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MULTIVARIABLE ISOMETRIES RELATED TO CERTAIN CONVEX DOMAINS

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ABSTRACT. Several interesting results exist in the literature on subnormal operator tuples having their spectral properties tied to the geometry of strictly pseudoconvex domains or to that of bounded symmetric domains in $\mathbb{C}^n$. We introduce a class $\Omega^{(n)}$ of convex domains in $\mathbb{C}^n$ which, for $n \geq 2$, is distinct from the class of strictly pseudoconvex domains and the class of bounded symmetric domains and which lends itself to the application of theories related to the abstract inner function problem and the $\overline{\partial}$-Neumann problem, allowing us to make a number of interesting observations about certain subnormal operator tuples associated with the members of the class $\Omega^{(n)}$.

1. Introduction. We use $\mathcal{B}(\mathcal{H})$ to denote the algebra of bounded linear operators on a complex infinite-dimensional separable Hilbert space $\mathcal{H}$ and I to denote the identity operator on $\mathcal{H}$. An n-tuple $S = (S_1, \ldots, S_n)$ of commuting operators S_i in $\mathcal{B}(\mathcal{H})$ is said to be *subnormal* if there exist a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ and an n-tuple $N = (N_1, \ldots, N_n)$ of commuting normal operators N_i in $\mathcal{B}(\mathcal{K})$ such that $N_i\mathcal{H} \subset \mathcal{H}$ and $N_i|\mathcal{H} = S_i$ for $1 \leq i \leq n$. Among all the normal extensions of a subnormal tuple S, there is a 'minimal normal extension,' which is unique up to unitary equivalence, see [**28**]. An n-tuple $T = (T_1, \ldots, T_n)$ of commuting operators T_i in $\mathcal{B}(\mathcal{H})$ is said to be *essentially normal* if the operators $T_i^*T_j - T_jT_i^*$ are compact for all i and j, while T is said to be *cyclic* if there exists a vector f in $\mathcal{H}$ (referred to as a *cyclic vector* for T) such that the linear span

$$\vee\{T_1^{k_1}T_2^{k_2}\cdots T_n^{k_n} f : k_i \text{ are non-negative integers}\}$$

is dense in $\mathcal{H}$. Several interesting results exist in the literature on subnormal operator tuples (and, in particular, on essentially normal and/or cyclic subnormal operator tuples) having their spectral properties tied

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to the geometry of strictly pseudoconvex domains or to that of bounded symmetric domains in $\mathbb{C}^n$ (refer, for example, to **[4, 6, 15, 16, 17, 18, 20, 21, 48]**). These results are largely manifestations of the functional calculus for subnormal operator tuples thriving upon some elegant function-theoretic results valid in the context of those two types of domains. (We refrain from referring to an endless list of papers that specifically deal with subnormal operator tuples related to the unit ball $\mathbb{B}_n$ in $\mathbb{C}^n$, which is a strictly pseudoconvex as well as a bounded symmetric domain).

In Section 2, we introduce a class $\Omega^{(n)}$ of convex domains in $\mathbb{C}^n$ whose members are parameterized by n-tuples p with the coordinates of p being tuples (of varying lengths) of positive integers subject to certain constraints. For $n \geq 2$, the class $\Omega^{(n)}$ of domains Ω_p turns out to be distinct from the class of strictly pseudoconvex domains and the class of bounded symmetric domains. The new class allows for the application of the theory related to the abstract inner function problem (refer to **[1, 17]**) as well as of the theory related to the $\overline{\partial}$-Neumann problem (refer to **[7, 22]**). The multiplication tuples associated with the Hardy-type function spaces associated with the domains Ω_p turn out to be so-called (regular) A-isometries. We record a few properties of the domains Ω_p that are relevant for the application of some known results in the literature to those A-isometries; these applications mostly result from the existence of an abundance of inner functions on the domains Ω_p as in the case of domains that are either strictly pseudoconvex or bounded symmetric (refer to **[1, 17]**).

In Section 3, we record parts of the theory related to the $\overline{\partial}$-Neumann problem and the tangential Neumann problem which are of interest to us. The $\overline{\partial}$-Neumann problem (respectively, tangential Neumann problem) will be seen to be of particular relevance in the context of the multiplication tuples $M_{\nu_p,z}$ (respectively, $M_{\sigma_p,z}$) associated with the Bergman (respectively, Hardy) spaces of the domains Ω_p. Indeed, among our concerns in Section 3 will be the compactness of the so-called $\overline{\partial}$-Neumann operator and that of the so-called tangential Neumann operator, since the compactness of the $\overline{\partial}$-Neumann operator (respectively, tangential Neumann operator) guarantees the essential normality of the tuple $M_{\nu_p,z}$ (respectively, $M_{\sigma_p,z}$).

In Section 4, we discuss multivariable isometries associated with certain convex domains Σ_p that are more general than the domains Ω_p, providing an intrinsic characterization of such multivariable isometries (referred to as $\partial\Sigma_p$-isometries). In particular, a succinct characterization of a $\partial\Sigma_p$-isometry is derived for a special type of Σ_p, which is an apt generalization of that of a 'spherical isometry,' see **[3]**. We also dwell there on the intertwining of a $\partial\Omega_p$-isometry with certain other subnormal tuples. Finally, we elaborate upon the significance of the domains Ω_p for some operator theoretic considerations that go beyond the topic of multivariable isometries.

For any terminology employed from the area of several complex variables and for any standard results quoted from there, the references **[29, 34, 41]** should be more than adequate.

2. Convex domains Ω_p. Let $p = (p_1, p_2, \ldots, p_n)$ be an n-tuple of m_i-tuples $p_i = (p_{i,1}, p_{i,2}, \ldots, p_{i,m_i})$ where, for each i satisfying $1 \leq i \leq n$, $p_{i,1}$, $p_{i,2}, \ldots, p_{i,m_i}$ (with $m_i \geq 2$) are relatively prime positive integers so that $\gcd\{p_{i,1}, p_{i,2}, \ldots, p_{i,m_i}\} = 1$. The subset Ω_p of $\mathbb{C}^n$ is defined by

$$\Omega_p = \left\{ z = (z_1, z_2, \ldots, z_n) \in \mathbb{C}^n : \sum_{i=1}^{n} \sum_{j=1}^{m_i} |z_i|^{2p_{i,j}} < 1 \right\}.$$

The set Ω_p is easily seen to be a convex complete Reinhardt domain in $\mathbb{C}^n$ with the real analytic boundary

$$\partial\Omega_p = \left\{ z = (z_1, z_2, \ldots, z_n) \in \mathbb{C}^n : \sum_{i=1}^{n} \sum_{j=1}^{m_i} |z_i|^{2p_{i,j}} = 1 \right\}.$$

Some of the results in Sections 2 and 3 as stated for the domains Ω_p also hold for certain domains more general than Ω_p; these will be pointed out explicitly in Section 4. We use the symbol $\Omega^{(n)}$ to denote the class of domains Ω_p in $\mathbb{C}^n$ parameterized by the tuples p as described above. For $z \in \mathbb{C}$, $\overline{z}$ denotes the complex conjugate of z and, for any complex-valued function ϕ, $\overline{\phi}$ is the function satisfying $\overline{\phi}(z) = \overline{\phi(z)}$.

Remark 2.1. For $n = 1$, the domains Ω_p reduce to the open unit disks in the plane (of various radii) centered at the origin for which the theme of the paper already stands well-explored (refer, for example, to **[10, 14]**). For that reason, and for the validity of certain assertions to follow, *we assume hereafter in any discussion involving Ω_p that $n \geq 2$.*

Remark 2.2. The domain Ω_p equals $\{z \in \mathbb{C}^n : u(z) < 0\}$, where

$$u(z) = \sum_{i=1}^{n} \sum_{j=1}^{m_i} |z_i|^{2p_{i,j}} - 1.$$

For $b \in \partial\Omega_p$, let

$$\mathcal{T}_b(\partial\Omega_p) = \left\{ X = (X_1, \ldots, X_n) \in \mathbb{C}^n : \sum_{j=1}^{n} \frac{\partial u}{\partial z_j}(b) X_j = 0 \right\}$$

be the complex tangent space to $\partial\Omega_p$ at b. The Levi form

$$\mathcal{L}u(b, X) = \sum_{j,k=1}^{n} \frac{\partial^2 u}{\partial z_j \partial \overline{z}_k}(b) X_j \overline{X}_k$$

is non-negative for every $b \in \partial\Omega_p$ and $X \in \mathcal{T}_b(\partial\Omega_p)$. However, for some permissible choices of p, the Levi form $\mathcal{L}u(b, X)$ is zero for some $b \in \partial\Omega_p$ and some non-zero $X \in \mathcal{T}_b(\partial\Omega_p)$. Thus, Ω_p (although a pseudoconvex domain) is not in general strictly pseudoconvex at every point of its boundary $\partial\Omega_p$, and the class $\Omega^{(n)}$ is distinct from the class of strictly pseudoconvex domains in $\mathbb{C}^n$.

Remark 2.3. By a result of Cartan [**8**], every bounded symmetric domain D in $\mathbb{C}^n$ is homogeneous in the sense that the automorphism group of D acts transitively on D. Also, by a result of Pinchuk [**37**], every bounded homogeneous domain in $\mathbb{C}^n$ with smooth boundary is biholomorphically equivalent to the unit ball $\mathbb{B}_n$ in $\mathbb{C}^n$. If Ω_p were to be a bounded symmetric domain, it would thus be biholomorphically equivalent to $\mathbb{B}_n$. A result of Sunada [**46**], however, states that two Reinhardt domains D_1 and D_2 in $\mathbb{C}^n$ that contain the origin are biholomorphically equivalent if and only if there exist positive numbers $r_1, \ldots, r_n$ and a permutation σ of $\{1, \ldots, n\}$ such that

$$D_2 = \{(r_1 z_{\sigma(1)}, \ldots, r_n z_{\sigma(n)}) : (z_1, \ldots, z_n) \in D_1\}.$$

It follows that the class $\Omega^{(n)}$ is distinct from the class of bounded symmetric domains in $\mathbb{C}^n$.

Let $K \subset \mathbb{C}^n$ be compact, and let A be a unital closed subalgebra of $C(K)$ containing n-variable complex polynomials. The *Shilov boundary* of A is defined to be the smallest closed subset S of K such that

$$\|f\|_{\infty,K} = \|f\|_{\infty,S}, \quad f \in A.$$

Of special interest to us is the subalgebra $A(\Omega_p) = \{f \in C(\overline{\Omega}_p) : f$ is holomorphic on $\Omega_p\}$ of $C(\overline{\Omega}_p)$, where

$$\overline{\Omega}_p = \left\{ z = (z_1, z_2, \dots, z_n) \in \mathbb{C}^n : \sum_{i=1}^{n} \sum_{j=1}^{m_i} |z_i|^{2p_{i,j}} \leq 1 \right\}$$

is the closure of Ω_p. If $\mathbb{O}(\overline{\Omega}_p)$ is the vector space of functions f such that f is holomorphic on an open neighborhood U_f of $\overline{\Omega}_p$, then (referring to the first line of Remark 3.2) it is easy to see that the closure of $\mathbb{O}(\overline{\Omega}_p)$ in the sup norm with respect to $\overline{\Omega}_p$ is $A(\Omega_p)$.

Proposition 2.4. *The Shilov boundary of $A(\Omega_p)$ coincides with the topological boundary $\partial\Omega_p$ of Ω_p.*

Proof. Since Ω_p is a bounded pseudoconvex domain in $\mathbb{C}^n$ with smooth boundary, it follows from [**36,** Folgerung 5] (see also [**24**]) that the Shilov boundary of $A(\Omega_p)$ is the closure of the set of strictly pseudoconvex points in $\partial\Omega_p$. It is easy to see that any point $b = (b_1, \dots, b_n)$ of $\partial\Omega_p$ for which each b_i is non-zero is a point of strict pseudoconvexity. But such points are dense in $\partial\Omega_p$ so that the Shilov boundary of $A(\Omega_p)$ is $\partial\Omega_p$. □

Let K be a compact subset of $\mathbb{C}^n$, let A be a closed subspace of $C(K)$, and let η be a positive regular Borel measure on K. The triple (A, K, η) is said to be *regular* (in the sense of [**1**]) if, for any positive function ϕ in $C(K)$, there exists a sequence of functions $\{\phi_m\}_{m\geq 1}$ in A such that $|\phi_m| < \phi$ on K and $\lim_{m\to\infty} |\phi_m| = \phi$ η-almost everywhere.

Proposition 2.5. *For any positive regular Borel measure μ_p on $\overline{\Omega}_p$ with* $\text{supp}(\mu_p) \subset \partial\Omega_p$*, the triple $(A(\Omega_p), \overline{\Omega}_p, \mu_p)$ is regular as is the triple $(A(\Omega_p)|\partial\Omega_p, \partial\Omega_p, \mu_p)$.*

Proof. For $p = (p_1, p_2, \dots, p_n)$ with $p_i = (p_{i,1}, p_{i,2}, \dots, p_{i,m_i})$, let $N = m_1 + \cdots + m_n$. Consider

$$f(z) = ({z_1}^{p_{1,1}}, \dots, {z_1}^{p_{1,m_1}}, \dots, {z_n}^{p_{n,1}}, \dots, {z_n}^{p_{n,m_n}}), \quad z \in \overline{\Omega}_p.$$

Clearly, f maps $\partial\Omega_p$ into the topological boundary $\partial\mathbb{B}_N$ of the unit ball $\mathbb{B}_N$ of $\mathbb{C}^N$. Thus, the regularity of the triple $(A(\Omega_p), \overline{\Omega}_p, \mu)$ will follow from [**17,** Proposition 2.5] provided we verify f to be injective. The regularity of the triple $(A(\Omega_p)|\partial\Omega_p, \partial\Omega_p, \mu)$ will then be an easy consequence of the Tietze extension theorem. Thus, let $z = (z_1, \ldots, z_n)$ and $w = (w_1, \ldots, w_n)$ be distinct points of $\overline{\Omega}_p$ so that $z_i \neq w_i$ for some i. If only one of z_i and w_i is non-zero, then clearly $f(z) \neq f(w)$. Thus, suppose that both z_i and w_i are non-zero. Using the coprimality of $p_{i,1}, p_{i,2}, \ldots, p_{i,m_i}$, we choose integers $n_{i,1}, n_{i,2}, \ldots, n_{i,m_i}$ such that

$$\sum_{j=1}^{m_i} n_{i,j} p_{i,j} = 1.$$

If one were to have ${z_i}^{p_{i,j}} = {w_i}^{p_{i,j}}$ for every j such that $1 \leq j \leq m_i$, then that would clearly force the contradiction $z_i = w_i$. Therefore, ${z_i}^{p_{i,j}} \neq {w_i}^{p_{i,j}}$ for some j satisfying $1 \leq j \leq m_i$, showing that $f(z) \neq f(w)$. $\square$

Here, we refer the reader to [**18,** Section 2]. If μ is a scalar spectral measure of the minimal normal extension $N \in \mathcal{B}(\mathcal{K})^n$ of a subnormal tuple $S \in \mathcal{B}(\mathcal{H})^n$, then there is an isomorphism Ψ_N of the von Neumann algebra $L^\infty(\mu)$ onto the von Neumann algebra $W^*(N)$ generated by $N_i \in \mathcal{B}(\mathcal{K})$. The *restriction algebra*

$$\mathcal{R}_S = \{f \in L^\infty(\mu) : \Psi_N(f)\mathcal{H} \subset \mathcal{H}\}$$

is a weak* closed subalgebra of $L^\infty(\mu)$. Let $K \subset \mathbb{C}^n$ be compact, and let A be a unital closed subalgebra of $C(K)$ containing n-variable complex polynomials. Following [**20**], we call a subnormal tuple S an *A-isometry* if the spectral measure of the minimal normal extension N of S is supported on the Shilov boundary of A and if A is contained in $\mathcal{R}_S$. Given a normalized positive regular Borel measure μ_p supported on $\partial\Omega_p$, we let $H^2(\mu_p)$ be the closure of $A(\Omega_p)$ in $\mathcal{L}^2(\mu_p)$. Letting σ_p denote the normalized surface area measure on $\partial\Omega_p$, we refer to $H^2(\sigma_p)$ as the *Hardy space* of Ω_p. In view of Proposition 2.4 and the discussion in [**18,** Section 2], the multiplication tuple $M_{\mu_p,z} = (M_{\mu_p,z_1}, \ldots, M_{\mu_p,z_n})$ of multiplications by the coordinate functions z_i on $H^2(\mu_p)$ is an $A(\Omega_p)$-isometry (and has the multiplication tuple $N_{\mu_p,z} = (N_{\mu_p,z_1}, \ldots, N_{\mu_p,z_n})$ associated with $L^2(\mu_p)$ as its minimal normal extension); also, in light of Proposition 2.5, $M_{\mu_p,z}$ is *regular* in the sense of [**20**], that is, in the sense of [**18,** Definition 2.6].

The preceding observations allow us to bring all of the results in [**16, 18, 20, 21**] related to a regular A-isometry to bear upon the multiplication tuple $M_{\mu_p,z}$; we highlight in Remarks 2.6 and 2.7 below a few implications of the results in those references. We also point out that some of those results are derived exploiting Prunaru's work in [**38**].

Remark 2.6. Let P_{μ_p} be the orthogonal projection of $\mathcal{L}^2(\mu_p)$ onto $H^2(\mu_p)$, and let, for $\phi \in L^\infty(\mu_p)$, $N_{\mu_p,\phi}$ denote the operator of multiplication by ϕ on $L^2(\mu_p)$. We let $T_{\mu_p,\phi}$ stand for $P_{\mu_p}N_{\mu_p,\phi}|H^2(\mu_p)$ and refer to $\mathcal{T}(M_{\mu_p,z}) = \{T_{\mu_p,\phi} : \phi \in L^\infty(\mu_p)\}$ as the set of $M_{\mu_p,z}$-*Toeplitz operators*. In addition, we use $H^\infty_{A(\Omega_p)}(\mu_p)$ to denote the weak* closure of $A(\Omega_p)$ in $L^\infty(\mu_p)$ and refer to any member θ of $H^\infty_{A(\Omega_p)}(\mu_p)$ satisfying $|\theta| = 1$ μ_p-almost everywhere as a μ_p-*inner function*. It follows from [**18,** Corollary 3.3] that $\mathcal{T}(M_{\mu_p,z})$ equals the set

$$\{X \in \mathcal{B}(H^2(\mu_p)) : T_{\mu_p,\bar{\theta}}XT_{\mu_p,\theta} = X \text{ for every } \mu_p\text{-inner function } \theta\}.$$

Further, if $C^*(\mathcal{T}(M_{\mu_p,z}))$ is the C^*-subalgebra of $\mathcal{B}(H^2(\mu_p))$ generated by $\mathcal{T}(M_{\mu_p,z})$, $\mathcal{SC}(M_{\mu,z})$ the two-sided closed ideal in $C^*(\mathcal{T}(M_{\mu_p,z}))$ generated by semicommutators $T_{\mu_p,\phi}T_{\mu_p,\psi} - T_{\mu_p,\phi\psi}$ $(\phi, \psi \in L^\infty(\mu_p))$, and $(N_{\mu_p,z})'$ the commutant in $\mathcal{B}(\mathcal{L}^2(\mu_p))$ of $\{N_{\mu_p,z_1}, \dots, N_{\mu_p,z_n}\}$, then [**18,** Corollary 3.7] yields the existence of a short exact sequence of C*-algebras

$$0 \longrightarrow \mathcal{SC}(M_{\mu_p,z}) \xrightarrow{\iota} C^*(\mathcal{T}(M_{\mu_p,z})) \xrightarrow{\pi} (N_{\mu_p,z})' \longrightarrow 0,$$

where ι is the inclusion map and π is a unital $*$-homomorphism which is, in fact, a left inverse of the compression map

$$\rho : (N_{\mu_p,z})' \longrightarrow \mathcal{B}(H^2(\mu_p)),$$

given by

$$\rho(Y) = P_{\mu_p}Y|H^2(\mu_p),$$

$Y \in (N_{\mu_p,z})'$.

Remark 2.7.

(a) Let $\mathcal{A}_{M_{\mu_p,z}}$ be the weak*-closed subalgebra of $\mathcal{B}(H^2(\mu_p))$ generated by M_{μ_p,z_i}, $1 \le i \le n$, and the identity operator on $H^2(\mu_p)$. It is a consequence of [**20,** Corollary 6] that the weak operator topology and the weak* operator topology coincide on $\mathcal{A}_{M_{\mu_p,z}}$ and that every unital weak*-closed subalgebra of $\mathcal{A}_{M_{\mu_p,z}}$ is reflexive; in particular, $M_{\mu_p,z}$ is reflexive (refer to [**20**] for the relevant definitions).

(b) It is a consequence of [**16,** Corollary 2] that the set $\mathcal{T}(M_{\mu_p,z})$ of $M_{\mu_p,z}$-Toeplitz operators is 2-hyperreflexive with the 2-hyperreflexivity constant $\kappa_2(\mathcal{T}(M_{\mu_p,z}))$ being less than or equal to 2 (refer to [**31**] for the relevant definitions).

As the results in [**18, 21**] show, some extra mileage may be obtained out of the notion of a regular A-isometry T under the additional assumption that T is essentially normal. We plan to explore the essential normality of the multiplication tuple $M_{\sigma_p,z}$ associated with the Hardy space $H^2(\sigma_p)$ of Ω_p, and, for that purpose, we invoke in the next section the theory related to the famous $\overline{\partial}$-Neumann problem.

3. $\overline{\partial}$-Neumann operator and the tangential Neumann operator. While a basic reference for the material in this section is [**22**], we find, in addition, [**48**] to be a convenient reference for our purposes (see also [**44**]). Indeed, some of the arguments in [**48**] are adaptations and extensions of the arguments in [**22**] to the context of the Hardy and Bergman spaces of strictly pseudoconvex domains, and our task here is to push through the analogs of those arguments in the context of the domains Ω_p.

Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$ with its boundary $\partial\Omega = \{z \in \mathbb{C}^n : \rho(z) = 0\}$ defined by a smooth function

$$\rho : \mathbb{C}^n \longrightarrow \mathbb{R}$$

satisfying $d\rho(z) \neq 0$ if $\rho(z) = 0$.

For $0 \leq q \leq n$ (≥ 2), let $C_q^\infty(\overline{\Omega})$ be the vector space of $(0,q)$-forms with coefficients in $C^\infty(\overline{\Omega})$, the vector space of complex-valued functions f such that f is infinitely differentiable on an open neighborhood U_f of $\overline{\Omega}$. The Cauchy-Riemann operator $\overline{\partial}$ gives rise to (a special version of) the *Dolbeault complex* (or the *Cauchy-Riemann complex*)

$$0 \longrightarrow C_0^\infty(\overline{\Omega}) \xrightarrow{\bar{\partial}_0} C_1^\infty(\overline{\Omega}) \longrightarrow \cdots \xrightarrow{\bar{\partial}_{n-1}} C_n^\infty(\overline{\Omega}) \longrightarrow 0.$$

Using the normalized volumetric measure ν on $\overline{\Omega}$, an inner product on $C_q^\infty(\overline{\Omega})$ may be defined in a natural way (refer to [**48,** Chapter 2, Section 2.1]). Let $L_q^2(\Omega)$ be the Hilbert space completion of $C_q^\infty(\overline{\Omega})$ in this inner product, with the corresponding norm on $L_q^2(\Omega)$ being denoted by $\|\cdot\|$ (for any q). The closure of $\overline{\partial}_q$ will still be denoted by $\overline{\partial}_q$; thus, $\overline{\partial}_q$ is a densely defined closed (linear) operator from $L_q^2(\Omega)$ into $L_{q+1}^2(\Omega)$. The Hilbert space adjoint of $\overline{\partial}_q$ will be denoted by $\overline{\partial}_{q+1}^*$ (unlike $\overline{\partial}_q^*$ in **[48,** (2.1.13)] which, in view of the subsequent formulas employed there, is a notational inaccuracy). The (qth) $\overline{\partial}$-*Neumann Laplacian* is defined by $\Box_q = \overline{\partial}_{q-1}\overline{\partial}_q^* + \overline{\partial}_{q+1}^*\overline{\partial}_q$ (with $\overline{\partial}_n$, $\overline{\partial}_{-1}$, $\overline{\partial}_{n+1}^*$ and $\overline{\partial}_0^*$ being interpreted as zero operators). For $1 \le q \le n$, $\Box_q$ turns out to be invertible with a bounded inverse N_q (refer to **[22, 27]**); the operator N_q is referred to as the (qth) $\overline{\partial}$-*Neumann operator*.

For $0 \le q \le n$ (≥ 2), let $R_q^\infty(\partial\Omega)$ be the vector space obtained by restricting the members of $C_q^\infty(\overline{\Omega})$ to $\partial\Omega$. If

$$f = \sum_{i_1<\cdots<i_q} \phi_{i_1,\ldots,i_q} z_{i_1} \wedge \cdots \wedge z_{i_q}$$

and

$$g = \sum_{i_1<\cdots<i_q} \psi_{i_1,\ldots,i_q} z_{i_1} \wedge \cdots \wedge z_{i_q}$$

(in the standard notation) are in $R_q^\infty(\partial\Omega)$, then f is said to be *pointwise orthogonal* to g if

$$\sum_{i_1<\cdots<i_q} \phi_{i_1,\ldots,i_q}(b)\overline{\psi_{i_1,\ldots,i_q}(b)} = 0$$

for every $b \in \partial\Omega$ (notation: $f \perp g$). If $N_q^\infty(\partial\Omega)$ is the vector space

$$\{f \in R_q^\infty(\partial\Omega) : f \wedge (\overline{\partial}\rho|\partial\Omega) = 0\},$$

then we declare $C_q^\infty(\partial\Omega)$ to be the vector space

$$\{f \in R_q^\infty(\partial\Omega) : f \perp g \text{ for all } g \in N_q^\infty(\partial\Omega)\};$$

it is to be noted that $C_n^\infty(\partial\Omega) = \{0\}$. The Cauchy-Riemann operator $\overline{\partial}$ induces the tangential Cauchy-Riemann operator $\overline{\partial}_b$ (refer to **[22, 33]**) that gives rise to (a special version of) the *Kohn-Rossi complex* (or the *tangential Cauchy-Riemann complex*)

$$0 \longrightarrow C_0^\infty(\partial\Omega) \xrightarrow{\overline{\partial}_{b,0}} C_1^\infty(\partial\Omega) \longrightarrow \cdots \xrightarrow{\overline{\partial}_{b,n-2}} C_{n-1}^\infty(\partial\Omega) \longrightarrow 0.$$

The vector space $C_q^\infty(\partial\Omega)$ can be naturally equipped with an inner product by using the normalized surface area measure σ on $\partial\Omega$ (refer to [**48,** Chapter 2, subsection 2.2]). Let $L_q^2(\partial\Omega)$ be the Hilbert space completion of $C_q^\infty(\partial\Omega)$ in this inner product. The closure of $\overline{\partial}_{b,q}$ will still be denoted by $\overline{\partial}_{b,q}$; thus, $\overline{\partial}_{b,q}$ is a densely defined closed (linear) operator from $L_q^2(\partial\Omega)$ into $L_{q+1}^2(\partial\Omega)$. The Hilbert space adjoint of $\overline{\partial}_{b,q}$ will be denoted by $\overline{\partial}_{b,q+1}^*$ (with the notational inaccuracy in [**48,** (2.2.9)] noted). The (qth) *Kohn Laplacian* is defined by

$$\Box_{b,q} = \overline{\partial}_{b,q-1}\overline{\partial}_{b,q}^* + \overline{\partial}_{b,q+1}^*\overline{\partial}_{b,q}$$

(with $\overline{\partial}_{b,n-1}$, $\overline{\partial}_{b,-1}$, $\overline{\partial}_{b,n}^*$ and $\overline{\partial}_{b,0}^*$ being interpreted as zero operators). For $1 \leq q \leq n-1$, $\Box_{b,q}$ turns out to be invertible with a bounded inverse $N_{b,q}$ (refer to [**22, 32**]); the operator $N_{b,q}$ is referred to as the (qth) *complex Green operator* or the (qth) *tangential Neumann operator.*

Let $W_1^{-1}(\Omega)$ be the vector space of $(0,1)$-forms f with coefficients in the Sobolev space $W^{-1}(\Omega)$ of order -1, and let $\|f\|_{-1}^2$ be the sum of squares of the $W^{-1}(\Omega)$ norms of the coefficients of f. We say that a *compactness estimate* holds (for Ω) if, for every positive ϵ, there exists a $C(\epsilon)$ such that

$$\|f\|^2 \leq \epsilon\{\|\overline{\partial}_1 f\|^2 + \|\overline{\partial}_1^* f\|^2\} + C(\epsilon)\|f\|_{-1}^2$$

for all $(0,1)$-forms f that lie in Domain$(\overline{\partial}_1) \cap$ Domain$(\overline{\partial}_1^*)$ $(\subset L_1^2(\Omega) \subset W_1^{-1}(\Omega))$.

It is said that $\partial\Omega$ satisfies the *Catlin property* (P) if, for every positive M, there exists a plurisubharmonic function λ in $C^\infty(\overline{\Omega})$ with $0 \leq \lambda \leq 1$ such that

$$\sum_{j,k=1}^n \frac{\partial^2\lambda}{\partial z_j \partial \overline{z}_k}(b) t_j \overline{t}_k \geq M\{|t_1|^2 + \cdots + |t_n|^2\}$$

for all points $t = (t_1, \ldots, t_n)$ in $\mathbb{C}^n$ and for all points b of $\partial\Omega$.

Remark 3.1. If a bounded pseudoconvex domain Ω has real analytic boundary $\partial\Omega$, then it follows from [**9,** Theorem 2] and [**19,** Lemma 2] that $\partial\Omega$ satisfies the Catlin property (P); in particular, $\partial\Omega_p$ satisfies the Catlin property (P).

The closure of $A(\Omega_p)$ in $L^2(\nu_p)$, where ν_p is the normalized volumetric measure on $\overline{\Omega}_p$, will be referred to as the *Bergman space* of Ω_p and will be denoted by $A^2(\nu_p)$. The tuple of multiplications by the coordinate functions z_i on $A^2(\nu_p)$ will be denoted by $M_{\nu_p,z}$. Let $\widetilde{P}_{\nu_p}$ be the orthogonal projection of $\mathcal{L}^2(\nu_p)$ onto $A^2(\nu_p)$, and let, for $\phi \in L^\infty(\nu_p)$, $\widetilde{N}_{\nu_p,\phi}$ denote the operator of multiplication by ϕ on $L^2(\nu_p)$. We let $\widetilde{T}_{\nu_p,\phi}$ stand for $\widetilde{P}_{\nu_p}\widetilde{N}_{\nu_p,\phi}|A^2(\nu_p)$ and refer to $\widetilde{T}_{\nu_p,\phi}$ as a *Bergman-Toeplitz operator*. The adjoint of the Bergman-Toeplitz operator $\widetilde{T}_{\nu_p,\phi}$ (respectively, $M_{\mu_p,z}$-Toeplitz operator $T_{\mu_p,\phi}$ of Remark 2.6) equals $\widetilde{T}_{\nu_p,\bar{\phi}}$ (respectively, $T_{\mu_p,\bar{\phi}}$).

Remark 3.2. The domain Ω_p is starlike with respect to the origin, and any $f \in A(\Omega_p)$ can be uniformly approximated on $\overline{\Omega}_p$ by the sequence $\{f_m\}$ of functions f_m in $\mathbb{O}(\overline{\Omega}_p)$ where $f_m(z) = f((1-(1/m))z)$. Further, $\overline{\Omega}_p$ is polynomially convex so that any function such as f_m that is holomorphic on an open neighborhood of $\overline{\Omega}_p$ is the uniform limit (on $\overline{\Omega}_p$) of polynomials by the Oka-Weil approximation theorem, see [**41,** Chapter VI, Theorem 1.5]. It is then clear that the Hardy space $H^2(\sigma_p)$ (respectively, Bergman space $A^2(\nu_p)$) as previously defined is really the closure of polynomials in $L^2(\sigma_p)$ (respectively, $L^2(\nu_p)$) with the constant function 1 in $H^2(\sigma_p)$ (respectively, $A^2(\nu_p)$) being a cyclic vector for $M_{\sigma_p,z}$ (respectively, $M_{\nu_p,z}$). The multiplication tuple $M_{\sigma_p,z}$ (respectively, $M_{\nu_p,z}$) can be looked upon as a multivariable weighted shift, with the positive weights of $M_{\sigma_p,z}$ (respectively, $M_{\nu_p,z}$) computed by checking the action of each M_{σ_p,z_i} (respectively, M_{ν_p,z_i}) on the members of the orthonormal basis obtained by applying the Gram-Schmidt process to the constant function 1 and the powers of z_i in the Hardy space $H^2(\sigma_p)$ (respectively, Bergman space $A^2(\nu_p)$) (refer to [**30**]). For an arbitrary Ω_p, such computations can turn out to be formidable as can be gathered, for example, by referring to similar computations carried out in [**12**] in the context of 'complex ellipsoids' in $\mathbb{C}^n$.

Proposition 3.3. *The semicommutator $\widetilde{T}_{\nu_p,\phi}\widetilde{T}_{\nu_p,\psi} - \widetilde{T}_{\nu_p,\phi\psi}$ of the Bergman-Toeplitz operators $\widetilde{T}_{\nu_p,\phi}$ and $\widetilde{T}_{\nu_p,\psi}$ is compact for any continuous functions ϕ and ψ on $\overline{\Omega}_p$.*

Proof. In view of Remark 3.1, $\partial\Omega_p$ satisfies the Catlin property (P). The Catlin property (P) implies that a compactness estimate holds for Ω_p (refer to [**9,** Theorem 1]). That, in turn, implies that the $\overline{\partial}$-Neumann operator N_1 corresponding to Ω_p is compact (refer to [**7,** Lemma 11].) Now, arguing exactly as in [**48,** Lemma 2.1.24], it may be proven that $\overline{\partial}_1^* N_1$ is a compact operator. (The symbol $\overline{\partial}_0^*$ in the proof of [**48,** Lemma 2.1.24] should be corrected to $\overline{\partial}_2^*$.) Next, using the compactness of $\overline{\partial}_1^* N_1$ and arguing as in [**48,** Lemma 2.1.22, Theorem 4.1.18], it may be proven that

$$(I - \widetilde{P}_{\nu_p})\widetilde{N}_{\nu_p,\phi}|A^2(\nu_p) : A^2(\nu_p) \longrightarrow L^2(\nu_p)$$

is compact for any ϕ that is continuous on $\overline{\Omega}_p$. And, as in [**48,** Corollary 4.1.21], that leads to the compactness of the semicommutator $\widetilde{T}_{\nu_p,\phi}\widetilde{T}_{\nu_p,\psi} - \widetilde{T}_{\nu_p,\phi\psi}$ for any continuous functions ϕ and ψ on $\overline{\Omega}_p$. $\square$

Corollary 3.4. *The commutator $\widetilde{T}_{\nu_p,\phi}\widetilde{T}_{\nu_p,\psi} - \widetilde{T}_{\nu_p,\psi}\widetilde{T}_{\nu_p,\phi}$ of the Bergman-Toeplitz operators $\widetilde{T}_{\nu_p,\phi}$ and $\widetilde{T}_{\nu_p,\psi}$ is compact for any continuous functions ϕ and ψ on $\overline{\Omega}_p$; in particular, the multiplication tuple $M_{\nu_p,z}$ is essentially normal.*

Proposition 3.5. *Let $n \geq 3$. For $\Omega_p \subset \mathbb{C}^n$, the semicommutator $T_{\sigma_p,\phi}T_{\sigma_p,\psi} - T_{\sigma_p,\phi\psi}$ of the $M_{\sigma_p,z}$-Toeplitz operators $T_{\sigma_p,\phi}$ and $T_{\sigma_p,\psi}$ is compact for any continuous functions ϕ and ψ on $\partial\Omega_p$.*

Proof. In view of Remark 3.1, $\partial\Omega_p$ satisfies the Catlin property (P). It follows from [**40,** Theorem 1.4] that the tangential Neumann operator $N_{b,1}$ corresponding to Ω_p is compact. Now, arguing exactly as in the Bergman case, an analog of [**48,** Lemma 2.2.19] may be proven to obtain that $\overline{\partial}_{b,1}^* N_{b,1}$ is a compact operator. Next, using the compactness of $\overline{\partial}_{b,1}^* N_{b,1}$ (and arguing as in the Bergman case) analogs of [**48,** Lemma 2.2.18, Theorem 4.2.17] may be established to obtain that

$$(I - P_{\sigma_p})N_{\sigma_p,\phi}|H^2(\sigma_p) : H^2(\sigma_p) \longrightarrow L^2(\sigma_p)$$

is compact for any ϕ that is continuous on $\partial\Omega_p$. And, that leads to an analog of [**48,** Corollary 4.2.20], yielding the compactness of the semicommutator $T_{\sigma_p,\phi}T_{\sigma_p,\psi} - T_{\sigma_p,\phi\psi}$ for any continuous functions ϕ and ψ on $\partial\Omega_p$. $\square$

Corollary 3.6. *Let $n \geq 3$. For $\Omega_p \subset \mathbb{C}^n$, the commutator $T_{\sigma_p,\phi}T_{\sigma_p,\psi} - T_{\sigma_p,\psi}T_{\sigma_p,\phi}$ of the $M_{\sigma_p,z}$-Toeplitz operators $T_{\sigma_p,\phi}$ and $T_{\sigma_p,\psi}$ is compact for any continuous functions ϕ and ψ on $\partial\Omega_p$; in particular, the multiplication tuple $M_{\sigma_p,z}$ is essentially normal.*

We do not know whether the tangential Neumann operator $N_{b,1}$ corresponding to an arbitrary $\Omega_p \subset \mathbb{C}^2$ is compact; as such, a different strategy is adopted below to prove the essential normality of the multiplication pair $M_{\sigma_p,z} \in (\mathcal{B}(H^2(\sigma_p))^2$ for any $\Omega_p \subset \mathbb{C}^2$.

Proposition 3.7. *For $\Omega_p \subset \mathbb{C}^2$, the multiplication pair $M_{\sigma_p,z}$ is essentially normal.*

Proof. Since Ω_p $(\subset \mathbb{C}^2)$ is a pseudoconvex complete Reinhardt domain with real analytic boundary, it follows from the work of Sheu [**45**] that there is a $*$-isomorphism Ψ of the C*-algebra $\mathbb{A}$ generated by the set

$$\{\widetilde{T}_{\nu_p,\phi} : \phi \text{ is continuous on } \overline{\Omega}_p\}$$

with the C*-algebra $\mathbb{B}$ generated by the set

$$\{T_{\sigma_p,\phi} : \phi \text{ is continuous on } \partial\Omega_p\}.$$

In view of Remark 3.2 and [**30,** Corollary 13], the C*-algebras $\mathbb{A}$ and $\mathbb{B}$ are irreducible. Let $\mathcal{K}(A^2(\nu_p))$ (respectively, $\mathcal{K}(H^2(\sigma_p)))$ be the C*-algebra of compact operators on $A^2(\nu_p)$ (respectively, $H^2(\sigma_p)$). Since $\mathbb{A}$ has, by Corollary 3.4, a non-trivial intersection with $\mathcal{K}(A^2(\nu_p))$ and, since $\mathbb{A}$ is irreducible, $\mathbb{A}$ contains $\mathcal{K}(A^2(\nu_p))$ (refer to [**11**]). Consider

$$\Psi|\mathcal{K}(A^2(\nu_p)) : \mathcal{K}(A^2(\nu_p)) \longrightarrow \mathcal{B}(H^2(\sigma_p)).$$

Since $\Psi(\mathcal{K}(A^2(\nu_p)))$ is an ideal of $\mathbb{B}$ and since $\mathbb{B}$ is irreducible, $(\Psi|\mathcal{K}(A^2(\nu_p), H^2(\sigma_p))$ is an irreducible representation of $\mathcal{K}(A^2(\nu_p))$. Then, it follows from [**11,** Corollary 16.12] that $\Psi(\mathcal{K}(A^2(\nu_p)) = \mathcal{K}(H^2(\sigma_p))$. Letting $T_i = \Psi^{-1}(M_{\sigma_p,z_i})$, it is clear that the compactness of

$$M^*_{\sigma_p,z_i}M_{\sigma_p,z_j} - M_{\sigma_p,z_j}M^*_{\sigma_p,z_i} \in \mathcal{B}(H^2(\sigma_p))$$

would follow from that of

$$T_i^*T_j - T_jT_i^* \in \mathcal{B}(A^2(\nu_p)).$$

However, the compactness of $T_i^*T_j - T_jT_i^* \in \mathcal{B}(A^2(\nu_p))$ can easily be deduced from the result of Proposition 3.3 and the fact that the uniform limit of compact operators is compact. $\square$

The results of Corollary 3.6 and Proposition 3.7 allow us to bring all of the results in [**18, 21**] related to an essentially normal regular A-isometry to bear upon the multiplication tuple $M_{\sigma_p,z}$; we highlight in Remark 3.8 below a couple of implications of the results in those works.

Remark 3.8.

(a) Let $\mathcal{T}_a(M_{\sigma_p,z})$ be the set

$$\{T_{\sigma_p,\phi} : \phi \in H^\infty_{A(\Omega_p)}(\sigma_p)\}.$$

For $\phi \in L^\infty$, let $H_{\sigma_p,\phi}$ be the *Hankel operator* from $H^2(\sigma_p)$ to

$$H^2(\sigma_p)^\perp = L^2(\sigma_p) \ominus H^2(\sigma_p),$$

defined by $H_{\sigma_p,\phi} = (I - P_{\sigma_p})N_{\sigma_p,\phi}|H^2(\sigma_p)$. In view of the observations in the proof of [**18,** Corollary 3.3], and in view of [**21,** Corollary 5.1], an operator $S \in \mathcal{B}(H^2(\sigma_p))$ is in the essential commutant of $\mathcal{T}_a(M_{\sigma_p,z})$ if and only if S equals $T_{\sigma_p,\phi} + K$ for some compact operator K on $H^2(\sigma_p)$ and some ϕ in $L^\infty(\sigma_p)$ for which the Hankel operator $H_{\sigma_p,\phi}$ is compact.

(b) From [**18,** Proposition 3.10], the existence of a short exact sequence of C*-algebras

$$0 \longrightarrow \mathcal{K}(H^2(\sigma_p)) \xrightarrow{\iota} \mathbb{B} \xrightarrow{\pi} C(\partial\Omega_p) \longrightarrow 0$$

may be deduced where $\mathcal{K}(H^2(\sigma_p))$ and $\mathbb{B}$ are as in the proof of Proposition 3.7, ι is the inclusion map and π is a unital $*$-homomorphism satisfying $\pi(T_{\sigma_p,\phi}) = \phi$ for any $\phi \in C(\partial\Omega_p)$.

4. $\partial\Sigma_p$-isometries. Let $p = (p_1, p_2, \ldots, p_n)$ be an n-tuple of m_i-tuples $p_i = (p_{i,1}, p_{i,2}, \ldots, p_{i,m_i})$ where $p_{i,1}, \ldots, p_{i,m_i}$ (with $m_i \geq 1$) are positive integers. The subset Σ_p of $\mathbb{C}^n$ is defined by

$$\Sigma_p = \left\{ z = (z_1, z_2, \ldots, z_n) \in \mathbb{C}^n : \sum_{i=1}^{n} \sum_{j=1}^{m_i} |z_i|^{2p_{i,j}} < 1 \right\}.$$

The set Σ_p is easily seen to be a convex complete Reinhardt domain in $\mathbb{C}^n$ with the real analytic boundary

$$\partial\Sigma_p = \left\{ z = (z_1, z_2, \ldots, z_n) \in \mathbb{C}^n : \sum_{i=1}^{n} \sum_{j=1}^{m_i} |z_i|^{2p_{i,j}} = 1 \right\}.$$

We use the symbol $\Sigma^{(n)}$ to denote the class of domains Σ_p in $\mathbb{C}^n$. Trivially, $\Sigma^{(n)}$ is a superclass of the class $\Omega^{(n)}$. The domain Σ_p is a so-called *complex ellipsoid* in case $m_i = 1$ for each i; we also note that, for any n, $\mathbb{B}_n \in \Sigma^{(n)} \setminus \Omega^{(n)}$.

Definition 4.1. If $S = (S_1, \dots, S_n)$ is a subnormal n-tuple of (commuting) operators S_i in $\mathcal{B}(\mathcal{H})$ such that the spectral measure ρ_N of the minimal normal extension N of S is supported on $\partial\Sigma_p$, then S is called a $\partial\Sigma_p$*-isometry*.

Remark 4.2. The statements (and proofs) of Propositions 2.4, 3.3, 3.5 and 3.7, along with those of Corollaries 3.4 and 3.6 hold, and the contents of Remarks 3.1 and 3.2 remain applicable with Σ_p in the place of Ω_p and with the obvious corresponding interpretations of $A(\Sigma_p)$, σ_p, ν_p, $H^2(\sigma_p)$, $A^2(\nu_p)$, $M_{\sigma_p,z}$, $M_{\nu_p,z}$, $\mathbb{A}$ and $\mathbb{B}$. We also note that any Σ_p-isometry $S \in \mathcal{B}(\mathcal{H})^n$ is an $A(\Sigma_p)$-isometry. (Indeed, if μ_p is a scalar spectral measure of the minimal normal extension N of S, then μ_p is supported on $\partial\Sigma_p$ where $\partial\Sigma_p$ is the Shilov boundary of $A(\Sigma_p)$ by the analog of Proposition 2.4 for Σ_p; thus, we need only check that $A(\Sigma_p)$ is contained in the restriction algebra $\mathcal{R}_S$ of S. Let $f \in A(\Sigma_p)$. Choosing f_m as in Remark 3.2 and using the Taylor functional calculus for S (refer to [**47**]), it is obtained that

$$f_m(N)|\mathcal{H} = f_m(S) \in \mathcal{B}(\mathcal{H}).$$

Since the sequence $\{f_m\}$ converges to f uniformly on $\partial\Sigma_p$, it is clear that $f(N)\mathcal{H} \equiv \Psi_N(f)\mathcal{H}$ is contained in $\mathcal{H}$.) Thus, Σ_p-isometries, like the less general Ω_p-isometries, are examples of essentially normal A-isometries, but Ω_p-isometries come with an added bonus of regularity.

Remark 4.3. The weak* closure $H^\infty_{A(\Sigma_p)}(\sigma_p)$ of $A(\Sigma_p)$ in $L^\infty(\sigma_p)$ can be identified with the algebra $H^\infty(\sigma_p)$ of the non-tangential boundary limits of the members of $H^\infty(\Sigma_p)$, where $H^\infty(\Sigma_p)$ is the algebra of bounded holomorphic functions on Σ_p. Indeed, $H^\infty(\Sigma_p)$ can be shown to be a weak*-closed subalgebra of $L^\infty(\sigma_p)$ and the map

$$\widetilde{r}_{\sigma_p} : H^\infty(\Sigma_p) \longrightarrow L^\infty(\sigma_p)$$

that associates with any $f \in H^\infty(\Sigma_p)$ its non-tangential boundary limit can be shown to be an isometric and a weak*-continuous algebra homomorphism as in the argument provided in the discussion preceding

[**18,** Corollary 4.8]; further, also as per the argument there, the inclusion

$$H^\infty_{A(\Sigma_p)}(\sigma_p) \subset H^\infty(\sigma_p)(= \widetilde{r}_{\sigma_p}(H^\infty(\Sigma_p))$$

holds. For the other direction of the inclusion, we use that $\widetilde{r}_{\sigma_p}$ is weak*-continuous and that any function $f \in H^\infty(\Sigma_p)$ can be approximated in the weak* topology of $L^\infty(\sigma_p)$ by the sequence $\{f_m\}$ where f_m are as in Remark 3.2.

An intrinsic characterization of $\partial\Sigma_p$-isometries can be provided using the results of [**5**]. If

$$q(z,w) = \sum_{\alpha,\beta} a_{\alpha,\beta} z^\alpha w^\beta$$

is a polynomial in the variables $z = (z_1, \ldots, z_n)$ and $w = (w_1, \ldots, w_n)$ with real coefficients $a_{\alpha,\beta}$, then, for any n-tuple $T = (T_1, \ldots, T_n)$ of commuting operators in $\mathcal{B}(\mathcal{H})$, we interpret $(q(z,w))(T,T^*)$ to be the operator

$$\sum_{\alpha,\beta} a_{\alpha,\beta} T^{*\beta} T^\alpha.$$

Since the Taylor spectrum of the minimal normal extension of a $\partial\Sigma_p$-isometry S is contained in $\partial\Sigma_p$, it follows by a result of Curto [**13**] that the Taylor spectrum of S is contained in the polynomial convex hull of $\partial\Sigma_p$, which is the closure $\overline{\Sigma}_p$ of Σ_p. Since $\overline{\Sigma}_p$ is contained in the closed unit polydisk in $\mathbb{C}^n$ centered at the origin, the spectral projection property of the Taylor spectrum implies that any coordinate S_i of S has its spectrum contained in the unit disk in $\mathbb{C}$ centered at the origin so that the spectral radius r_{S_i} of S_i cannot exceed 1. Since S_i is subnormal, the norm of S_i must equal r_{S_i} (refer to [**10**]), and hence, S_i is a contraction. The following result is now a consequence of [**5,** Proposition 7] and the observations in the proof of [**5,** Proposition 8].

Proposition 4.4. *Let $S = (S_1, \ldots, S_n)$ be an n-tuple of commuting operators S_i in $\mathcal{B}(\mathcal{H})$. Statements* (i) *and* (ii) *below are equivalent*:

(i) *S is a $\partial\Sigma_p$-isometry.*
(ii) (a) $(\Pi_{i=1}^n [1 - z_i w_i]^{k_i})(S, S^*) \geq 0$ *for all integers* $k_i \geq 0$.

(b)
$$\Big(1-\sum_{i=1}^{n}\sum_{j=1}^{m_i} z_i^{p_{i,j}} w_i^{p_{i,j}}\Big)(S,S^*)=0.$$

Condition (ii)(b) of Proposition 4.4 can simply be written as

$$I-\sum_{i=1}^{n}\sum_{j=1}^{m_i} S_i^{*p_{i,j}} S_i^{p_{i,j}}=0$$

and, as shown below, characterizes by itself a $\partial\Sigma_p$-isometry for a special type of Σ_p. We consider those Σ_p (with $p=(p_1,\dots,p_n)$) for which each p_i has at least one integer coordinate equal to 1; we use the symbol $\widetilde{\Sigma}_p$ to denote any such Σ_p and note that $\widetilde{\Sigma}_p$ is strictly pseudoconvex. The unit ball $\mathbb{B}_n=\Sigma_{(p_1,\dots,p_n)}$ with $p_i=(1)$ for every i is a special example of such a domain; we note that $\partial\mathbb{B}_n$-isometries are precisely spherical isometries. The next proposition provides a characterization of a $\partial\widetilde{\Sigma}_p$-isometry that is a generalization of that of a spherical isometry.

Proposition 4.5. *Let $S=(S_1,\dots,S_n)$ be an n-tuple of commuting operators S_i in $\mathcal{B}(\mathcal{H})$. Statements* (i) *and* (ii) *below are equivalent*:

(i) *S is a $\partial\widetilde{\Sigma}_p$-isometry.*

(ii)
$$\Big(1-\sum_{i=1}^{n}\sum_{j=1}^{m_i} z_i^{p_{i,j}} w_i^{p_{i,j}}\Big)(S,S^*)=0.$$

Proof. The implication (i) $\Rightarrow$ (ii) is trivial. In order to prove (ii) $\Rightarrow$ (i), we need only show that condition (ii) of Proposition 4.5 guarantees condition (ii)(a) of Proposition 4.4, viz., $(\Pi_{i=1}^n[1-z_iw_i]^{k_i})(S,S^*)\geq 0$ for all integers $k_i\geq 0$. We assume, without any loss of generality, that $p_{i,1}=1$ for each i. Let

$$q_i(z,w)=\sum_{j=2}^{m_i} z_i^{p_{i,j}} w_i^{p_{i,j}}+\sum_{k\neq i}\sum_{j=1}^{m_k} z_k^{p_{k,j}} w_k^{p_{k,j}}.$$

That the condition (ii)(a) of Proposition 4.4 holds follows by observing that $(\Pi_{i=1}^n [1 - z_i w_i]^{k_i})(S, S^*)$ can be written as

$$\left(\Pi_{i=1}^n \left[\left\{1 - \sum_{i=1}^{n} \sum_{j=1}^{m_i} z_i^{p_{i,j}} w_i^{p_{i,j}}\right\} + q_i(z, w)\right]^{k_i}\right)(S, S^*). \qquad \square$$

We now turn to examining the intertwining of two $\partial\Omega_p$-isometries. By choosing $S = T$ in the following proposition, one obtains a commutant lifting theorem for a $\partial\Omega_p$-isometry.

Proposition 4.6. *Let $S = (S_1, \ldots, S_n) \in \mathcal{B}(\mathcal{H})^n$ and $T = (T_1, \ldots, T_n) \in \mathcal{B}(\mathcal{K})^n$ be $\partial\Omega_p$-isometries, and let $M = (M_1, \ldots, M_n) \in \mathcal{B}(\widetilde{\mathcal{H}})^n$ and $N = (N_1, \ldots, N_n) \in \mathcal{B}(\widetilde{\mathcal{K}})^n$ be the minimal normal extensions of S and T, respectively. If*

$$X : H \longrightarrow K$$

is a bounded linear map intertwining S and T so that $XS_i = T_iX$ for all i, then X lifts to a bounded linear map

$$\widetilde{X} : \widetilde{H} \longrightarrow \widetilde{K}$$

intertwining M and N and satisfying $\|\widetilde{X}\| = \|X\|$.

Proof. Since the Taylor spectra of M and N are contained in $\partial\Omega_p$, from a result of Curto **[13]** (mentioned previously) the Taylor spectra of S and T are contained in the polynomial convex hull of $\partial\Omega_p$, which is $\overline{\Omega}_p$. Let $f \in A(\Omega_p)$. For any positive integer $m \geq 2$, f_m defined by $f_m(z) = f((1 - 1/m)z)$ is holomorphic on an open neighborhood of $\overline{\Omega}_p$. If X intertwines S and T, then it follows by the Taylor functional calculus (see [**47,** Proposition 4.5]) that $Xf_m(S) = f_m(T)X$. If ρ_M (respectively, ρ_N) is the spectral measure of M (respectively, N), then $\rho_S = P_{\mathcal{H}}\rho_M|\mathcal{H}$ (respectively, $\rho_T = P_{\mathcal{K}}\rho_N|\mathcal{K}$) is the semi-spectral measure of S (respectively, T) with $P_{\mathcal{H}}$ and $P_{\mathcal{K}}$ being appropriate projections, and, for any $u \in \mathcal{H}$ and any $v \in \mathcal{K}$, we have

$$\|f_m(S)u\|^2 = \int |f_m(z)|^2 d\langle\rho_S(z)u, u\rangle$$

and

$$\|f_m(T)v\|^2 = \int |f_m(z)|^2 d\langle\rho_T(z)v, v\rangle.$$

Letting $v = Xu$, and using $Xf_m(S) = f_m(T)X$, we obtain

$$\int |f_m(z)|^2 d\langle \rho_T(z)Xu, Xu\rangle \leq \|X\|^2 \int |f_m(z)|^2 d\langle \rho_S(z)u, u\rangle,$$

which, upon letting m tend to infinity, yields

$$\int |f(z)|^2 d\langle \rho_T(z)Xu, Xu\rangle \leq \|X\|^2 \int |f(z)|^2 d\langle \rho_S(z)u, u\rangle.$$

Consider

$$\eta(\cdot) = \langle \rho_T(\cdot)Xu, Xu\rangle + \langle \rho_S(\cdot)u, u\rangle.$$

We have by Proposition 2.5 that $(A(\Omega_p)|\partial\Omega_p, \partial\Omega_p, \eta)$ is a regular triple. Thus, if ϕ is any positive continuous function on $\partial\Omega_p$, then a sequence of functions $\{\phi_m\}_{m\geq 1}$ exists in $A(\Omega_p)$ such that $|\phi_m| < \sqrt{\phi}$ on $\partial\Omega_p$ and $\lim_{m\to\infty} |\phi_m| = \sqrt{\phi}$ η-almost everywhere. Replacing f by ϕ_m in the last integral inequality and letting m tend to infinity, we obtain

$$\int \phi(z)\, d\langle \rho_T(z)Xu, Xu\rangle \leq \|X\|^2 \int \phi(z)\, d\langle \rho_S(z)u, u\rangle,$$

which yields

$$\langle \rho_T(\cdot)Xu, Xu\rangle \leq \|X\|^2 \langle \rho_S(\cdot)u, u\rangle \quad \text{for every } u \text{ in } \mathcal{H}.$$

Using [**35,** Lemma 4.1] yields the desired conclusion. □

Remark 4.7. Requiring X to be of a special type in Proposition 4.6 may guarantee the lift $\widetilde{X}$ of X also to be of that special type. Indeed, arguing as in [**35,** Theorem 5.2], we can establish the following facts: if X is isometric, then so is $\widetilde{X}$; if X has dense range, then so has $\widetilde{X}$; if X is bijective, then so is $\widetilde{X}$. If a bounded linear map X that intertwines n-tuples S and T is invertible (respectively, unitary), then we refer to S and T as being *similar* (respectively, *unitarily equivalent*). It follows from [**3,** Lemma 1] and Proposition 4.6 above that, if $\partial\Omega_p$-isometries S and T are intertwined by a bounded linear map X that is injective and has dense range (that is, if S and T are *quasisimilar*), then the minimal normal extensions of S and T are unitarily equivalent (cf., [**3,** Proposition 9]).

In light of Remark 3.2, it is natural to investigate analogs of Proposition 4.6 for a pair of subnormal tuples, one of which is a cyclic $\partial\Omega_p$-

isometry. It is a standard fact of subnormal operator theory (refer, for example, to [**25**]) that any cyclic subnormal tuple S is, up to unitary equivalence, a multiplication tuple $M_{\theta,z}$ on the closure $P^2(\theta)$ of polynomials in $L^2(\theta)$ for some compactly supported positive regular Borel measure θ; in case S happens to be a cyclic $\partial\Omega_p$-isometry, θ must be supported on $\partial\Omega_p$.

Hereafter, $T = M_{\theta,z}$ stands for a fixed cyclic $\partial\Omega_p$-isometry with θ supported on $\partial\Omega_p$ and having no atoms on $\partial\Omega_p$.

In order to discuss subnormal tuples S quasisimilar to $T = M_{\theta,z}$, we need only consider $S = M_{\eta,z}$ for some compactly supported positive regular Borel measure η on $\mathbb{C}^n$ as is justified by [**2,** Proposition 1].

Arguing almost verbatim along the lines of [**4,** Section 4] (refer also to [**2**]), where the context was that of strictly pseudoconvex domains, Lemmas 4.8, 4.9 and Propositions 4.10, 4.11 below may be established. That one can use polynomials in the statements of those lemmas and propositions is a pleasant consequence of our observations in Remark 3.2. We point out that, as in the proof of [**4,** Lemma 4.5], an appeal in the proof of Lemma 4.9 below must be made to [**17,** Corollary 2.8], which is a consequence of some refinements in [**17**] of Aleksandrov's work in [**1**]; the requirement that θ have no atoms on $\partial\Omega_p$ stems from the necessity of applying [**17,** Corollary 2.8].

Lemma 4.8. *Let S be a cyclic subnormal tuple so that S can be identified with $M_{\eta,z}$ for some compactly supported positive regular Borel measure η on $\mathbb{C}^n$. If there exists a bounded linear map*

$$Y : P^2(\theta) \longrightarrow P^2(\eta)$$

with dense range such that $YM_{\theta,z} = M_{\eta,z}Y$, then there exists a cyclic vector g for $M_{\eta,z}$ such that

$$\int |p|^2|g|^2 d\eta \leq \int |p|^2 d\theta$$

for every polynomial p, and $\eta|\partial\Omega_p$ is absolutely continuous with respect to θ.

Lemma 4.9. *Let S be a cyclic subnormal tuple so that S can be identified with $M_{\eta,z}$ for some compactly supported positive regular Borel measure η on $\mathbb{C}^n$. Assume that* $\mathrm{supp}(\eta) \subset \overline{\Omega}_p$ *and η has no atoms on*

$\partial\Omega_p$. *If there exists a bounded linear map*

$$X : P^2(\eta) \longrightarrow P^2(\theta)$$

with dense range such that $XM_{\eta,z} = M_{\theta,z}X$, *then there exists a cyclic vector* h *for* $M_{\theta,z}$ *such that*

$$\int |p|^2 |h|^2 d\theta \leq \int |p|^2 d(\eta|\partial\Omega_p)$$

for every polynomial p, *and* θ *is absolutely continuous with respect to* $\eta|\partial\Omega_p$.

Proposition 4.10. *Let* S *be a cyclic subnormal tuple so that* S *can be identified with* $M_{\eta,z}$ *for some compactly supported positive regular Borel measure* η *on* $\mathbb{C}^n$. *Then* $(S =)M_{\eta,z}$ *is quasisimilar to* $M_{\theta,z}$ *if and only if*

(a) *there exists a cyclic vector* g *for* $M_{\eta,z}$ *such that*

$$\int |p|^2 |g|^2 d\eta \leq \int |p|^2 d\theta$$

for every polynomial p, *and*

(b) *there exists a cyclic vector* h *for* $M_{\theta,z}$ *such that*

$$\int |p|^2 |h|^2 d\theta \leq \int |p|^2 d(\eta|\partial\Omega_p)$$

for every polynomial p.

Proposition 4.11. *Let* S *be a cyclic subnormal tuple so that* S *can be identified with* $M_{\eta,z}$ *for some compactly supported positive regular Borel measure* η *on* $\mathbb{C}^n$. *Then* $(S =)M_{\eta,z}$ *is similar to* $M_{\theta,z}$ *if and only if there exist positive constants* c *and* d *such that*

$$\int |p|^2 d\eta \leq c \int |p|^2 d\theta$$

and

$$\int |p|^2 d\theta \leq d \int |p|^2 d(\eta|\partial\Omega_p)$$

for every polynomial p. Also, $(S =)M_{\eta,z}$ *is unitarily equivalent to* $M_{\theta,z}$ *if and only if* $d\eta = |h|^2 d\theta$ *for some cyclic vector h for* $M_{\theta,z}$*.*

It would be interesting to know whether the statements of Propositions 4.10 and 4.11 remain valid even when θ has atoms on $\partial\Omega_p$. Since the surface area measure σ_p on $\partial\Omega_p$ is not absolutely continuous with respect to the restriction $\nu_p|\partial\Omega_p$ of the volumetric measure ν_p to $\partial\Omega_p$, Lemma 4.9 shows, in particular, that $M_{\sigma_p,z}$ cannot be quasisimilar to $M_{\nu_p,z}$. This negative result can actually be extended to the multiplication tuples $M_{\sigma_p,z}$ and $M_{\nu_p,z}$ associated with the domains Σ_p. The next proposition generalizes [**6,** Proposition 3.4(d)] with an analogous proof; a complete proof is presented here for the reader's convenience.

Proposition 4.12. *There is no injective bounded linear map from* $A^2(\nu_p)$ *to* $H^2(\sigma_p)$ *that intertwines the multiplication tuples* $M_{\nu_p,z}$ *and* $M_{\sigma_p,z}$ *associated with* Σ_p*.*

Proof. We note that

$$\sum_{i=1}^{n} (M^*_{\sigma_p,z_i})^{p_{i,1}} (M_{\sigma_p,z_i})^{p_{i,1}} + \cdots + (M^*_{\sigma_p,z_i})^{p_{i,m_i}} (M_{\sigma_p,z_i})^{p_{i,m_i}}$$

is the identity operator on $H^2(\sigma_p)$ so that

$$S \equiv ((M_{\sigma_p,z_1})^{p_{1,1}}, \ldots, (M_{\sigma_p,z_n})^{p_{n,m_n}})$$

is a spherical isometry. It follows from [**3,** Proposition 2] that S is subnormal and that the minimal normal extension M of S has its spectral measure ρ_M supported on $\partial\mathbb{B}_Q$, where $Q = m_1 + \cdots + m_n$. It also follows from Taylor functional calculus (see [**47**]) and the spectral inclusion property for subnormal tuples (see [**39**]) that the minimal normal extension N of

$$T \equiv ((M_{\nu_p,z_1})^{p_{1,1}}, \ldots, (M_{\nu_p,z_n})^{p_{n,m_n}})$$

has its spectral measure ρ_N supported on the closure $\overline{\mathbb{B}}_Q$ of $\mathbb{B}_Q$. Suppose that there exists an injective bounded linear map

$$Y : A^2(\nu_p) \longrightarrow H^2(\sigma_p)$$

satisfying $YM_{\nu_p,z_i} = M_{\sigma_p,z_i}Y$ for all i. Then, Y also satisfies $YT_i = S_iY$ for all i. If 1_{ν_p} is the constant function of $A^2(\nu_p)$ taking the value 1, then for any m-variable polynomial $q \in \mathbb{C}[z]$, we have

$$\begin{aligned}\int_{\partial \mathbb{B}_Q} |q(z)|^2 d\|\rho_M(z) Y 1_{\nu_p}\|^2 = \|q(S) Y 1_{\nu_p}\|^2 &\le \|Y\|^2 \|q(T) 1_{\nu_p}\|^2 \\ &= \|Y\|^2 \int_{\bar{\mathbb{B}}_Q} |q(z)|^2 d\|\rho_N(z) 1_{\nu_p}\|^2.\end{aligned}$$

Appealing to [**42,** Theorem 3.5], we choose a sequence $\{q_n\}$ of polynomials in $\mathbb{C}[z]$ such that q_n are bounded in absolute value by 1, uniformly converge to 0 on compact subsets of $\mathbb{B}_Q$ and satisfy

$$\lim_{n \to \infty} |q_n(z)| = 1 \ z\text{-almost everywhere } [\|\rho_M(\cdot) Y 1_{\nu_p}\|^2].$$

Replacing q by q_n in the previous inequality, letting n tend to ∞ and noting that the measure $\|\rho_N(\cdot) 1_{\nu_p}\|^2$ vanishes on $\partial \mathbb{B}_Q$, we arrive at the absurdity $0 < \|Y 1_{\nu_p}\|^2 \le 0$. □

Remark 4.13. Combining [**43,** Theorem 2.3] with our observation in the proof of Proposition 3.5 that the $\overline{\partial}$-Neumann operator N_1 corresponding to Ω_p is compact, the short exact sequence of C*-algebras

$$0 \longrightarrow \mathcal{K}(A^2(\nu_p)) \xrightarrow{\iota} \mathbb{A} \xrightarrow{\pi} C(\partial \Omega_p) \longrightarrow 0$$

is obtained, where $\mathcal{K}(A^2(\nu_p))$ and $\mathbb{A}$ are as in the proof of Proposition 3.7, ι is the inclusion map and π is a unital $*$-homomorphism satisfying $\pi(\widetilde{T}_{\nu_p,\phi}) = \phi|\partial\Omega_p$ for any $\phi \in C(\overline{\Omega}_p)$. In view of Remark 4.2, even the $\overline{\partial}$-Neumann operator N_1 corresponding to Σ_p is compact; as such, [**43,** Theorem 2.3] yields that the short exact sequence as recorded here is obtained with Ω_p replaced by Σ_p (and with the associated symbols interpreted accordingly). On the other hand, the short exact sequence of Remark 3.8 (b) was derived by appealing to [**18,** Proposition 3.10] which necessitated that the multiplication tuple $M_{\sigma_p,z}$ there be regular; this, in turn, forced us to use the full strength of the definition of Ω_p via Proposition 2.5. It may then be asked, in particular, whether the short exact sequence of Remark 3.8 (b) is obtained with Ω_p replaced by Σ_p. Indeed, it is obtained if Σ_p is chosen to be a complex ellipsoid (see [**12,** Theorem 2.1]) and also if Σ_p is chosen to be $\widetilde{\Sigma}_p$ since a $\partial\widetilde{\Sigma}_p$-isometry is an essentially normal $A(\widetilde{\Sigma}_p)$-isometry (by Remark 4.2) and is, moreover, regular by the virtue of $\widetilde{\Sigma}_p$ being strictly pseudoconvex (refer to [**20**]).

While the main focus of the present paper has been on multivariable isometries associated with the domains Ω_p, Proposition 3.3 as well as the analysis in the present section suggest that even subnormal tuples that have the spectral measures of their minimal normal extensions supported on $\overline{\Omega}_p$ (and not just on $\partial\Omega_p$) are worth exploring. In order to corroborate that assertion, we first proceed to verify that the domains Ω_p satisfy the properties (F1), (F2), (F3) and (F4) as enunciated in [**17,** Section 1]. (It will also be clear that the domains Σ_p satisfy the properties (F1), (F2) and (F4)).

(F1) The closure $\overline{\Omega}_p$ of Ω_p is a Stein compactum of $\mathbb{C}^n$: this follows from the fact that $\overline{\Omega}_p$ is a compact convex subset of $\mathbb{C}^n$ (refer to [**41,** Chapter 3]).

(F2) $\mathbb{O}(\overline{\Omega}_p)$, the vector space of functions f such that f is holomorphic on an open neighborhood U_f of $\overline{\Omega}_p$, is weak*-dense in $H^\infty(\Omega_p)$: this follows from our observation in the last line of Remark 4.3.

It may be recalled that $A(\Omega_p)$ is the closure of $\mathbb{O}(\overline{\Omega}_p)$ in the sup norm with respect to $\overline{\Omega}_p$.

(F3) There exist a natural number N and an injective mapping $f \in A(\Omega_p)^N$ such that the image of the Shilov boundary of $A(\Omega_p)$ is contained in the topological boundary of the unit ball $\mathbb{B}_N$: this follows from Proposition 2.4 and our observations in the proof of Proposition 2.5.

(F4) There exists a positive regular Borel measure μ supported on the Shilov boundary of Ω_p (which, as we know, is $\partial\Omega_p$) such that the canonical map r_μ from

$$\mathbb{O}(\overline{\Omega}_p) \longrightarrow L^\infty(\mu)$$

extends to an algebra homomorphism

$$\widetilde{r}_\mu : H^\infty(\Omega_p) \longrightarrow L^\infty(\mu)$$

that is isometric and weak*-continuous (which is the same as calling μ a 'faithful Henkin measure'). Since the non-tangential boundary limit of any $f \in \mathbb{O}(\overline{\Omega}_p)$ is the restriction of f to $\partial\Omega_p$, the normalized surface area measure σ_p on $\partial\Omega_p$ is a faithful Henkin measure in light of Remark 4.3.

Remark 4.14. The preceding observations allow us to apply [**17,** Theorem 1.4] to those operator tuples $T \in \mathcal{B}(\mathcal{H})^n$ that possess an isometric and a weak*-continuous $H^\infty(\Omega_p)$-functional calculus

$$\Phi_T : H^\infty(\Omega_p) \longrightarrow \mathcal{B}(\mathcal{H})$$

(satisfying $\Phi_T(1) = I$ and $\Phi_T(z_i) = T_i$ for all i) so that, for such tuples T, we have the following: the weak operator topology and the weak* operator topology coincide on the algebra $\Phi_T(H^\infty(\Omega_p))$, and any unital weak*-closed subalgebra of $\Phi_T(H^\infty(\Omega_p))$ is reflexive (cf., Remark 2.7 (a)).

Let $T \in \mathcal{B}(\mathcal{H})^n$ be an operator tuple possessing a contractive and a weak*-continuous $H^\infty(\Omega_p)$-functional calculus Φ_T. Suppose, further, that T has its Taylor spectrum *dominating* in Ω_p so that the sup norm of any $f \in H^\infty(\Omega_p)$ equals the supremum of $|f|$ over the intersection of Ω_p with the Taylor spectrum $\sigma(T)$ of T. Since Ω_p is a bounded convex domain with smooth boundary, Ω_p satisfies the 'Gleason property' so that, for any $a \in \Omega_p$ and any $f \in H^\infty(\Omega_p)$, we have

$$f(z) - f(a) = \sum_{i=1}^{n} (z_i - a_i) f_i(z), \quad z \in \Omega_p,$$

where the so-called Leibenzon divisors f_i are given by

$$f_i(z) = \int_0^1 \frac{\partial f}{\partial z_i}(a + t(z-a))\, dt$$

and are in $H^\infty(\Omega_p)$ (refer to [**23, 26**]). Using this and arguing exactly as in [**15,** Lemma 2.3.6], it may be proven that, for any $f \in H^\infty(\Omega_p)$, $f(\sigma(T) \cap \Omega_p)$ is contained in the Taylor spectrum of $\Phi_T(f)$. That easily leads to the sup norm of f with respect to Ω_p being less than or equal to $\|\Phi_T(f)\|$. Thus, in this case, the functional calculus Φ_T is indeed isometric.

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