close to 1.

In order to finish Step 1, we will prove the following claim. *If $w_{\lambda,u}$, $w_{\lambda,v} \ge 0, w_{\lambda,u}$ or $w_{\lambda,v} \not\equiv 0$ in Σ_λ, then $w_{\lambda,u}, w_{\lambda,v} > 0$ in Σ_λ.*

By way of contradiction, suppose that $w_{\lambda,v} \not\equiv 0$ in Σ_λ, and there exist $x_0, y_0 \in \Sigma_\lambda$, such that

$$w_{\lambda,v}(x_0) = 0, \qquad w_{\lambda,u}(y_0) = 0.$$

On one hand, from the monotonicity hypothesis of f and g, we have

$$\begin{aligned}(-\Delta)^\alpha w_{\lambda,v}(x_0) &= (-\Delta)^\alpha v_\lambda(x_0) - (-\Delta)^\alpha v(x_0)\\ &= f(u_\lambda(x_0)) - f(u(x_0)) + g(x^\lambda) - g(x) \ge 0.\end{aligned} \tag{3.14}$$

On the other hand, defining $B = \{(x_1, x') \in \mathbf{R}^n | x_1 > \lambda\}$, and recalling $w_{\lambda,v}(x_0) = 0$, we can conclude

$$\begin{aligned}(-\Delta)^\alpha w_{\lambda,v}(x_0) &= -c(n,\alpha)\int_B \frac{w_{\lambda,v}(y)}{|x_0-y|^{n+2\alpha}}\,dy\\ &\quad - c(n,\alpha)\int_{\mathbf{R}^n\setminus B} \frac{w_{\lambda,v}(y)}{|x_0-y|^{n+2\alpha}}\,dy\\ &= -c(n,\alpha)\int_B \frac{w_{\lambda,v}(y)}{|x_0-y|^{n+2\alpha}}\,dy\\ &\quad - c(n,\alpha)\int_B \frac{w_{\lambda,v}(y^\lambda)}{|x_0-y^\lambda|^{n+2\alpha}}\,dy\\ &= -c(n,\alpha)\int_B w_{\lambda,v}(y)\left(\frac{1}{|x_0-y|^{n+2\alpha}} - \frac{1}{|x_0-y^\lambda|^{n+2\alpha}}\right)dy.\end{aligned} \tag{3.15}$$

Since $|x_0 - y| < |x_0 - y^\lambda|$, $w_{\lambda,v}(y) \geq 0$ and $w_{\lambda,v}(y) \not\equiv 0$ for $y \in \Sigma_\lambda$, we see

$$(-\Delta)^\alpha w_{\lambda,v}(x_0) < 0,$$

which is impossible by (3.14). Therefore, $w_{\lambda,v} > 0$ in Σ_λ for $0 < \lambda < 1$.

Noting that $w_{\lambda,u}(x)$ arrives at the minima at y_0, and applying the maximum principle, we have

$$(-\Delta) w_{\lambda,u}(y_0) \leq 0; \tag{3.16}$$

however, from (3.2), it follows that

$$(-\Delta) w_{\lambda,u}(y_0) = (-\Delta) u_\lambda(y_0) - (-\Delta) u(y_0) = v_\lambda(y_0) - v(y_0) > 0,$$

which is impossible by (3.16). Thus, $w_{\lambda,u} > 0$ in Σ_λ for $0 < \lambda < 1$.

Assume that $w_{\lambda,u} \not\equiv 0$ in Σ_λ and there exist $\xi, \eta \in \Sigma_\lambda$ such that

$$w_{\lambda,u}(\xi) = 0, \qquad w_{\lambda,v}(\zeta) = 0.$$

Using (3.2) and the maximum principle, we conclude that $w_{\lambda,u} \equiv 0$ in Σ_λ, which is impossible since $u > 0$ in B_1 and $u = 0$ in $\mathbb{R}^n \setminus B_1$. Thus, $w_{\lambda,u} > 0$ in Σ_λ. Deducing similarly to (3.14) and (3.15), we obtain that $(-\Delta)^\alpha w_{\lambda,v}(\zeta) \geq 0$ and $(-\Delta)^\alpha w_{\lambda,v}(\zeta) < 0$. Therefore, $w_{\lambda,v} > 0$ in Σ_λ for $0 < \lambda < 1$. The proof of the claim is complete.

Step 2. Let

$$\lambda_0 = \inf\{\lambda \in (0,1) \mid w_{\lambda,u}(x), w_{\lambda,v}(x) > 0, x \in \Sigma_\lambda\}.$$

We will prove that $\lambda_0 = 0$. If not, then $\lambda_0 > 0$. We know that $w_{\lambda_0,u}(x)$, $w_{\lambda_0,v}(x) \geq 0$ and $w_{\lambda_0,u}(x), w_{\lambda_0,v}(x) \not\equiv 0$ for $x \in \Sigma_{\lambda_0}$. The claim in Step 1 infers that $w_{\lambda_0,u}(x), w_{\lambda_0,v}(x) > 0$. For $0 < \mu < 1$ sufficiently small, set

$$D_\mu = \{x \in \Sigma_{\lambda_0} \mid \operatorname{dist}(x, \Sigma_{\lambda_0}) \geq \mu\}.$$

Then, there exists an $m_0 > 0$ such that, for $x \in D_\mu$,

$$w_{\lambda_0,u}(x), w_{\lambda_0,v}(x) \geq m_0.$$

By continuity of $w_{\lambda,u}, w_{\lambda,v}$ with respect to λ, we have that, for $\epsilon > 0$ small enough, and $x \in D_\mu$,

$$w_{\lambda_0-\varepsilon,u}(x), w_{\lambda_0-\epsilon,v}(x) \geq 0.$$

Since

$$\Sigma^-_{\lambda_0-\varepsilon,u}, \Sigma^-_{\lambda_0-\varepsilon,v} \subset \Sigma_{\lambda_0-\varepsilon} \setminus D_\mu,$$

for μ and λ sufficiently small, we use the same argument in Step 1 to prove that

$$\Sigma^-_{\lambda_0-\varepsilon,u}, \Sigma^-_{\lambda_0-\varepsilon,v} = \emptyset,$$

that is,

$$w_{\lambda_0-\varepsilon,u}(x), w_{\lambda_0-\epsilon,v}(x) \geq 0$$

for $x \in \Sigma_{\lambda_0-\varepsilon}$. This is a contradiction with the definition of λ_0. Hence, $\lambda_0 = 0$, which implies that

$$u(-x_1, x') \geq u(x_1, x')$$

and

$$v(-x_1, x') \geq v(x_1, x')$$

for $x_1 > 0$. Similarly, we can move the plane T_λ from -1 to the right and obtain

$$u(x_1, x') \geq u(-x_1, x') \quad \text{and} \quad v(x_1, x') \geq v(-x_1, x')$$

for $x_1 > 0$. Since the x_1 direction can be arbitrarily changed, we have shown that $u(x)$ and $v(x)$ are radially symmetric.

For $0 < x_1 < \overline{x}_1 < 1$, let $\lambda = (x_1 + \overline{x}_1)/2$. Then, $w_{\lambda,u}(x), w_{\lambda,v}(x) > 0$ in Σ_λ, and thus, $u(x_1, x') > u(\overline{x}_1, x')$ and $v(x_1, x') > v(\overline{x}_1, x')$. Hence, monotonicity of u and v follow from the radial symmetry. $\square$

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