ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 48, Number 2, 2018

RAMANUJAN-NAGELL CUBICS

MARK BAUER AND MICHAEL A. BENNETT

ABSTRACT. A well-known result of Beukers **[3]** on the generalized Ramanujan-Nagell equation has, at its heart, a lower bound on the quantity $|x^2 - 2^n|$. In this paper, we derive an inequality of the shape $|x^3 - 2^n| \geq x^{4/3}$, valid provided $x^3 \neq 2^n$ and $(x, n) \neq (5, 7)$, and then discuss its implications for a variety of Diophantine problems.

1. Introduction. Surfing the internet one day, the second author came across a conversation on a physics forum **[10]**, in which a Diophantine problem was proposed. The proposer wished to find a proof of his conjecture, to the effect that the Diophantine equation

$$x^3 - x + 8 = 2^k \tag{1.1}$$

has no solutions in integers $x > 8$. Since it, indeed, has solutions with

$$x \in \{-2, -1, 0, 1, 3, 5, 8\},$$

such a result would be, in some sense, best possible.

That equations such as (1.1) have at most finitely many solutions is an immediate consequence of a classical result of Siegel **[12]**, and, in fact, if we denote by $P(m)$ the greatest prime factor of a nonzero integer m, it may be shown **[7]** that

$$P(f(x)) \geq c_1 \cdot \log\log\max\{|x|, 3\}. \tag{1.2}$$

Here, $c_1 = c_1(f) > 0$ and f is, say, an irreducible polynomial with integer coefficients and degree at least two. Further, if $F(x, y)$ is an irreducible binary form, i.e., homogeneous polynomial, with integer coefficients and degree at least 3, work of Mahler **[8]**, as extended by

---

2010 AMS *Mathematics subject classification.* Primary 11A63, Secondary 11D61, 11J68.

Keywords and phrases. Polynomial-exponential equations, digital problems.

This research was supported in part by grants from the NSERC.

Received by the editors on November 4, 2016, and in revised form on February 6, 2017.

DOI:10.1216/RMJ-2018-48-2-385

Shorey, et al., **[11]** implies that

$$P(F(x,y)) \geq c_2 \cdot \log\log\max\{|x|,|y|,3\}, \tag{1.3}$$

with $c_2 = c_2(F) > 0$, so that, given primes $p_1, p_2, \ldots, p_n$, the Thue-Mahler equation

$$F(x,y) = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_n^{\alpha_n} \tag{1.4}$$

has at most finitely many solutions in coprime integers x and y and nonnegative integers $\alpha_1, \ldots, \alpha_n$. In particular, the equation

$$x^3 - xy^2 + 8y^3 = 2^k$$

can be satisfied by at most finitely many coprime integers x and y and nonnegative k; those with $y = 1$ provide the solutions to (1.1). Statements (1.2) and (1.3) can both be made effective. The interested reader is directed to **[13]** for details.

The motivation for studying such an equation in **[10]** was, apparently, to find a cubic analog of the Ramanujan-Nagell equation $x^2 + 7 = 2^k$, which is known **[9]** to have precisely the integer solutions corresponding to $|x| = 1, 3, 5, 11$ and 181. This is extremal in the sense that there exists no monic quadratic $f(x)$ for which $f(x) = 2^k$ has more than ten solutions in integers x, via the following theorem of Beukers **[3]**.

Theorem 1.1. **[3]**. *If D is an odd integer, then the equation*

$$x^2 + D = 2^n$$

has at most five solutions in positive integers x.

The only monic irreducible cubics f we know for which the equation $f(x) = 2^k$ has more solutions than the seven to (1.1) are those corresponding to the polynomial $f(x) = x^3 - 13x + 20$ and its translates, each with 8 solutions. The results of this paper make it a routine matter to solve such an equation for any monic cubic (and the machinery we employ readily generalizes to certain non-monic cases, although we omit the details in the interest of keeping our exposition reasonably simple). An example of what we prove is the following.

Proposition 1.2. *Let b and c be integers, and suppose that x and k are integers for which*

$$|x^3 + bx + c| = 2^k. \tag{1.5}$$

Then, either $b = c = 0$, $x = -c/b$, $(x, b, c, k) = (5, 0, 3, 7)$, $(-5, 0, -3, 7)$, or we have

$$|x| \leq \max\{|2b|^3, |2c|^{3/4}\}. \tag{1.6}$$

Applying this to equation (1.1) implies that its solutions satisfy $|x| \leq 8$, whereby a routine check leads to the desired conclusion.

Our real motivation in writing this paper is to emphasize the unnaturally large influence a single numerical "fluke" can have upon certain results in explicit Diophantine approximation. Underlying [**3,** Theorem 1.1] is an inequality of the shape $|x^2 - 2^n| \gg 2^{0.1n}$, valid for odd n and derived through Padé approximation through appeal to the identity $181^2 + 7 = 2^{15}$ (which implies that $|\sqrt{2} - (181/2^7)|$ is "small"). Proposition 1.2 follows from a rather similar approach and fundamentally depends upon the relation $5^3 + 3 = 2^7$ (which implies that $|\sqrt[3]{2} - (5/2^2)|$ is also "small"). The fact that we are able to prove an effective inequality of the shape

$$\left|\sqrt[3]{2^r} - \frac{p}{2^k}\right| \gg 2^{-\lambda k},$$

for $r \in \{1, 2\}$ and, crucially, $\lambda < 2$, is what enables us to derive results like Proposition 1.2. It is worth observing that techniques based upon lower bounds for linear forms in logarithms lead to upper bounds for the heights of solutions to much more general equations than (1.5), but have the disadvantage of these bounds being significantly worse than exponential in the coefficients b and c.

The outline of this paper is as follows. In Section 2, we begin by stating our results, expressed both in terms of explicit rational approximation to certain algebraic numbers by rational numbers with restricted denominators, and also as results about corresponding Diophantine equations. In Sections 3 and 4, we collect the necessary technical lemmata regarding Padé approximation to binomial functions (at least in terms of Archimedean valuations). Section 5 contains the proof of our main result, Theorem 2.1, modulo an arithmetic result on the coefficients of our Padé approximants, which we provide in Section 6.

Sections 7, 8 and 9 consist of the proofs of Corollary 2.2, Theorems 2.3 and 2.4, respectively. Finally, in Section 10, we make a few remarks concerning more general Thue-Mahler equations.

2. Statements of our results. Our results are of two closely related types. First, we have an explicit lower bound for approximation to $\sqrt[3]{2}$ or $\sqrt[3]{4}$ by rational numbers with denominators restricted to being essentially a power of 2. From this, with basically elementary arguments, we are able to derive a number of results on both the heights and the number of solutions of equations of the shape $F(x) = 2^n$, for monic cubic polynomial $F \in \mathbb{Z}$. We begin with the following:

Theorem 2.1. *Suppose that r, p, s and k are integers with $r \in \{1, 2\}$, $s \in \{1, 3\}$ and $k > 12$. Then, we have*

$$\left| 2^{r/3} - \frac{p}{s2^k} \right| > 2^{-1.62k}. \tag{2.1}$$

As noted earlier, this "restricted irrationality measure" to $\sqrt[3]{2}$ and $\sqrt[3]{4}$ has a number of straightforward consequences for certain Diophantine equations. We will begin by stating an almost immediate corollary that will prove a useful form for later applications to Diophantine equations.

Corollary 2.2. *If x and n are integers with $x^3 \neq 27 \cdot 2^n$, then either*

$$x \in \{4, 5, 8, 15, 19, 38, 121\}$$

or

$$|x^3 - 27 \cdot 2^n| \geq 3^{5/3} \cdot x^{4/3}. \tag{2.2}$$

From the standpoint of explicit solution of Diophantine equations, our main result is the following (from which Proposition 1.2 is an immediate corollary, taking $a = 0$).

Theorem 2.3. *Let a, b and c be integers, and suppose that x and n are integers for which we have*

$$|x^3 + ax^2 + bx + c| = 2^n. \tag{2.3}$$

Then, either

- $(a, b, c) = (3t, 3t^2, t^3)$ *for some integer* t, *or*
- $(a, b, c, x, n) = (3t, 3t^2, t^3+3, 5-t, 7)$ *or* $(3t, 3t^2, t^3-3, -5-t, 7)$, *for some integer* t, *or*
- $x = (a^3 - 27c)/(27b - 9a^2)$,

or we have

$$|x| \leq \max\{8|b-a^2/3|^3 + |a/3|, |4a^3/27 + 2c - 2ab/3|^{3/4} + |a/3|\}. \tag{2.4}$$

Finally, as an analog of Theorem 1.1, we have

Theorem 2.4. *Let* D *be an odd integer. Then, the number of solutions to the equation*

$$x^3 + D = 2^n \tag{2.5}$$

in pairs of integers (x, n) *is at most three.*

This last result is sharp since equation (2.5) with $D = 3$ has precisely three solutions $(x, n) = (-1, 1)$, $(1, 2)$ and $(5, 7)$. In fairness, we should point out that this is not an analog of comparable generality to Theorem 1.1 in that the latter provides an upper bound for the number of solutions to the equation $f(x) = 2^n$, for all monic quadratic polynomials f, while the same is not true of Theorem 2.4 for monic cubics.

3. Padé approximants to $(1-x)^\nu$**.** All of our results in this paper have, at their heart, Padé approximations to the algebraic function $(1-x)^\nu$, where $\nu \in \mathbb{Q}/\mathbb{Z}$. Recall that an $[n_1/n_2]$-Padé approximant to a function $f(x)$ is a rational function $p(x)/q(x)$, where the numerator and denominator are polynomials with, say, integer coefficients, of degrees n_1 and n_2, respectively, such that $f(x)$ and $p(x)/q(x)$ have the same MacLaurin series expansion up to degree $n_1 + n_2$, i.e., such that $p(x)/q(x)$ is a good approximation to $f(x)$ in a neighborhood of $x = 0$.

Since the function $p(x)/q(x)$ is a rational function with rational coefficients, it takes rational values for rational choices of its argument. In this way, we will be able to take a single suitably good approximation to, in our case, a particular algebraic number, and generate an infinite sequence of “good” approximations to the same number. In order to obtain sharp estimates for the quality of the approximations that are generated with these functions, we will use representations coming from contour integrals, as well as explicit descriptions of the Padé approximants as polynomials. Define

$$I_{n_1,n_2}(x) = \frac{1}{2\pi i}\int_\gamma \frac{(1-zx)^{n_2}(1-zx)^\nu}{z^{n_1+1}(1-z)^{n_2+1}}\,dz,$$

where n_1 and n_2 are positive integers, γ is a closed, positively oriented contour enclosing $z=0$ and $z=1$, and $|x|<1$. A straightforward application of Cauchy's theorem yields that

$$I_{n_1,n_2}(x) = P_{n_1,n_2}(x) - (1-x)^\nu Q_{n_1,n_2}(x), \tag{3.1}$$

where $P_{n_1,n_2}(x)$ and $Q_{n_1,n_2}(x)$ are polynomials with rational coefficients of degrees n_1 and n_2, respectively. In fact, examining the relevant residues, it is possible to show that

$$P_{n_1,n_2}(x) = \sum_{k=0}^{n_1}\binom{n_2+\nu}{k}\binom{n_1+n_2-k}{n_2}(-x)^k \tag{3.2}$$

and

$$Q_{n_1,n_2}(x) = \sum_{k=0}^{n_2}\binom{n_1-\nu}{k}\binom{n_1+n_2-k}{n_1}(-x)^k. \tag{3.3}$$

In particular, if we choose $\nu \in \{1/3, 2/3\}$, we have that $P_{n_1,n_2}(x)$, $Q_{n_1,n_2}(x) \in \mathbb{Z}[1/3][x]$.

Our goal in the following sections will be twofold. First, we will derive estimates for the size of $|I_{n_1,n_2}(x)|$ and $|P_{n_1,n_2}(x)|$ using contour integral representations. Second, we will find lower bounds for the size of the greatest common divisor of the coefficients involved in $|P_{n_1,n_2}(x)|$ and $|Q_{n_1,n_2}(x)|$.

4. Bounding our approximants. Here and henceforth, given a positive integer n_1, let us set $n_2 = 4n_1 - \delta$, where $\delta \in \{0,1\}$, and write $\nu = r/3$, where $r \in \{1,2\}$. Define

$$P_{n_1,\delta} = P_{n_1,4n_1-\delta}(3/128),$$

$$Q_{n_1,\delta} = Q_{n_1,4n_1-\delta}(3/128)$$

and

$$I_{n_1,\delta} = I_{n_1,4n_1-\delta}\,(3/128),$$

so that, from (3.1),

$$I_{n_1,\delta} = P_{n_1,\delta} - \left(\frac{5}{2^{7/3}}\right)^r Q_{n_1,\delta}. \tag{4.1}$$

We further define $F(z)$ by

$$F(z) = \frac{(1-3z/128)^4}{z(1-z)^4}. \tag{4.2}$$

It is easy to show, via calculus, that $F(z)$ attains its minimum for $z \in (0,1)$ at the value

$$\tau = \frac{1}{6}\big(631 - 5\sqrt{15865}\big) = 0.20304\ldots, \tag{4.3}$$

where we have

$$\begin{aligned} F := F(\tau) &= \frac{243\big(75-\sqrt{15865}\big)^4}{34359738368\big(631-5\sqrt{15865}\big)\big(125-\sqrt{15865}\big)^4} \\ &= 11.97804\ldots. \end{aligned}$$

Our argument will require upper bounds upon $|P_{n_1,\delta}|$, $|Q_{n_1,\delta}|$ and $|I_{n_1,\delta}|$; from (4.1), it suffices to bound one of the first two of these, together with the last.

Lemma 4.1. *We have*

$$|P_{n_1,\delta}| < 1.26 \cdot F^{n_1} \quad \textit{and} \quad |I_{n_1,\delta}| < \frac{2\sqrt{3}}{\pi}\left(2^{35}\,3^{-5}\,F\right)^{-n_1}.$$

Proof. We begin by separating the integral defining $I_{n_1,\delta}(3/128)$ into two pieces, one involving a closed contour containing only the pole at $z=0$. Taking τ as in (4.3), we may write

$$P_{n_1,\delta} = \frac{1}{2\pi i}\int_\Gamma \frac{(1-3z/128)^{4n_1-\delta+r/3}}{z^{n_1+1}(1-z)^{4n_1-\delta+1}}\,dz,$$

where Γ is the closed, positively oriented contour with $|z| = \tau$. Using the transformation $z = \tau e^{i\theta}$, we have that

$$|P_{n_1,\delta}| \leq \frac{1}{2\pi} \int_0^{2\pi} \left| \frac{(1-3z/128)^{4n_1-\delta+r/3}}{z^{n_1+1}(1-z)^{4n_1-\delta+1}} \right| \tau \, d\theta$$

and, thus,

$$|P_{n_1,\delta}| \leq \frac{1}{\tau^{n_1}} \max_{0\leq\theta\leq 2\pi} \left| \frac{(1-3\tau e^{i\theta}/128)^{4n_1-\delta+r/3}}{(1-\tau e^{i\theta})^{4n_1-\delta+1}} \right|,$$

whereby

$$|P_{n_1,\delta}| \leq \frac{(1-3\tau/128)^{r/3}}{\tau^{n_1}(1-\tau)} \left(\frac{1-3\tau/128}{1-\tau} \right)^{4n_1-\delta}.$$

Since $r \in \{1, 2\}$, the desired bound for $|P_{n_1,\delta}|$ follows.

In order to bound $|I_{n_1,\delta}|$, we argue as in **[1]** to arrive at the identity

$$(4.4) \quad |I_{n_1,\delta}| = \frac{(3/128)^{5n_1-\delta+1}}{\pi} \sqrt{3}/2 \int_0^1 \frac{v^{4n_1-\delta+r/3}(1-v)^{n_1-r/3}dv}{(1-3(1-v)/128)^{4n_1-\delta+1}}.$$

We may thus rewrite the integrand here as

$$f_{\delta,r}(v) \left(\frac{v^4(1-v)}{(1-3(1-v)/128)^4} \right)^{n_1-1}$$

where

$$f_{\delta,r}(v) = \frac{v^{4+r/3-\delta}(1-v)^{1-r/3}}{(1-3(1-v)/128)^{5-\delta}},$$

$\delta \in \{0, 1\}$ and $r \in \{1, 2\}$. Changing variables via $v = 1 - z$, we thus have

$$|I_{n_1,\delta}| \leq \frac{(3/128)^{5n_1+1-\delta}}{\pi} \sqrt{3}/2 \max\{f_{\delta,r}(v) : v \in (0,1)\} \, F^{1-n_1}.$$

Since a little calculus reveals that, in all cases, $\max\{f_{\delta,r}(v) : v \in (0, 1)\} < 1/3$ and since we have $F < 12$, it follows that

$$|I_{n_1,\delta}| < \frac{2\sqrt{3}}{\pi} (2^{35} \, 3^{-5} \, F)^{-n_1},$$

as desired. $\square$

5. Proof of a (restricted) irrationality measure: Theorem 2.1. We suppose that $r \in \{1,2\}$, $s \in \{1,3\}$, p and k are integers, and write $m = 3k + r$. From (3.2), (3.3) and the fact that

$$\binom{n \pm r/3}{j} 3^{[3j/2]} \in \mathbb{Z}$$

for all positive integers n and j, it follows that

$$3^{[n_1/2]} \, 2^{7n_1} P_{n_1,\delta} \quad \text{and} \quad 3^{2n_1-\delta} \, 2^{7(4n_1-\delta)} Q_{n_1,\delta}$$

are both integers. Here, n_1 is a positive integer to be chosen later. We set

$$\Pi_{n_1,\delta,r} = \gcd\{3^{[n_1/2]} \, 2^{7n_1} P_{n_1,\delta}, 3^{2n_1-\delta} \, 2^{28n_1-7\delta} Q_{n_1,\delta}\}$$

such that

$$A = \frac{3^{[n_1/2]} \, 2^{7n_1}}{\Pi_{n_1,\delta,r}} P_{n_1,\delta} \quad \text{and} \quad B = \frac{3^{2n_1-\delta} \, 2^{28n_1-7\delta}}{\Pi_{n_1,\delta,r}} Q_{n_1,\delta}$$

are integers. Equation (3.1) therefore implies that

$$\Pi^{-1}_{n_1,\delta,r} \, |I_{n_1,\delta}| = \left| \frac{A}{3^{[n_1/2]} \, 2^{7n_1}} - \left(\frac{5}{2^{7/3}}\right)^r \frac{B}{3^{2n_1-\delta} \, 2^{28n_1-7\delta}} \right|.$$

If we define

$$\Omega = \left| 1 - \frac{p}{s \, 2^{m/3}} \right|$$

and

$$\Lambda = \left| \frac{p}{s \, 2^{m/3}} - \left(\frac{5}{2^{7/3}}\right)^r \frac{B}{A \, 2^{7(3n_1-\delta)} \, 3^{2n_1-[n_1/2]-\delta}} \right|,$$

we have that

$$\Lambda < \Omega + \Pi^{-1}_{n_1,\delta,r} \, A^{-1} \, 3^{[n_1/2]} \, 2^{7n_1} \, |I_{n_1,\delta}|$$

(note that the nonvanishing of A is a consequence of the contour integral representation for $P_{n_1,n_2}(x)$). From [**3,** Lemma 4], for one of our two choices of $\delta \in \{0,1\}$, we have $\Lambda \neq 0$; we fix δ to be this value, and choose

$$n_1 = 1 + \left[\frac{m - 7r + 21\delta}{63} \right], \tag{5.1}$$

so that

$$7r/3 + 7(3n_1 - \delta) > m/3.$$

Since $s \mid 3$, we thus have

$$\Lambda \geq (A\, 2^{7(3n_1-\delta+r/3)}\, 3^{2n_1-[n_1/2]-\delta})^{-1}.$$

Combining our upper and lower bounds for Λ, we find that

$$1 < \Omega\Lambda_1 + \Lambda_2, \tag{5.2}$$

where, upon substituting for A,

$$\Lambda_1 = \Pi_{n_1,\delta,r}^{-1}\, 3^{2n_1-\delta}\, 2^{7(4n_1-\delta+r/3)}\, |P_{n_1,\delta}|$$

and

$$\Lambda_2 = \Pi_{n_1,\delta,r}^{-1}\, 3^{2n_1-\delta}\, 2^{7(4n_1-\delta+r/3)}\, |I_{n_1,\delta}|.$$

Applying Lemma 4.1, we thus have

$$\Lambda_1 < 1.26\, \Pi_{n_1,\delta,r}^{-1}\, 2^{14/3}\, (3^2\, 2^{28}\, F)^{n_1}$$

and

$$\Lambda_2 < \frac{2\sqrt{3}}{\pi}\, \Pi_{n_1,\delta,r}^{-1}\, 2^{14/3} \left(\frac{3^7}{2^7\, F}\right)^{n_1},$$

whereby, from (5.2),

$$\Omega = \left|1 - \frac{p}{s\, 2^{m/3}}\right| > \frac{\Pi_{n_1,\delta,r} - (2\sqrt{3}/\pi)\, 2^{14/3}\, (3^7/(2^7\, F))^{n_1}}{1.26\, 2^{14/3}\, (3^2\, 2^{28}\, F)^{n_1}}. \tag{5.3}$$

In order for inequality (5.3) to be nontrivial, it remains therefore to show that

$$\liminf_{n_1 \to \infty} \frac{1}{n_1} \log \Pi_{n_1,\delta,r} > \log\left(\frac{3^7}{2^7\, F}\right). \tag{5.4}$$

In the next section, we will in fact prove the following result.

Proposition 5.1. *For $r \in \{1,2\}$, $n_1 \geq 497$ and $\delta \in \{0,1\}$, we have that*

$$\Pi_{n_1,\delta,r} > 1.8^{n_1}.$$

Since we have that $3^7/(2^7 F) < 1.427$, inequality (5.4) follows as desired. Assuming Proposition 5.1, and further, that $k \geq 11000$ so that $m \geq 33000$, (5.1) thus implies that $n_1 \geq 497$. We, therefore, have

$$\Pi_{n_1,\delta,r} - \frac{2\sqrt{3}}{\pi} 2^{14/3} \left(\frac{3^7}{2^7 F}\right)^{n_1} > 1.79^{n_1},$$

and hence, from (5.3),

$$\left|1 - \frac{p}{s\, 2^{m/3}}\right| > 16166467234^{-n_1}.$$

From (5.1), we have that $n_1 \leq (m+77)/63$, and thus,

$$\left|1 - \frac{p}{s\, 2^{m/3}}\right| > 16166467234^{-(m+77)/63} > 1.454^{-m},$$

where we have appealed to the fact that $m \geq 33000$. Since $m = 3k + r$, we thus have

$$\left|2^{r/3} - \frac{p}{s2^k}\right| > 2^{r/3}\, 1.454^{-3k-r} > 2^{-1.62k},$$

for $r \in \{1, 2\}$, $k, p \in \mathbb{Z}$ and $s \in \{1, 3\}$.

For $k \leq 100$, a brute force search shows that the only approximations in this range that fail to satisfy the desired bound (2.1) have (r, s, p, k) in the following list (where we assume that $\gcd(p, s2^k) = 1$ to avoid redundancy):

$$\begin{gathered}(1,1,1,0), (1,1,2,0), (1,1,3,1), (1,1,5,2), (1,3,4,0), (1,3,7,1),\\ (1,3,31,3), (1,3,61,4), (1,3,121,5), (2,1,1,0), (2,1,2,0),\\ (2,1,3,1), (2,1,3251,11), (2,3,5,0), (2,3,19,2), (2,3,305,6).\end{gathered}$$

In particular, in every case, we have $k < 12$.

All that remains is to verify the inequality for, say, $100 < k < 11000$. We do this by considering the binary expansions of $2^{r/3}$ and $3 \cdot 2^{r/3}$, for $r \in \{1, 2\}$, and searching for either unusually long strings of zeros or unusually long strings of ones (each of which would correspond to a very good approximation to $2^{r/3}$ or $3 \cdot 2^{r/3}$ by a rational with denominator a power of two). Such an argument is described in detail in [**1,** Lemma 9.1]. The fact that no such strings occur completes the proof of Theorem 2.1 (again, assuming Proposition 5.1).

6. Arithmetical properties of the coefficients. We now turn our attention to proving Proposition 5.1. In order to do this, we first require a good understanding of the content of the polynomials $P_{n_1,n_2}(x)$ and $Q_{n_1,n_2}(x)$.

Lemma 6.1. *Let n_1 be a positive integer, $n_2 = 4n_1 - \delta$ for $\delta \in \{0,1\}$ and $r \in \{1,2\}$. Suppose that p is prime, with*

$$p > \max\{\sqrt{3n_2+2}, 5\}. \tag{6.1}$$

Assume that

$$\left\{\frac{n_1}{p}\right\} \in \left(\frac{2}{3}, \frac{3}{4}\right) \cup \left(\frac{5}{6}, 1\right), \tag{6.2}$$

if $p \equiv r \pmod 3$, or

$$\left\{\frac{n_1}{p}\right\} \in \left(\frac{5}{12}, \frac{1}{2}\right) \cup \left(\frac{2}{3}, \frac{3}{4}\right) \cup \left(\frac{11}{12}, 1\right), \tag{6.3}$$

if $p \equiv -r \pmod 3$. Then, we have

$$\mathrm{ord}_p \binom{n_2 + r/3}{k} \binom{n_1 + n_2 - k}{n_2} \geq 1 \quad \textit{for } 0 \leq k \leq n_1$$

and

$$\mathrm{ord}_p \binom{n_1 - r/3}{k} \binom{n_1 + n_2 - k}{n_1} \geq 1 \quad \textit{for } 0 \leq k \leq n_2.$$

Proof. We begin by considering the case when $p \equiv r \pmod 3$. Observe that, from (6.2), we have that either

$$\left\{\frac{n_1}{p}\right\} \in \left[\frac{2}{3} + \frac{r}{3p}, \frac{3}{4}\right),$$

whereby

$$\left\{\frac{n_2}{p}\right\} = \left\{\frac{4n_1 - \delta}{p}\right\} \in \left[\frac{2}{3} + \frac{4r-3}{3p}, 1\right), \tag{6.4}$$

or that

$$\left\{\frac{n_1}{p}\right\} \in \left[\frac{5}{6} + \frac{4r-3}{6p}, 1\right),$$

whence

$$\left\{\frac{n_2}{p}\right\} = \left\{\frac{4n_1-\delta}{p}\right\} \in \left[\frac{1}{3}+\frac{8r-9}{3p}, 1\right). \tag{6.5}$$

We conclude, in either case, that

$$\left\{\frac{n_1}{p}\right\} + \left\{\frac{n_2}{p}\right\} \geq \frac{7}{6} + \frac{20r-21}{6p} \geq \frac{7}{6} - \frac{1}{6p} \geq 1 + \frac{1}{p}, \tag{6.6}$$

where the last inequality follows from the assumption that $p \geq 7$. We proceed to show that p satisfying the hypotheses of the lemma have positive valuation for the desired binomial coefficients. If $k=0$, then

$$\binom{n_2+r/3}{k}\binom{n_1+n_2-k}{n_2} = \binom{n_1-r/3}{k}\binom{n_1+n_2-k}{n_1} = \binom{n_1+n_2}{n_1}$$

and, since $n_1 < n_2$, our assumption that $p^2 > 3n_2+2 > 3n_2$ implies that

$$\operatorname{ord}_p \binom{n_1+n_2}{n_1} = \left\{\frac{n_1}{p}\right\} + \left\{\frac{n_2}{p}\right\} - \left\{\frac{n_1+n_2}{p}\right\}.$$

It follows that

$$\operatorname{ord}_p \binom{n_1+n_2}{n_1} \geq 1$$

if and only if $\{n_1/p\} + \{n_2/p\} \geq 1$, whereby, from (6.6), we conclude as desired.

Similarly, if $k=1$, then the fact that

$$\operatorname{ord}_p \binom{n_2+r/3}{k}\binom{n_1+n_2-k}{n_2} \geq 1$$

is a consequence of the inequalities

$$\left\{\frac{n_1-1}{p}\right\} + \left\{\frac{n_2}{p}\right\} \geq \left\{\frac{n_1}{p}\right\} + \left\{\frac{n_2}{p}\right\} - \frac{1}{p} \geq 1,$$

while

$$\operatorname{ord}_p \binom{n_1-r/3}{k}\binom{n_1+n_2-k}{n_1} \geq 1$$

follows from

$$\left\{\frac{n_1}{p}\right\} + \left\{\frac{n_2-1}{p}\right\} \geq \left\{\frac{n_1}{p}\right\} + \left\{\frac{n_2}{p}\right\} - \frac{1}{p} \geq 1.$$

Next, we suppose that $k \geq 2$. From Chudnovsky [**4,** Lemma 4.5], if $n \in \mathbb{N}$ and $p^2 > 3n + r$, we have

$$\operatorname{ord}_p \binom{n + r/3}{k} = \left[\frac{n-q}{p}\right] - \left[\frac{n-k-q}{p}\right] - \left[\frac{k}{p}\right]$$

where $q = (p - r)/3$. It follows that

$$\operatorname{ord}_p \binom{n_2 + r/3}{k} = \left\{\frac{n_2 - q - k}{p}\right\} + \left\{\frac{k}{p}\right\} - \left\{\frac{n_2 - q}{p}\right\} \tag{6.7}$$

and

$$\operatorname{ord}_p \binom{n_1 + n_2 - k}{n_2} = \left\{\frac{n_2}{p}\right\} + \left\{\frac{n_1 - k}{p}\right\} - \left\{\frac{n_1 + n_2 - k}{p}\right\}. \tag{6.8}$$

Let us suppose that $\operatorname{ord}_p \binom{n_2+r/3}{k}\binom{n_1+n_2-k}{n_2} = 0$ and seek to derive a contradiction. From (6.7), we have $\{(n_2 - q)/p\} \geq \{k/p\}$ which, with (6.4) and (6.5), implies that

$$\left\{\frac{n_2 - q}{p}\right\} = \left\{\frac{n_2}{p}\right\} - \frac{q}{p} = \left\{\frac{n_2}{p}\right\} - \frac{p-r}{3p} \geq \left\{\frac{k}{p}\right\}. \tag{6.9}$$

Similarly, from (6.8), we have

$$\left\{\frac{n_2}{p}\right\} + \left\{\frac{n_1 - k}{p}\right\} < 1, \tag{6.10}$$

i.e.,

$$\left\{\frac{n_2}{p}\right\} + \left\{\frac{n_1 - k}{p}\right\} \leq 1 - \frac{1}{p}.$$

If $\{n_1/p\} < \{k/p\}$, since $\{n_1/p\} \geq 2/3 + 1/(3p)$, we have from (6.9), that

$$\left\{\frac{n_2}{p}\right\} > \frac{2}{3} + \frac{1}{3p} + \frac{p-r}{3p} = 1 + \frac{1-r}{3p} \geq 1 - \frac{1}{3p},$$

an immediate contradiction. It follows that $\{n_1/p\} \geq \{k/p\}$, and hence,

$$\left\{\frac{n_1 - k}{p}\right\} = \left\{\frac{n_1}{p}\right\} - \left\{\frac{k}{p}\right\}.$$

We therefore have from (6.9) and (6.10) that

$$\left\{\frac{n_1}{p}\right\} + \left\{\frac{n_2}{p}\right\} \le 1 - \frac{1}{p} + \left\{\frac{k}{p}\right\} \le 1 - \frac{1}{p} + \left\{\frac{n_2}{p}\right\} - \frac{p-r}{3p},$$

whence

$$\left\{\frac{n_1}{p}\right\} \le \frac{2p-3+r}{3p} < \frac{2}{3},$$

contradicting (6.2).

Similarly, we may write

$$\operatorname{ord}_p \binom{n_1 - r/3}{k} = \left\{\frac{n_1-q-k}{p}\right\} + \left\{\frac{k}{p}\right\} - \left\{\frac{n_1-q}{p}\right\} \tag{6.11}$$

and

$$\operatorname{ord}_p \binom{n_1+n_2-k}{n_1} = \left\{\frac{n_1}{p}\right\} + \left\{\frac{n_2-k}{p}\right\} - \left\{\frac{n_1+n_2-k}{p}\right\}, \tag{6.12}$$

where now $q = (2p+r)/3$. We suppose that

$$\operatorname{ord}_p \binom{n_1 - r/3}{k}\binom{n_1+n_2-k}{n_1} = 0 \tag{6.13}$$

so that, in particular, we have

$$\left\{\frac{n_1-q}{p}\right\} = \left\{\frac{n_1}{p}\right\} - \frac{q}{p} = \left\{\frac{n_1}{p}\right\} - \frac{2p+r}{3p} \ge \left\{\frac{k}{p}\right\}. \tag{6.14}$$

It follows that we necessarily have

$$\left\{\frac{k}{p}\right\} < 1 - \frac{2p+r}{3p} = \frac{1}{3} - \frac{r}{3p} \le \left\{\frac{n_2}{p}\right\},$$

whereby

$$\left\{\frac{n_1}{p}\right\} + \left\{\frac{n_2-k}{p}\right\} = \left\{\frac{n_1}{p}\right\} + \left\{\frac{n_2}{p}\right\} - \left\{\frac{k}{p}\right\} \le 1 - \frac{1}{p}.$$

Arguing as before, we find that

$$\left\{\frac{n_2}{p}\right\} < 1 - \frac{1}{p} - \frac{2p+r}{3p} < \frac{1}{3} - \frac{1}{p},$$

again contradicting (6.4) and (6.5).

The argument for $p \equiv -r \pmod 3$ is essentially similar. Relation (6.3) implies that we have one of

$$\left\{\frac{n_1}{p}\right\} \in \left[\frac{5}{12} + \frac{9-4r}{12p}, \frac{1}{2}\right) \Longrightarrow \left\{\frac{n_2}{p}\right\} = \left\{\frac{4n_1-\delta}{p}\right\} \in \left[\frac{2}{3} + \frac{6-4r}{3p}, 1\right), \tag{6.15}$$

$$\left\{\frac{n_1}{p}\right\} \in \left[\frac{2}{3} + \frac{3-r}{3p}, \frac{3}{4}\right) \Longrightarrow \left\{\frac{n_2}{p}\right\} = \left\{\frac{4n_1-\delta}{p}\right\} \in \left[\frac{2}{3} + \frac{9-4r}{3p}, 1\right) \tag{6.16}$$

or

$$\left\{\frac{n_1}{p}\right\} \in \left[\frac{11}{12} + \frac{9-4r}{12p}, 1\right) \Longrightarrow \left\{\frac{n_2}{p}\right\} = \left\{\frac{4n_1-\delta}{p}\right\} \in \left[\frac{2}{3} + \frac{6-4r}{3p}, 1\right). \tag{6.17}$$

In each case, we may thus conclude that

$$\left\{\frac{n_1}{p}\right\} + \left\{\frac{n_2}{p}\right\} \geq \frac{13}{12} + \frac{33-20r}{12p} \geq \frac{13}{12} - \frac{7}{12p} \geq 1 + \frac{1}{p}. \tag{6.18}$$

Arguing as previously, (6.18) implies the desired conclusion almost immediately in case $k = 0$ or 1. Let us assume that $k \geq 2$. From [**4,** Lemma 4.5], we once again have (6.7) and (6.11), only this time with $q = (2p - r)/3$ and $q = (p + r)/3$, respectively. If we suppose that

$$\operatorname{ord}_p \binom{n_2 + r/3}{k} = 0,$$

then

$$\left\{\frac{n_2}{p} - \frac{2}{3} + \frac{r}{3p}\right\} \geq \left\{\frac{k}{p}\right\},$$

whereby, from (6.15), (6.16) and (6.17),

$$\left\{\frac{n_2}{p}\right\} \geq \frac{2}{3} - \frac{r}{3p} + \left\{\frac{k}{p}\right\}, \tag{6.19}$$

whence $\{k/p\} < 1/3 < \{n_1/p\}$. If, also,

$$\operatorname{ord}_p \binom{n_1 + n_2 - k}{n_2} = 0,$$

we again have (6.10), and thus, from (6.19),

$$\frac{2}{3} - \frac{r}{3p} + \left\{\frac{n_1}{p}\right\} \leq \left\{\frac{n_2}{p}\right\} + \left\{\frac{n_1}{p}\right\} - \left\{\frac{k}{p}\right\} \leq 1 - \frac{1}{p},$$

a contradiction. If, on the other hand, we assume (6.13), then both

$$\left\{\frac{n_1 - ((p+r)/3)}{p}\right\} \geq \left\{\frac{k}{p}\right\} \quad \text{and} \quad \left\{\frac{n_1}{p}\right\} + \left\{\frac{n_2 - k}{p}\right\} \leq 1 - \frac{1}{p}.$$

The first of these implies that

$$1 - \frac{1}{p} \geq \left\{\frac{n_1}{p}\right\} \geq \frac{1}{3} + \frac{r}{3p} + \left\{\frac{k}{p}\right\},$$

whence

$$\left\{\frac{k}{p}\right\} \leq \frac{2}{3} - \frac{r+3}{3p}$$

and thus, from (6.15), (6.16) and (6.17),

$$\begin{aligned} 1 - \frac{1}{p} &\geq \left\{\frac{n_1}{p}\right\} + \left\{\frac{n_2 - k}{p}\right\} \\ &= \left\{\frac{n_1}{p}\right\} + \left\{\frac{n_2}{p}\right\} - \left\{\frac{k}{p}\right\} \\ &\geq \frac{1}{3} + \frac{r}{3p} + \left\{\frac{n_2}{p}\right\}, \end{aligned}$$

whereby

$$\left\{\frac{n_2}{p}\right\} \leq \frac{2}{3} - \frac{r+3}{3p}.$$

The resulting contradiction (to (6.15), (6.16) and (6.17)) completes the proof. $\square$

In order to apply this result, we observe that, if $1 \leq c < d$ are integers, then we have $\{n_1/p\} > c/d$ precisely, when

$$p \in \bigcup_{k=0}^{\infty} \left(\frac{n_1}{k+1}, \frac{n_1}{k + c/d}\right).$$

We define

$$\theta(x, q, r) = \sum_{\substack{p \leq x \\ p \equiv r \pmod q}} \log p,$$

where the sum is over primes p. Fixing $r \in \{1, 2\}$, it follows from Lemma 6.1 that $\log \Pi_{n_1,\delta,r}$ is bounded below by

$$L_{r,n_1} = \sum_{k=0}^{k_0} (T_{1,k} + T_{2,k} + T_{3,k} + T_{4,k} + T_{5,k}), \tag{6.20}$$

where

$$\begin{aligned}
T_{1,k} &= \theta\left(\frac{n_1}{k+2/3}, 3, r\right) - \theta\left(\frac{n_1}{k+3/4}, 3, r\right),\\
T_{2,k} &= \theta\left(\frac{n_1}{k+5/6}, 3, r\right) - \theta\left(\frac{n_1}{k+1}, 3, r\right),\\
T_{3,k} &= \theta\left(\frac{n_1}{k+5/12}, 3, -r\right) - \theta\left(\frac{n_1}{k+1/2}, 3, -r\right),\\
T_{4,k} &= \theta\left(\frac{n_1}{k+2/3}, 3, -r\right) - \theta\left(\frac{n_1}{k+3/4}, 3, -r\right)
\end{aligned}$$

and

$$T_{5,k} = \theta\left(\frac{n_1}{k+11/12}, 3, -r\right) - \theta\left(\frac{n_1}{k+1}, 3, -r\right),$$

for $k_0 = [\sqrt{n_1/12}] - 2$ (we can actually, in most cases, use a slightly larger value for k_0; it is simply chosen so that inequality (6.1) is satisfied). In order to estimate L_{r,n_1} for large values of n_1, we will appeal to recent bounds on $\theta(x, 3, r)$ **[2]**. In particular, we use that

$$\left|\theta(x, 3, r) - \frac{x}{2}\right| < \begin{cases} 1.798158\sqrt{x} & \text{if } x \le 10^{13} \\ 0.001(x/\log x) & \text{if } x > 10^{13}. \end{cases} \tag{6.21}$$

Note that, together, these inequalities imply that we have

$$\left|\theta(x, 3, r) - \frac{x}{2}\right| < 0.001\frac{x}{\log x}$$

for all $x > 1.5 \cdot 10^9$. We will show that $L_{r,n_1} > 0.58779\, n_1$, which immediately implies Proposition 5.1.

Let us assume first that $n_1 > 3 \cdot 10^9$. Then,

$$T_{1,0} > \frac{n_1}{12} - 0.001\left(\frac{3n_1}{2\log(3n_1/2)} + \frac{4n_1}{3\log(4n_1/3)}\right) > 0.0832\, n_1,$$

$$T_{2,0} > \frac{n_1}{10} - 0.001\left(\frac{6n_1}{5\log(6n_1/5)} + \frac{n_1}{\log(n_1)}\right) > 0.0998\, n_1,$$

$$T_{3,0} > \frac{n_1}{5} - 0.001\left(\frac{12n_1}{5\log(12n_1/5)} + \frac{2n_1}{\log(2n_1)}\right) > 0.1998\, n_1,$$

$$T_{4,0} > \frac{n_1}{12} - 0.001\left(\frac{3n_1}{2\log(3n_1/2)} + \frac{4n_1}{3\log(4n_1/3)}\right) > 0.0832\, n_1,$$

$$T_{5,0} > \frac{n_1}{11} - 0.001\left(\frac{12n_1}{11\log(12n_1/11)} + \frac{n_1}{\log(n_1)}\right) > 0.0908\, n_1,$$

$$T_{1,1} > \frac{n_1}{70} - 0.001\left(\frac{3n_1}{5\log(3n_1/5)} + \frac{4n_1}{7\log(4n_1/7)}\right) > 0.0142\, n_1,$$

and

$$T_{3,1} > \frac{n_1}{51} - 0.001\left(\frac{12n_1}{17\log(12n_1/17)} + \frac{2n_1}{3\log(2n_1/3)}\right) > 0.0195\, n_1,$$

whereby $L_{r,n_1} > 0.58779\, n_1$, as desired. Next, suppose that $n_1 \leq 3 \cdot 10^9$. Then, for each $k \geq 0$, we have

$$T_{1,k} > \frac{n_1}{2(3k+2)(4k+3)} - 1.798158\left(\sqrt{\frac{n_1}{k+2/3}} + \sqrt{\frac{n_1}{k+3/4}}\right),$$

$$T_{2,k} > \frac{n_1}{2(6k+5)(k+1)} - 1.798158\left(\sqrt{\frac{n_1}{k+5/6}} + \sqrt{\frac{n_1}{k+1}}\right),$$

$$T_{3,k} > \frac{n_1}{(2k+1)(12k+5)} - 1.798158\left(\sqrt{\frac{n_1}{k+5/12}} + \sqrt{\frac{n_1}{k+1/2}}\right),$$

$$T_{4,k} > \frac{n_1}{2(3k+2)(4k+3)} - 1.798158\left(\sqrt{\frac{n_1}{k+2/3}} + \sqrt{\frac{n_1}{k+3/4}}\right)$$

and

$$T_{5,k} > \frac{n_1}{2(12k+11)(k+1)} - 1.798158\left(\sqrt{\frac{n_1}{k+11/12}} + \sqrt{\frac{n_1}{k+1}}\right).$$

If we suppose that $10^6 \leq n_1 \leq 3 \cdot 10^9$, then it is readily checked that the inequalities here are nontrivial (i.e., that the right-hand-sides are positive) for, in each case, $0 \le k \le 4$, whereby we find that, once again,

$$L_{r,n_1} \ge \sum_{k=0}^{4} (T_{1,k} + T_{2,k} + T_{3,k} + T_{4,k} + T_{5,k}) > 0.58779\, n_1.$$

It remains to treat values of $n_1 < 10^6$. Note that, if we have $a = 1$, $n_1 = 496$ and $n_2 = 4n_1 - 1$, then

$$\Pi_{n_1,\delta,r}^{1/n_1} = 1.79954218\ldots,$$

and hence, we cannot expect to extend Proposition 5.1 to smaller values of n_1. By direct (if slow) computation of $\Pi_{n_1,\delta,r}$, we find that the inequality of Proposition 5.1 is satisfied for each $r \in \{1, 2\}$, $497 \le n_1 \le 1000$ and $n_2 = 4n_1 - \delta$ with $\delta \in \{0, 1\}$. For larger values of n_1, instead of relying upon the definition of $\Pi_{n_1,\delta,r}$, we appeal to the bound $\log \Pi_{n_1,\delta,r} \ge L_{r,n_1}$, where L_{r,n_1} is as defined in (6.20). For $1000 < n_1 \le 10000$ and $r \in \{1, 2\}$, we check that, in each case, $\exp(L_{r,n_1}/n_1) > 1.8$. This takes roughly 20 minutes in `Maple` on an older Macbook Air. We find that the largest value of n_1 in this range for which we have $\exp(L_{r,n_1}/n_1) < 1.9$ corresponds to

$$\exp(L_{2,3319}/3319) = 1.89773\ldots.$$

This is unsurprising since, from (6.21), we have that

$$L := \lim_{n_1 \to \infty} L_{r,n_1}/n_1$$

is equal to

$$\sum_{k=0}^{\infty} \left(\frac{1}{(3k+2)(4k+3)} + \frac{1}{2(6k+5)(k+1)} + \frac{1}{(2k+1)(12k+5)} + \frac{1}{2(12k+11)(k+1)} \right).$$

Defining

$$\psi(x) = \frac{d}{dx} \ln \Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)},$$

we have that

$$\psi(x) = -\gamma + \sum_{n=0}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n+x} \right) \quad \text{for } x \ne 0, -1, -2, \ldots,$$

whence

$$L = \psi(1) + \psi(3/4) + \psi(1/2)/2 - \psi(2/3) \\ - \psi(5/6)/2 - \psi(5/12)/2 - \psi(11/12)/2.$$

Using known identities for ψ, we thus have

$$L = -\gamma - 3\ln(2) + \frac{\pi}{2} - \frac{5\pi\sqrt{3}}{12} + \frac{9\ln 3}{4} \\ - \frac{\psi(5/12)}{2} - \frac{\psi(11/12)}{2} = 0.70264\ldots,$$

and so,

$$\lim_{n_1 \to \infty} \exp(L_{r,n_1}/n_1) = 2.019084\ldots.$$

In order to finish the computation verifying the inequality $\exp(L_{r,n_1}/n_1) > 1.8$ for $10000 < n_1 < 10^6$, we employ the "bootstrapping" procedure described in detail in [**1,** Section 7], which exploits that $\lim_{n_1\to\infty} \exp(L_{r,n_1}/n_1)$ greatly exceeds 1.8; by way of example,

$$\exp(L_{1,10000}/10000) > 2.006523,$$

together with the fact that the difference between L_{r,n_1} and L_{r,n_1+k} is "small," provided n_1 is much larger than k. This enables us to significantly reduce the number of times we actually compute L_{r,n_1}. Full details are available from the authors on request. This completes the proof of Proposition 5.1.

7. Proof of Corollary 2.2. In order to proceed from Theorem 2.1 to Corollary 2.2 is straightforward. Suppose that x and n are integers with $x^3 \neq 27 \cdot 2^n$ and $x \notin \{4, 5, 8, 15, 19, 38, 121\}$. We may further suppose that x is positive since our desired conclusion is trivial otherwise. If $n \equiv 0 \pmod 3$, say $n = 3n_0$, we have

$$|x^3 - 27 \cdot 2^n| \geq x^3 - (x-1)^3 > 3x^3 - 3x^2 \geq 3^{5/3} \cdot x^{4/3},$$

where the last inequality holds for $x \geq 3$ (and, for $x = 1$ or 2, the desired result immediately follows). We may thus suppose that $n = 3n_0 + r$ for $r \in \{1, 2\}$, and therefore,

$$|x^3 - 27 \cdot 2^n| = 3 \cdot 2^{n_0} \left| 2^{r/3} - \frac{x}{3 \cdot 2^{n_0}} \right| (x^2 + 3 \cdot x 2^{n_0 + r/3} + 9 \cdot 2^{2n_0 + 2r/3}).$$

If $|x - 3 \cdot 2^{n_0+r/3}| > 1$, then, once again, we have

$$|x^3 - 27 \cdot 2^n| > 3x^3 - 3x^2 \geq 3^{5/3} \cdot x^{4/3}.$$

Otherwise, applying Theorem 2.1, we thus have

$$|x^3 - 27 \cdot 2^n| > 3 \cdot 2^{-0.62n_0}(x^2 + x(x-1) + x(x-1)^2),$$

at least provided $n_0 > 12$. Since

$$2^{0.62n_0} \leq \left(\frac{x+1}{3}\right)^{0.62},$$

we obtain inequality (2.2), after a little work. The values $n_0 \leq 12$ correspond to $n \leq 38$. For these, we readily check that (2.2) is satisfied, except for $x \in \{4, 5, 8, 15, 19, 38, 121\}$.

8. Proof of Theorem 2.3. We next proceed with the proof of Theorem 2.3. We suppose that a, b and c are given integers and that there exist integers x and n such that

$$x^3 + ax^2 + bx + c = \pm 2^n.$$

Writing $u = x + a/3$, we find that

$$u^3 + (b - a^2/3)u + (2a^3/27 + c - ab/3) = (-1)^\delta 2^n, \tag{8.1}$$

where now either u or $3u$ is an integer and $\delta \in \{0, 1\}$.

If $u^3 = (-1)^\delta 2^n$ then, from (8.1),

$$(b - a^2/3)(x + a/3) = ab/3 - c - 2a^3/27$$

so that

$$(b - a^2/3)x = -c + a^3/27.$$

If $b = a^2/3$, then, necessarily, we have that $a = 3t$ for some integer t, whereby $b = 3t^2$ and $c = t^3$ so that $x^3 + ax^2 + bx + c = (x + t)^3$. Otherwise, we conclude that

$$x = \frac{a^3 - 27c}{27b - 9a^2}.$$

Next, to treat the cases where $3 \mid a$ and

$$x = u - a/3 \quad \text{for } |u| \in \{4, 5, 8, 15, 19, 38, 121\},$$

let us suppose that, for one of these choices of x, we have

(8.2) $|x| > \max\{8|b-a^2/3|^3+|a/3|, |4a^3/27+2c-2ab/3|^{3/4}+|a/3|\}.$

Since $|x| \leq |a/3| + |u|$ we thus have

$$\max\{8|b-a^2/3|^3, |4a^3/27+2c-2ab/3|^{3/4}\} < |u|. \tag{8.3}$$

It follows that $b = a^2/3 + b_0$ for some integer b_0 with $|b_0| < (1/2)\,|u|^{1/3}$. Since we also have

$$\left| c - \frac{ab_0}{3} - \frac{a^3}{27} \right| < \frac{1}{2}\,|u|^{4/3},$$

we may write

$$c = \frac{ab_0}{3} + \frac{a^3}{27} + c_0,$$

where c_0 is an integer with $|c_0| < (1/2)\,|u|^{4/3}$. Writing $a = 3t$, for t an integer, we thus have

$$(u-t)^3 + 3t(u-t)^2 + (3t^2+b_0)(u-t) + t^3 + tb_0 + c_0 = \pm 2^n,$$

whence

$$u^3 + b_0 u + c_0 = \pm 2^n, \tag{8.4}$$

where

$$|u| \in \{4,\ 5,\ 8,\ 15,\ 19,\ 38,\ 121\},$$

$$|b_0| < \frac{1}{2}\,|u|^{1/3} \quad \text{and} \quad |c_0| < \frac{1}{2}\,|u|^{4/3}.$$

A short computation reveals that (8.4) is satisfied only for

$$(u, b_0, c_0) \in \{(\pm 4, 0, 0),\ \pm(5,0,3),\ (\pm 8, 0, 0)\},$$

the first and last of which correspond to $x^3 + ax^2 + bx + c = (x+t)^3$. The case $(u, b_0, c_0) = \pm(5,0,3)$ leads to $(a,b,c,x) = (3t, 3t^2, t^3+3, 5-t)$ and $(3t, 3t^2, t^3-3, -5-t)$.

If we suppose finally that $u^3 \neq (-1)^\delta 2^n$ and $x \neq u - a/3$ for any u satisfying $|u| \in \{4,5,8,15,19,38,121\}$, from the fact that

$$|u^3 - (-1)^\delta 2^n| = |(b-a^2/3)u + (2a^3/27 + c - ab/3)| > 0$$

we thus have, in all cases, applying Corollary 2.2 to $||3u|^3 - 27 \cdot 2^n|$, that

$$\begin{aligned} u^{4/3} &\le |(b - a^2/3)u + (2a^3/27 + c - ab/3)| \\ &\le 2 \max\{|b - a^2/3||u|, |2a^3/27 + c - ab/3|\} \end{aligned}$$

and thus,

$$|u| \le \max\{8|b - a^2/3|^3, |4a^3/27 + 2c - 2ab/3|^{3/4}\},$$

whereby

$$|x| \le \max\{8|b - a^2/3|^3 + |a/3|, |4a^3/27 + 2c - 2ab/3|^{3/4} + |a/3|\}.$$

This completes the proof of Theorem 2.3.

9. Proof of Theorem 2.4. In this section, we will prove Theorem 2.4. We begin by supposing that D is an odd integer and that we have

$$x_i^3 + D = 2^{k_i}, \quad i \in \{1, 2, 3, 4\},$$

with

$$x_1 < x_2 < x_3 < x_4 \quad \text{and} \quad k_1 < k_2 < k_3 < k_4.$$

Then, for each $i \in \{1, 2, 3\}$,

$$(x_{i+1} - x_i)(x_{i+1}^2 + x_{i+1}x_i + x_i^2) = x_{i+1}^3 - x_i^3 = 2^{k_i}(2^{k_{i+1}-k_i} - 1),$$

whereby we may write

$$x_{i+1} = x_i + a_i \cdot 2^{k_i},$$

with a_i a positive integer. Substituting this into $2^{k_{i+1}} = x_{i+1}^3 + D$, we find that

$$2^{k_{i+1}-k_i} = 1 + 3a_i x_i^2 + 3a_i^2 2^{k_i} x_i + a_i^3 2^{2k_i}. \tag{9.1}$$

First, we suppose that D is positive and write $x_i = -D^{1/3} + y_i$ for y_i a positive real number. From (9.1), we have

$$\begin{aligned} 2^{k_{i+1}-k_i} &= 1 + a_i \cdot (3D^{2/3} - 3a_i 2^{k_i} D^{1/3} + a_i^2 2^{2k_i}) \\ &\quad + 3a_i y_i (y_i + a_i 2^{k_i} - 2D^{1/3}). \end{aligned}$$

Applying the arithmetic-geometric mean inequality to the first bracketed term on the right-hand-side of this equation, we thus have that

$$2^{k_{i+1}-k_i} \geq 1 + (2\sqrt{3}-3)a_i^2 D^{1/3} 2^{k_i} + 3a_i y_i (y_i + a_i 2^{k_i} - 2D^{1/3}). \tag{9.2}$$

Note that

$$2^{k_i} = x_i^3 + D = 3y_i D^{2/3} - 3y_i^2 D^{1/3} + y_i^3,$$

and hence,

$$y_i + a_i 2^{k_i} - 2D^{1/3} = y_i + 3a_i y_i D^{2/3} - 3a_i y_i^2 D^{1/3} + a_i y_i^3 - 2D^{1/3}.$$

If we have $y_1 < 1$, then, since

$$y_2 - y_1 = x_2 - x_1 = a_1 \cdot 2^{k_1} \geq 1,$$

we thus have $y_2 > 1$, and hence, in all cases, may write $y_2 = D^\theta$ for $\theta > 0$. Since, again by the arithmetic-geometric mean inequality, the function

$$f(\theta) = 3D^{2/3+\theta} - 3D^{1/3+2\theta} + D^{3\theta}$$

is monotone increasing as a function of θ. We thus have

$$2^{k_2} > 3D^{2/3} - 3D^{1/3} + 1 > 2D^{2/3},$$

at least assuming that $D \geq 27$. From (9.2), it follows that

$$\begin{aligned} 2^{k_3} &> 2D^{2/3}((2\sqrt{3}-3)2D + 3(2D^{2/3} - 2D^{1/3})) \\ &> 4(2\sqrt{3}-3)D^{5/3} > 1.8 \cdot D^{5/3}. \end{aligned}$$

We, therefore, have $x_3 > D^{5/9}$, and thus, again applying (9.2), we find that $2^{k_4} > 1.5 \cdot D^{11/3}$ and

$$x_4^3 > 1.5 \cdot D^{11/3} - D > D^{11/3},$$

since we may assume that $D \geq 2$. We thus have, from Corollary 2.2 (where we take $x = 3x_4$), that

$$D = |x_4^3 - 2^{k_4}| \geq x_4^{4/3} > (D^{11/3})^{4/9} = D^{44/27},$$

a contradiction.

Suppose next that $D < 0$ (so that $x_2 > x_1 > |D|^{1/3}$). From (9.1), we have that $2^{k_2} > 2\,|D|^{2/3}$, and hence,

$$2^{k_3} > 2^{k_2}(3x_2^2 + 3 \cdot 2^{k_2} x_2 + 2^{2k_2}) > 8\,|D|^2.$$

A final appeal to (9.1) implies that

$$2^{k_4} > 2^{k_3}(3x_3^2 + 3 \cdot 2^{k_3} x_3 + 2^{2k_3}) > 512\,|D|^6.$$

We thus have, again appealing to Corollary 2.2, that

$$|D| = |x_4^3 - 2^{k_4}| \geq x_4^{4/3} = (2^{k_4} + |D|)^{4/9} > (512|D|^6)^{4/9} > |D|^{8/3},$$

a contradiction. This completes the proof of Theorem 2.4.

It is perhaps worthwhile noting that we know of only three odd values of D for which the equation $x^3 + D = 2^n$ even has as many as two solutions in integers x and n, namely,

$$D = -215, \quad \text{with } (x, n) = (6, 0) \text{ and } (7, 7),$$
$$D = 1, \quad \text{with } (x, n) = (0, 0) \text{ and } (1, 1),$$

and

$$D = 3, \quad \text{with } (x, n) = (-1, 1), (1, 2) \text{ and } (5, 7).$$

10. Thue-Mahler equations. As noted earlier, the equation

$$x^3 - xy^2 + 8y^3 = 2^k, \tag{10.1}$$

which generalizes (1.1), has, itself, at most finitely many solutions in integers (x, y, n), which may be determined effectively following arguments of **[13]**, based upon lower bounds for linear forms in complex and p-adic logarithms, together with computational techniques from Diophantine approximation. Hambrook **[6]** has an implementation of such an approach which works well in the case of few primes and low degree forms (i.e., precisely the situation in which we find ourselves); appealing to his Thue-Mahler solver, the only coprime solutions to (10.1) are with (x, y) one of

$$(-113, 53), (-19, 9), (-2, 1), (-1, 1), (-1, 5), (0, 1), (1, 0),$$
$$(1, 1), (3, -1), (3, 1), (5, 1), (7, -3), (8, 1), (13, -6).$$

Similarly, the equation

$$x^3 - 13xy^2 + 20y^3 = 2^k$$

has corresponding solutions with (x, y) among

$$(-21,5),(-4,1),(-3,1),(-1,1),(1,0),(1,1),(2,1),$$
$$(3,1),(4,1),(7,3),(11,5),(13,1),(19,-3).$$

There exist completely general bounds for the number of solutions to Thue-Mahler equations that depend only upon the degree of the given form F and the number of primes on the right hand side of the equation $F(x,y) = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$. Along these lines, let us note that Evertse [**5**] has shown that, if F is an irreducible cubic form and p is a fixed prime, then the equation

$$|F(x,y)| = p^n$$

has at most $7^{60} + 6 \cdot 7^4$ solutions in integers. This bound, while admirably uniform, exceeds 10^{50}.

REFERENCES

1. M. Bauer and M. Bennett, *Application of the hypergeometric method to the generalized Ramanujan-Nagell equation*, Ramanujan J. **6** (2002), 209–270.

2. M. Bennett, G. Martin, K. O'Bryant and A. Rechnitzer, *Explicit bounds for primes in arithmetic progressions*, to appear.

3. F. Beukers, *On the generalized Ramanujan-Nagell equation* I, Acta Arith. **38** (1980/81), 389–410.

4. G.V. Chudnovsky, *On the method of Thue-Siegel*, Ann. Math. **117** (1983), 325–382.

5. J.-H. Evertse, *Upper bounds for the numbers of solutions of Diophantine equations*, Masters thesis, Amsterdam, 1983.

6. K. Hambrook, *Implementation of a Thue-Mahler solver*, `http://www.math.rochester.edu/people/faculty/khambroo/research/ubc_2011_fall_hambrook_kyle_updated.pdf`

7. S.V. Kotov, *The greatest prime factor of a polynomial*, Mat. Z. **13** (1973), 515–522.

8. K. Mahler, *Zur Approximation algebraischer Zahlen* II, Math. Ann. **108** (1933), 37–55.

9. T. Nagell, *The diophantine equation* $x^2 + 7 = 2^n$, Ark. Mat. **4** (1961), 185–187.

10. Physics forum, `http://www.physicsforums.com/showthread.php?t=443958`.

11. T.N. Shorey, A.J. van der Poorten, R. Tijdeman and A. Schinzel, *Applications of the Gel'fond-Baker method to diophantine equations*, in *Transcendence theory*: *Advances and applications*, Academic Press, London, 1977.

12. C.L. Siegel (under the pseudonym X), *The integer solutions of the equation* $y^2 = ax^n + bx^{n-1} + \cdots + k$, J. Lond. Math. Soc. **1** (1926), 66–68.

13. N. Tzanakis and B.M.M. de Weger, *How to explicitly solve a Thue-Mahler equation*, Compos. Math. **84** (1992), 223–288.

University of Calgary, Department of Mathematics, Calgary, AB, T2N 1N4, Canada
Email address: **mbauer@math.ucalgary.ca**

University of British Columbia, Department of Mathematics, Vancouver, BC V6T 1Z2 Canada
Email address: **bennett@math.ubc.ca**