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ON A GENERALIZATION OF WOLFF'S IDEAL THEOREM FOR CERTAIN SUBALGEBRAS OF $H^\infty(\mathbb{D})$

D.P. BANJADE, M. EPHREM, A. INCOGNITO AND M. WILKERSON

ABSTRACT. We settle an open question proposed in [**2**] to generalize Wolff's ideal theorem on certain uniformly closed subalgebras of $H^\infty(\mathbb{D})$. Also, we discuss some subalgebras where Wolff's ideal theorem holds without the additional condition $F(0) \neq 0$.

1. Introduction. In 1962, Carleson [**3**] proved his celebrated corona theorem characterizing when a finitely generated ideal of $H^\infty(\mathbb{D})$ is all of $H^\infty(\mathbb{D})$. In functional notation, Carleson's corona theorem can be expressed as the ideal $\mathcal{I}$ generated by a finite set of functions

$$\{f_i\}_{i=1}^n \subset H^\infty(\mathbb{D})$$

is the entire space $H^\infty(\mathbb{D})$, provided that there exists a $\delta > 0$ such that

$$\left(\sum_{i=1}^n |f_i(z)|^2\right)^{1/2} \geq \delta \quad \text{for all } z \in \mathbb{D}. \tag{1.1}$$

In light of this, Wolff [**13**] later attempted to generalize the corona theorem by proposing the following question: does

$$\left(\sum_{i=1}^n |f_i(z)|^2\right)^{1/2} \geq |h(z)| \quad \text{for all } z \in \mathbb{D},\ f_j, h \in H^\infty(\mathbb{D}) \tag{1.2}$$

imply $h^p \in \mathcal{I}$? Unfortunately, the answer is negative for $p = 1$ (see Rao's example in Garnett [**4,** page 369, Ex-3] and Treil showed with a counterexample that condition (1.2) is insufficient for $p = 2$ as well [**11**]. Wolff provided the following for all $p \geq 3$:

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Theorem 1.1 ([**13**]). *If $f_j \in H^\infty(\mathbb{D})$, $j = 1, 2, \ldots, n$, $h \in H^\infty(\mathbb{D})$ and*

$$\left(\sum_{j=1}^{n} |f_j(z)|^2\right)^{1/2} \geq |h(z)| \quad \textit{for all } z \in \mathbb{D},$$

then, $h^p \in \mathcal{I}(\{f_j\}_{j=1}^n)$, $p \geq 3$.

(See Garnett [**4,** page 319, Theorem 2.3].)

Wolff's theorem has been extended to various subalgebras of $H^\infty(\mathbb{D})$. For example, in [**2**], the authors obtained an analogous result to Wolff's theorem on the subalgebra of $H^\infty(\mathbb{D})$ given by

$$\mathbb{C} + BH^\infty(\mathbb{D}) = \{\alpha + Bg : \alpha \in \mathbb{C},\ g \in H^\infty(\mathbb{D})\},$$

where B is a fixed Blaschke product. A similar result was obtained in [**1**] for the subalgebra

$$H^\infty_{\mathbb{I}}(\mathbb{D}) = \{f \in H^\infty(\mathbb{D}) : f = c + \phi,\ c \in \mathbb{C} \text{ and } \phi \in \mathbb{I}\},$$

where $\mathbb{I}$ is a proper ideal of $H^\infty(\mathbb{D})$. Moreover, in [**5**], the authors extended Wolff's result to matrix cases. It was proven there that a similar result holds true for semi-infinite matrices on any algebra of functions in which Wolff's ideal theorem is already known to hold.

For this paper, we consider the following subalgebra: Let $K \subset \mathbb{Z}_+$, and define

$$H^\infty_K(\mathbb{D}) = \{f \in H^\infty(\mathbb{D}) : f^{(j)}(0) = 0 \text{ for all } j \in K\}.$$

We consider those sets K for which $H^\infty_K(\mathbb{D})$ is a subalgebra of $H^\infty(\mathbb{D})$ under the usual product of functions.

For the remainder of the paper we use f and f_i to represent complex-valued scalar functions and F to denote a vector-valued function. Similarly, we let c and c_i denote complex numbers and C a complex vector. For $\{f_j\}_{j=1}^\infty \subset H^\infty(\mathbb{D})$, if we let

$$F(z) = (f_1(z), f_2(z), \ldots),$$

we use $F(z)^*$ to denote the adjoint of $F(z)$.

We also let $H^\infty_{l^2}(\mathbb{D})$ denote the Hilbert space of bounded, analytic functions that map $\mathbb{D}$ to l^2, that is, an element $F \in H^\infty_{l^2}(\mathbb{D})$ is an infinite-dimensional row vector, the entries of which consist of functions $f_i \in H^\infty(\mathbb{D})$ such that

$$\|F\|_\infty^2 = \sum_{i=1}^{\infty} \sup_{z\in\mathbb{D}} |f_i(z)|^2 < \infty.$$

2. The subalgebra $H_K^\infty(\mathbb{D})$. Not every set K defines an algebra; for example, $K = \{2\}$. Although there is not a complete characterization of the set K for which $H_K^\infty(\mathbb{D})$ is an algebra, Ryle and Trent **[9]** have given certain criteria which the set K must meet.

Lemma 2.1 ([**9**])**.** *Let $K \subseteq \mathbb{N}$ be such that $H_K^\infty(\mathbb{D})$ is an algebra. Then*:

(i) *$k_0 \notin K$ if and only if $z^{k_0} \in H_K^\infty(\mathbb{D})$.*

(ii) *If $j, k \notin K$, then $j + k \notin K$.*

(iii) *Suppose $k_0 \in K$. If $1 < j < k_0$ satisfies $j \notin K$, then $k_0 - j \in K$.*

We will assume that K is finite due to the consequences of the next lemma.

Lemma 2.2 ([**10**])**.** *If $H_K^\infty(\mathbb{D})$ is an algebra, then there exist $d \in \mathbb{N}$, a finite set $\{n_i\}_{i=1}^p \subset \mathbb{N}$ with $n_1 < \cdots < n_p$ and $\gcd(n_1, \ldots, n_p) = 1$, and a positive integer $N_0 > n_p$ such that*

$$\mathbb{N} - K = \{n_1 d, n_2 d, \ldots, n_p d, N_0 d, (N_0 + j)\, d : j \in \mathbb{N}\}.$$

Lemma 2.2 states that the nontrivial sets $K \subset \mathbb{N}$ for which $H_K^\infty(\mathbb{D})$ is an algebra are the sets K for which there exist $l_1 < \cdots < l_r$ in $\mathbb{N}$ with $\gcd(l_1, \ldots, l_r) = d$ such that $\mathbb{N} - K$ is the semigroup of $\mathbb{N}$ generated by $\{l_1, \ldots, l_r\}$ under addition.

Thus, the elements of $H_K^\infty(\mathbb{D})$ have the form

$$f(z) = c_0 + c_1 z^{n_1 d} + \cdots + c_j z^{n_j d} + c_{j+1} z^{(n_j+1)d} + c_{j+2} z^{(n_j+2)d} + \cdots,$$

where $c_i \in \mathbb{C}$. Letting $w = z^d$ yields

$$g(w) = c_0 + c_1 w^{n_1} + \cdots + c_{j-1} w^{n_{j-1}} + \sum_{k=0}^{\infty} c_{j+k} w^{n_j + k}.$$

Therefore, $g(w)$ is contained in the algebra $H^\infty_{K_1}(\mathbb{D})$, where

$$K_1 = \{1, \ldots, n_1 - 1, n_1 + 1, \ldots, n_2 - 1, n_2 + 1, \ldots, n_j - 1\}$$

is a finite set.

The above argument suggests that finding a solution to the ideal problem in $H^\infty_K(\mathbb{D})$ where K is infinite can be reduced to a simpler problem involving two steps. First, solve the corresponding problem in $H^\infty_{K_1}(\mathbb{D})$, where K_1 is finite as above. Then, take those solutions in $H^\infty_{K_1}(\mathbb{D})$ and compose them with z^d in order to obtain the solution in $H^\infty_K(\mathbb{D})$.

We define algebras comprised of vectors with entries in $H^\infty_K(\mathbb{D})$ as follows:

$$\mathcal{H}^\infty_{K,n}(\mathbb{D}) = \Big\{ \{f_j\}_{j=1}^n : f_j \in H^\infty_K(\mathbb{D}) \text{ for } j = 1, 2, \ldots, n \\ \text{and } \sup_{z\in\mathbb{D}} \sum_{j=1}^n |f_j(z)|^2 < \infty \Big\}.$$

Here, multiplication is entrywise, and n can either be a positive integer or ∞. We write the elements of $\mathcal{H}^\infty_{K,n}(\mathbb{D})$ as row vectors such that $F \in \mathcal{H}^\infty_{K,n}(\mathbb{D})$.

In [**2**], Banjade, et al., established the partial analogues of Wolff's theorem in this algebra. The results there required the additional assumption that $F(0) \neq \mathbf{0}$.

Theorem 2.3 ([**2**])**.** *Let*

$$F = (f_1, f_2, \ldots) \in \mathcal{H}^\infty_{K,n}(\mathbb{D})$$

and $h \in H^\infty_K(\mathbb{D})$, *with*

$$1 \geq \sqrt{F(z)F(z)^*} \geq |h(z)| \quad \textit{for all } z \in \mathbb{D}.$$

Also, assume that $F(0) \neq \mathbf{0}$. *Then, there exists a*

$$V = (v_1, v_2, \ldots) \in \mathcal{H}^\infty_{K,n}(\mathbb{D})$$

such that

$$F(z)V(z)^T = h^3(z) \quad \textit{for all } z \in \mathbb{D}$$

and

$$\|V\|_\infty \leq C_0 + \frac{\|G^{(k_p)}(0)\|_{l^2}}{k_p!\|F(0)\|_{l^2}}.$$

Here, k_p is the largest element of K, and G is an H^∞ solution obtained as in [**12**].

3. Main results. In [**2**], the authors proposed the following open question: can Wolff's theorem be fully extended to the subalgebras $H_K^\infty(\mathbb{D})$ of $H^\infty(\mathbb{D})$ without the additional assumption that $F(0) \neq \mathbf{0}$? In this paper, we prove that the answer is negative. We also discuss the subalgebras on which the assumption $F(0) \neq \mathbf{0}$ can be removed to establish Wolff's theorem.

Proposition 3.1. *Let $F = (f_1, f_2, \ldots) \in \mathcal{H}_{K,n}^\infty(\mathbb{D})$ and $h \in H_K^\infty(\mathbb{D})$, with $1 \geq \sqrt{F(z)F(z)^*} \geq |h(z)|$ for all $z \in \mathbb{D}$. If $F(0) = \mathbf{0}$, then the existence of $V = (v_1, v_2, \ldots) \in \mathcal{H}_{K,n}^\infty(\mathbb{D})$ such that*

$$F(z)V(z)^T = h^3(z) \quad \textit{for all } z \in \mathbb{D}$$

cannot be guaranteed.

Proof. We proceed by counterexample. If we consider the set

$$K = \{1, 2, 3, 6, 7, 11\},$$

then there exist $f, h \in H_K^\infty$ such that

$$|h(z)| \leq |f(z)| \quad \text{for all } z \in \mathbb{D}.$$

However, as is seen below, there is not necessarily a g in H_K^∞ such that $h^3 = fg$.

It is clear from Lemma 2.1 that $H_K^\infty(\mathbb{D})$ is an algebra. Also, any element f in $H_K^\infty(\mathbb{D})$ looks like:

$$f(z) = c_0 + c_4z^4 + c_5z^5 + c_8z^8 + c_9z^9 + c_{10}z^{10} + c_{12}z^{12} + \cdots$$

for some collection of $c_i \in \mathbb{C}$.

If we take $h(z) = z^4$ and $f(z) = 2z^4 + z^5$, we clearly see that $h, f \in H_K^\infty(\mathbb{D})$. In addition, they satisfy $|h(z)| \leq |f(z)|$ for all $z \in \mathbb{D}$ and $f(0) = 0$.

If there were a g such that $h^3 = fg$, then we would have

$$z^{12} = (2z^4 + z^5)g.$$

This suggests that g is of the form

$$g(z) = \frac{z^8}{2+z} = \frac{z^8}{2}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{2}\right)^n = \frac{z^8}{2} - \frac{z^9}{4} + \frac{z^{10}}{8} - \frac{z^{11}}{16} + \cdots,$$

which implies that $g \notin H_K^\infty(\mathbb{D})$ for $K = \{1,2,3,6,7,11\}$ since there is a nonzero coefficient on the z^{11} term. Thus, there is no g in H_K^∞ such that $h^3 = fg$. $\square$

This proves that the answer to the question asked in **[2]** is, indeed, negative. However, there are some algebras in which Theorem 2.3 is true without the additional condition that $F(0) \neq \mathbf{0}$.

Let $h \in H_K^\infty(\mathbb{D})$ be such that

$$[F(z)F(z)^*]^{1/2} \geq |h(z)| \quad \text{for all } z \in \mathbb{D}.$$

$F(0) = \mathbf{0}$ implies $F_0 = \mathbf{0}$, and thus, $h_0 = 0$. Let r be the smallest element of K such that $F_r \neq \mathbf{0}$. This means that $F(z)$ and $h(z)$ can be written as $F(z) = z^r F_H(z)$ and $h(z) = z^r H(z)$, $F_H(z)$ and $H(z) \in H_{\mathcal{K}}^\infty(\mathbb{D})$, where

$$\mathcal{K} = \{k - r : k \in K \text{ and } k - r > 0\}.$$

Theorem 3.2. *Let $F = (f_1, f_2, \ldots) \in \mathcal{H}_{K,n}^\infty(\mathbb{D})$ and $h \in H_K^\infty(\mathbb{D})$, with $1 \geq \sqrt{F(z)F(z)^*} \geq |h(z)|$ for all $z \in \mathbb{D}$. If $H_{\mathcal{K}}^\infty(\mathbb{D})$ is a subalgebra of $H^\infty(\mathbb{D})$, then there exists a $V = (v_1, v_2, \ldots) \in \mathcal{H}_{K,n}^\infty(\mathbb{D})$ such that*

$$F(z)V(z)^T = h^3(z) \quad \textit{for all } z \in \mathbb{D}$$

and

$$\|V\|_\infty \leq C_0 + \frac{\|G^{(k_p)}(0)\|_{l^2}}{k_p!\|F(0)\|_{l^2}}.$$

Here, k_p is the largest element of K, and G is an H^∞ solution obtained as in **[12]**.

Proof. Let $h \in H_K^\infty(\mathbb{D})$ be such that

$$\sqrt{F(z)F(z)^*} \geq |h(z)| \quad \text{for all } z \in \mathbb{D}.$$

Since $F(0) = \mathbf{0}$ implies $F_0 = \mathbf{0}$, we assume that r is the smallest element of K such that $F_r \neq \mathbf{0}$, which means that F and H can be expressed as $F(z) = z^r F_H(z)$ and $h(z) = z^r H(z)$, $F_H(z)$ and $H(z) \in H_{\mathcal{K}}^\infty(\mathbb{D})$. Moreover,

$$|H(z)| \leq \sqrt{F_H(z)F_H(z)^*} \quad \text{for all } z \in \mathbb{D}.$$

The condition $F_r \neq \mathbf{0}$ implies that $F_H(0) \neq \mathbf{0}$. In addition, since $H_{\mathcal{K}}^\infty(\mathbb{D})$ is an algebra, Theorem 2.3 implies that there exists a $G \in H_{\mathcal{K}}^\infty(\mathbb{D})$ such that $H^3 = F_H G$. Therefore,

$$h^3 = (z^r H)^3 = (z^r F_H)(z^{2r} G) = FV,$$

where $V = z^{2r}G = z^r(z^r G) \in H_K^\infty(\mathbb{D})$ as $z^r, z^r G \in H_K^\infty(\mathbb{D})$. Also, since $|V(z)| \leq |G(z)|$ for all $z \in \mathbb{D}$, the same norm estimate from **[2]** also works well here. $\square$

Example 3.3. Theorem 3.2 may be demonstrated with this example. Let K be the set of all positive odd integers. Then, $H_K^\infty(\mathbb{D})$ is an algebra, and the elements of $H_K^\infty(\mathbb{D})$ are of the form

$$F(z) = C_0 + C_1 z^2 + C_4 z^4 + C_6 z^6 + C_8 z^8 + \cdots .$$

If $F(0) = \mathbf{0}$, then

$$F(z) = z^2(C_1 + C_4 z^2 + C_6 z^4 + C_8 z^6 + \cdots).$$

Thus, the set $\mathcal{K}$ associated to K will again be the set of all consecutive odd numbers, i.e., $\mathcal{K} = \{1, 3, 5, \ldots\}$. This means that the result holds true on $H_K^\infty(\mathbb{D})$ without the condition $F(0) \neq \mathbf{0}$.

However, it is unnecessary that all the subsets $\mathcal{K}$ of K form a subalgebra $H_{\mathcal{K}}^\infty(\mathbb{D})$. For example, if we take $K = \{1, 2, 5\}$, $H_K^\infty(\mathbb{D})$ is an algebra whose elements are of the form

$$f(z) = c_0 + c_3 z^3 + c_4 z^4 + \cdots + c_n z^n \quad \text{for } n \geq 6.$$

In this case, $f(0) = 0$ implies that

$$f(z) = z^3(c_3 + c_4 z + c_6 z^3 + \cdots + c_n z^{n-3}) \quad \text{for } n \geq 3.$$

Therefore, the subset $\mathcal{K}$ corresponding to the function

$$\widetilde{f} = c_3 + c_4 z + c_6 z^3 + \cdots + c_n z^{n-3}$$

is $\mathcal{K} = \{2\}$. As discussed above, $H^\infty_{\mathcal{K}}(\mathbb{D})$ is not an algebra for $\mathcal{K} = \{2\}$ which means that our Theorem 3.2 cannot be applied on the subalgebras $H^\infty_K(\mathbb{D})$ associated to such a K.

Theorem 3.4. *Let $F = (f_1, f_2, \ldots) \in \mathcal{H}^\infty_{K,n}(\mathbb{D})$ and $h \in H^\infty_K(\mathbb{D})$, with $1 \geq \sqrt{F(z)F(z)^*} \geq |h(z)|$ for all $z \in \mathbb{D}$. If $2r > k_p$, then there exists a $V = (v_1, v_2, \ldots) \in \mathcal{H}^\infty_{K,n}(\mathbb{D})$ such that*

$$F(z)V(z)^T = h^3(z) \quad \textit{for all } z \in \mathbb{D}$$

and

$$\|V\|_\infty \leq C_0 + \frac{\|G^{(k_p)}(0)\|_{l^2}}{k_p!\|F(0)\|_{l^2}}.$$

We note that this theorem is clear from Theorem 3.2 without the condition $2r > k_p$ if $H^\infty_{\mathcal{K}}(\mathbb{D})$ is an algebra.

Proof. If the subset $\mathcal{K}$ of the original set K does not constitute an algebra, we take the maximal subset, say K_1, of $\mathcal{K}$ such that $H^\infty_{K_1}(\mathbb{D})$ is an algebra. Although $H^\infty_{\mathcal{K}}(\mathbb{D})$ is not an algebra, it holds true that $H^\infty_{\mathcal{K}}(\mathbb{D}) \subset H^\infty_{K_1}(\mathbb{D})$.

We now have that $h, F \in H^\infty_K(\mathbb{D})$, where $h = z^r H$, $F = z^r F_H, F_H$,

$$H \in H^\infty_{\mathcal{K}}(\mathbb{D}) \subset H^\infty_{K_1}(\mathbb{D}).$$

Moreover, $|h(z)| \leq \sqrt{F(z)F(z)^*}$ implies $|H(z)| \leq \sqrt{F_H(z)F_H(z)^*}$. In addition, since H, $F_H \in H^\infty_{K_1}(\mathbb{D})$ and $F_H(0) \neq \mathbf{0}$, there exists a $G \in H^\infty_{K_1}(\mathbb{D})$ such that $H^3 = F_H G$. Therefore,

$$h^3 = (z^r H)^3 = (z^r F_H)(z^{2r} G),$$

and $2r > k_p$ implies that $z^{2r} G \in H^\infty_K(\mathbb{D})$. This completes the proof. $\square$

Example 3.5. If we consider the set $K = \{1, 2, 5\}$, the maximal subset K_1 of $\mathcal{K} = \{2\}$, such that $H^\infty_{K_1}(\mathbb{D})$ is an algebra, is simply just $K_1 = \Phi$. Thus, $H^\infty_{K_1}(\mathbb{D}) = H^\infty(\mathbb{D})$. Therefore, $|H(z)| \leq \sqrt{F_H(z)F_H(z)^*}$, and

$F_H(0) \neq \mathbf{0}$ implies that there exists a $G \in H^\infty_{K_1}(\mathbb{D}) = H^\infty(\mathbb{D})$ such that $H^3 = F_H G$. Thus,

$$h^3 = (z^3 H)^3 = (z^3 F_H)(z^6 G).$$

It is obvious that $z^6 G$ belongs to $H^\infty_K(\mathbb{D})$ as $G \in H^\infty(\mathbb{D})$.

4. Future research. In this paper, we showed that Wolff's theorem does not hold true in general on $H^\infty_K(\mathbb{D})$ without the additional condition $F(0) \neq 0$, and we also partially characterized subalgebras where the theorem holds without this additional condition. We consider this result an introductory stepping stone towards developing a complete characterization of subalgebras of $H^\infty(\mathbb{D})$ where Wolff's theorem holds.

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Coastal Carolina University, Department of Mathematics and Statistics, P.O. Box 261954, Conway, SC 29528
Email address: **dpbanjade@coastal.edu**

Coastal Carolina University, Department of Mathematics and Statistics, P.O. Box 261954, Conway, SC 29528
Email address: **menassie@coastal.edu**

Coastal Carolina University, Department of Mathematics and Statistics, P.O. Box 261954, Conway, SC 29528
Email address: **aincogni@coastal.edu**

Coastal Carolina University, Department of Mathematics and Statistics, P.O. Box 261954, Conway, SC 29528
Email address: **mwilkerso@coastal.edu**