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ON THE τ-LI COEFFICIENTS FOR AUTOMORPHIC L-FUNCTIONS

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ABSTRACT. In this paper, we extend the Li coefficients for automorphic L-functions and the Li criterion for the Riemann hypothesis to yield a necessary and sufficient condition for the existence of zero-free strips for automorphic L-functions inside the critical strip. Next, we give an arithmetical and asymptotical formula for these coefficients. Finally, we show that there exists an entire function of exponential type that interpolates the extended Li coefficients (or the τ-Li coefficients) at integer values. The results of this paper arise from ideas of the author [**15**], Freitas [**8**], Lagarias [**11**] and Odzǎk and Smajlovič [**17**].

1. Introduction. The Riemann hypothesis is the subject of several studies and research papers. Most of them provide new reformulations and numerical evidence for this hypothesis. In the literature, there exist various formulations of the Riemann hypothesis [**10, 12**]. We believe that most of them may be extended to automorphic L-functions and to the framework of the Selberg class [**19, 21, 26**] or subclass. In this paper, we specifically deal with the formulation of the Li criterion for the Riemann hypothesis in the context of automorphic L-functions. The Li criterion for the Riemann hypothesis, see [**12**], is a necessary and sufficient condition for the sequence

$$\lambda_n = \sum_{\rho} \left[1 - \left(1 - \frac{1}{\rho}\right)^n\right]$$

to be non-negative for all $n \in \mathbb{N}$ and where ρ runs over the non-trivial zeros of $\zeta(s)$. This criterion holds for a large class of Dirichlet series,

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the so-called Selberg class, as given in [**19, 21**], and for automorphic L-functions as given in [**11**]. More recently, Omar and Bouanani [**18**] extended the Li criterion for function fields and established an explicit and asymptotic formula for the Li coefficients.

In this paper, our main objective is to extend the Li coefficients and the Li criterion for the Riemann hypothesis to automorphic L-functions. Next, we give an arithmetical and asymptotical formula for these coefficients. Finally, we show that there exists an entire function of exponential type that interpolates the extended Li coefficients (or the τ-Li coefficients) at integer values.

2. Automorphic L-functions. First, we recall basic facts regarding principal L-functions $L(s,\pi)$ attached to irreducible cuspidal unitary automorphic representations of $\mathrm{GL}(N)$, as in Rudnick and Sarnak [**25**] (here, we use the same notation as in the Lagarias paper [**11,** Section 2]). These L-functions are associated to $\mathrm{GL}(n,\mathbb{Q}) \mid \mathrm{GL}(n,\mathbb{A}_{\mathbb{Q}})$, and they are denoted by $L(s,\pi,\rho)$ in which the Langland L-group $l_G = \mathrm{GL}(N,\mathbb{C})$ and $\rho : l_G \to \mathrm{GL}(N,\mathbb{C})$ is the standard representation. For the trivial representation π_{triv} of $\mathrm{GL}(1)$, we have the completed automorphic L-function $\Lambda(s,\pi_{\mathrm{triv}}) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$. This function has simple poles at $s=0$ and $s=1$. Aside from this representation, all other $\Lambda(s,\pi)$ are entire functions. Each completed automorphic L-function $\Lambda(s,\pi)$ has an Euler product factorization:

$$\Lambda(s,\pi) = Q(\pi)^{s/2} L_\infty(s,\pi) L(s,\pi).$$

Here, $Q(\pi)$ is a positive integer, called the *conductor* of the representation π, and the *Archimedean factor* is

$$L_\infty(s,\pi) = \prod_{k=1}^{N} \Gamma_{\mathbb{R}}(s + k_j(\pi)),$$

in which $k_j(\pi)$ are certain constants and

$$\Gamma_{\mathbb{R}}(s) := \pi^{-s/2}\,\Gamma(s/2).$$

The L-function $L(s,\pi)$ is given by an Euler product over the finite places

$$L(s,\pi) = \prod_p \prod_{j=1}^{N} (1 - \alpha_{p,j}(\pi) p^{-s})^{-1} = \sum_{n=1}^{\infty} a_n(\pi) n^{-s}.$$

The Euler product and its associated Dirichlet series absolutely converge in the half-plane $\mathrm{Re}(s) > 1$. The Ramanujan-Peterson conjecture states that the local parameters $|\alpha_{p,j}(\pi)|$ are of absolute value at most one. We have, for all $n \geq 1$, that $|a_n(\pi)| \leq d(n)n^{N/2}$, where $d(n)$ is the number of divisors of n, see [**9,** subsections 5.11, 5.12]. The function $\Lambda(s, \pi)$ satisfies a functional equation

$$\Lambda(s, \pi) = \epsilon(\pi)\Lambda(1 - s, \check{\pi}),$$

in which $\epsilon(\pi)$ is a constant of absolute value one, and $\check{\pi}$ denotes the contragredient representation. The archimedean factors $\Gamma_{\mathbb{R}}(s + k_j(\pi))$ in the Euler product $\Lambda(s, \pi)$ satisfy $\mathrm{Re}(k_j(\pi)) > -1/2$ and $L_\infty(s, \pi) = \overline{L_\infty(\overline{s}, \pi)}$, see [**11**]. The zeros of $\Lambda(s, \pi)$ all lie in the open critical strip $0 < \mathrm{Re}(s) < 1$. In particular, $\Lambda(s, \pi)$ is non-vanishing on the lines $\mathrm{Re}(s) = 0$ and $\mathrm{Re}(s) = 1$, see [**9,** Theorem 5.42]. Letting $N_\pi(T)$ be the number of zeros $\rho = \beta + i\gamma$ of $L(s, \pi)$ such that $0 \leq \beta \leq 1$ and $0 < \gamma \leq T$, then, see [**9,** Theorem 5.8],

$$N_\pi(T) = \frac{N}{2\pi} T \log T + C(\pi)T + O(\log T), \tag{2.1}$$

where $C(\pi) = (1/\pi)\log Q(\pi) - N(1 + \log 2\pi)/2$. We define the ξ-function $\xi(s, \pi)$ associated to π by

$$\xi(s, \pi) = s^{-e(0,\pi)}(s - 1)^{-e(1,\pi)}\left(\frac{1}{\sqrt{(-1)^{e(1/2,\pi)}\epsilon(\pi)}}\Lambda(s, \pi)\right),$$

where $e(s_0, \pi)$ denotes the order of a zero or pole of $\Lambda(s, \pi)$ at $s = s_0$, with poles assigned negative orders. We have $e(0, \pi) = e(1, \pi)$ by the functional equation, and this definition ensures that $\xi(s, \pi)$ is holomorphic and has no zero at $s = 0$ nor at $s = 1$. In this definition, the square roots must be consistently chosen so that

$$\sqrt{(-1)^{e(1/2,\pi)}\epsilon(\pi)}\sqrt{(-1)^{e(1/2,\pi)}\epsilon(\check{\pi})} = 1.$$

The choice of sign under the square root in the last equation may be removed by the requirement that $\xi(1/2 + it, \pi) > 0$ holds for small positive t, as justified by Lagarias [**11,** Theorem 2.1, assertion (5)]. For the trivial representation π_{triv} on $\mathrm{GL}(1)$, we have $e(0, \pi_{\mathrm{triv}}) = e(1, \pi_{\mathrm{triv}}) = -1$, and $\xi(s, \pi_{\mathrm{triv}}) = 2\xi(s)$. This convention is forced if we wish to obtain an entire function in all cases for which we must remove the poles at $s = 0$ and $s = 1$ for the case π_{triv}. The function

$\xi(s,\pi)$ satisfies the functional equation

$$\xi(s,\pi) = (-1)^k \xi(1-s,\check{\pi}),$$

where $k = e(1/2,\pi) = e(1/2,\check{\pi})$ is the order of the zero of $\xi(s,\pi)$ at $s = 1/2$. It is real-valued on the critical line $\mathrm{Re}(s) = 1/2$. Let $Z(\pi)$ denote the multi-set of zeros of $\xi(s,\pi)$ (counted with multiplicity) which is the same as that of $\Lambda(s,\pi)$ except possibly at $s = 0$ and $s = 1$. The multi-set $Z(\pi)$ is invariant under the map $\rho \mapsto 1-\overline{\rho}$. The function $\xi(s,\pi)$ is an entire function of order one and maximal type. It is bounded in vertical strips $-B < \mathrm{Re}(s) < B$ for any finite B and has a rapid decrease there as $|\Im(s)| \to \infty$. The function $\xi(s,\pi)$ yields the Hadamard factorization

$$\xi(s,\pi) = e^{a(\pi)+b(\pi)s} \prod_{\rho\in Z(\pi)} \left(1-\frac{s}{\rho}\right) e^{s/\rho},$$

where the constant $b(\pi)$ in the last equation satisfies (see [**9,** Theorem 5.6]).[1]

$$b(\pi) = -\sum_{\rho\in Z(\pi)}{}' \frac{1}{\rho}.$$

We refer to [**27,** Proposition 3.4] for the existence of the last sum, where Smajlovič's class of L-functions unconditionally contains all automorphic L-functions attached to irreducible cuspidal unitary representations of $\mathrm{GL}(N)$, also see [**11,** Lemma 2.1].

3. Li coefficients for automorphic L-functions. In this section, we recall some results on the Li criterion and the Li coefficients as in Lagarias's paper [**11**]. Let $Z(\pi)$ denote the multi-set of zeros of $\xi(s,\pi)$ (counted with multiplicity) which is the same as that of $\Lambda(s,\pi)$, except possibly at $s = 0$ and $s = 1$. Lagarias [**11,** Lemma 2.1] proved that, for any principal L-functions $L(s,\pi)$ for $\mathrm{GL}(N)$, the power sums

$$\sigma_n(\pi) := \sum_{\rho\in Z(\pi)}{}' \frac{1}{\rho^n}, \quad n \ge 1,$$

are absolutely convergent for $n \ge 2$, *-convergent for $n = 1$, and the real parts of these sums are absolutely convergent for all $n \ge 1$. Furthermore, for all irreducible unitary automorphic representations on $\mathrm{GL}(N)$, the sum

$$\lambda_n(\pi) = \sum_{\rho \in Z(\pi)} 1 - \left(1 - \frac{1}{\rho}\right)^n$$

is *-convergent for all $n \in \mathbb{Z}$. This is shown by

$$\lambda_n(\pi) = \sum_{j=1}^{n} (-1)^{j-1} \tbinom{n}{j} \sigma_j(\pi),$$

with $\lambda_0(\pi) = 0$, which also satisfies $\lambda_{-n}(\pi) = \overline{\lambda_n(\pi)}$. Lagarias **[11]** states a general version of the Li criterion.

Theorem 3.1 ([**11,** Theorem 2.2]). *Let π be an irreducible cuspidal unitary automorphic representation of* $\mathrm{GL}(n)$. *The following conditions are each equivalent to the Riemann hypothesis for $\xi(s,\pi)$.*

- (a) *For all $n \geq 1$,* $\mathrm{Re}(\lambda_n(\pi)) \geq 0$.
- (b) *For each $\epsilon > 0$, there is a positive constant $C(\epsilon)$ such that*

$$\mathrm{Re}(\lambda_n(\pi)) \geq -C(\epsilon) e^{\epsilon n} \quad \textit{for all } n \geq 1.$$

- (c) *The generalized Li coefficients $\lambda_n(\pi)$ satisfy*

$$\lim_{n \to \infty} |\lambda_n(\pi)|^{1/n} \leq 1.$$

In addition, Lagarias **[11]** gave an arithmetical interpretation of the coefficients in terms of the logarithmic derivative of $\xi(s,\pi)$, expanded around the point $s = 1$, and obtained an asymptotic formula for $\lambda_n(\pi)$.

Theorem 3.2 ([**11,** Theorem 1.1]). *Let π be an irreducible cuspidal unitary automorphic representation for* $\mathrm{GL}(N)$ *over* $\mathbb{Q}$. *For $n \geq 1$, the following holds*:

$$\lambda_n(\pi) = \frac{N}{2} \log n + C_1(\pi) n - \lambda_n(\sqrt{n}, \pi) + O(\sqrt{n} \log n),$$

in which $C_1(\pi)$ is real-valued, $\lambda_n(\sqrt{n}, \pi)$ denotes the incomplete Li coefficient[2] *at height $\sqrt{n}$, and the implied constant in the O-notation depends upon π. If the Riemann hypothesis holds for $L(s,\pi)$, then the* *incomplete Li coefficient* $\lambda_n(\sqrt{n}, \pi) = O(\sqrt{n} \log n)$ *such that, for* $n \geq 1$,

$$\lambda_n(\pi) = \frac{N}{2} \log n + C_1(\pi) n + O(\sqrt{n} \log n),$$

where the implied constant in the O*-notation depends upon* π.

4. The τ-Li coefficients. We introduce, in a similar manner as defined by Freitas [**8**], the extended Li coefficient or the τ-Li coefficients. For $\tau \geq 1$ and $n \in \mathbb{N}$, let

$$\lambda_n(\pi, \tau) := \sum_{\rho \in Z(\pi)}' \left(1 - \left(\frac{\rho}{\rho - \tau}\right)^n\right).$$

The definition of $\lambda_n(\pi, \tau)$ for $\tau = 1$ is not identical to that given by Lagarias [**11**]. Note that $\lambda_n(\pi, 1) = \lambda_{-n}(\pi)$ according to the notation. The convergence of the sum on the right is justified by the next lemma.

Lemma 4.1. *Let* $\tau \geq 1$. *Then, we have*

- (a) $\lambda_n(\pi, \tau)$ *is *-convergent for all* $n \in \mathbb{N}$.
- (b) *The real parts of the sums are absolutely convergent for all* $n \geq 1$.
- (c)
$$\sum_{\rho \in Z(\pi, \tau)} \frac{1 + |\mathrm{Re}(\rho)|}{(1 + |\rho|)^2} < \infty.$$

Proof. We define the set

$$Z(\pi, \tau) = \left\{ \frac{\rho}{\tau} \,/\, \rho \in Z(\pi) \right\},$$

where ρ/τ and ρ have the same multiplicity. Recall that the sum

$$\sum_{\rho \in Z(\pi)}' \frac{1}{\rho}$$

is *-convergent. Then the sum

$$\sum_{\rho \in Z(\pi, \tau)}' \frac{1}{\rho} = \sum_{\rho \in Z(\pi)}' \frac{\tau}{\rho} = \tau \sum_{\rho \in Z(\pi)}' \frac{1}{\rho}$$

is also *-convergent. Since $\xi(s,\pi)$ is an entire function of order one, the sum $\sum'_{\rho\in Z(\pi)} 1/|\rho|^2$ is absolutely convergent. Furthermore, for $\tau \geq 1$, we have

$$\begin{aligned}\frac{1}{\tau}\sum_{\rho\in Z(\pi,\tau)} \frac{1+|\mathrm{Re}(\rho)|}{(1+|\rho|)^2} &= \frac{1}{\tau}\sum_{\rho\in Z(\pi)} \frac{1+|\mathrm{Re}(\rho/\tau)|}{(1+|\rho/\tau|)^2}\\ &= \sum_{\rho\in Z(\pi)} \frac{\tau+|\mathrm{Re}(\rho)|}{(\tau+|\rho|)^2}.\end{aligned}$$

Since $0\leq \mathrm{Re}(\rho)\leq 1$, we obtain

$$\frac{1}{\tau}\sum_{\rho\in Z(\pi,\tau)} \frac{1+|\mathrm{Re}(\rho)|}{(1+|\rho|)^2} < (\tau+1)\sum_{\rho\in Z(\pi)} \frac{1}{|\rho|^2}.$$

Therefore, the sum

$$\sum_{\rho\in Z(\pi,\tau)} \frac{1+|\mathrm{Re}(\rho)|}{(1+|\rho|)^2}$$

is absolutely convergent simultaneously with the sum

$$\sum_{\rho\in Z(\pi)} \frac{1+|\mathrm{Re}(\rho)|}{(1+|\rho|)^2}.$$

Using that the sum $\sum'_{\rho\in Z(\pi,\tau)} 1/\rho$ is *-convergent,

$$\sum_{\rho\in Z(\pi,\tau)} \frac{1+|\mathrm{Re}(\rho)|}{(1+|\rho|)^2} < \infty,$$

and that $0,1\notin Z(\pi,\tau)$, we conclude using Bombieri and Lagarias's lemma [**2,** Lemma 1] and the fact that

$$\begin{aligned}\lambda_n(\pi,\tau) &= \sum_{\rho\in Z(\pi)}{}' \left(1-\left(\frac{\rho}{\rho-\tau}\right)^n\right)\\ &= \sum_{\rho\in Z(\pi)}{}' \left(1-\left(\frac{\rho/\tau}{\rho/\tau-1}\right)^n\right)\\ &= \sum_{\rho\in Z(\pi,\tau)}{}' \left(1-\left(\frac{\rho}{\rho-1}\right)^n\right)\end{aligned}$$

is *-convergent for all $n\geq 1$. The two assertions (a) and (b) hold. $\square$

For $\tau \geq 1$ and $n \in \mathbb{N}$, consider

$$\alpha_n(\pi,\tau) := \frac{1}{(n-1)!}\frac{d^n}{ds^n}[s^{n-1}\log\xi(s,\pi)]_{s=\tau}.$$

Let $s_0 \neq 1$ be a real number. Denote $\varphi(s,\pi) = \xi(1/(s-1),\pi)$ and $d_n(s_0,\pi)$ the power series coefficient of the logarithmic derivative of $\varphi(s,\pi)$ around s_0.

$$\frac{\varphi'(s,\pi)}{\varphi(s,\pi)} = \sum_{n=0}^{+\infty} d_n(s_0,\pi)(s-s_0)^n.$$

Note that, for $\tau \geq 1$, if $0 \notin Z(\pi)$, then $\tau \notin Z(\pi)$. Therefore, for all $n \in \mathbb{N}$, we have

$$\begin{aligned}
\alpha_n(\pi,\tau) &= \frac{1}{(n-1)!}\,\frac{d^n}{ds^n}[s^{n-1}\log\xi(s,\pi)]_{s=\tau}\\
&= \frac{1}{(n-1)!}\left[\sum_{k=0}^{n-1}\binom{k}{n}\left(\frac{d^k}{ds^k}s^{n-1}\right)\left(\frac{d^{n-k}}{ds^{n-k}}\log\xi(s,\pi)\right)\right]_{s=\tau}\\
&= \frac{1}{(n-1)!}\left[\sum_{k=0}^{n-1}\binom{k}{n}\frac{(n-1)!}{(n-1-k)!}s^{n-1-k}\left(\frac{d^{n-k}}{ds^{n-k}}\log\xi(s,\pi)\right)\right]_{s=\tau}.
\end{aligned}$$

Recall that, for $s \notin Z(\pi)$,

$$\frac{\xi'(s,\pi)}{\xi(s,\pi)} = b(\pi) + \sum_{\rho\in Z(\pi)}{}' \left[\frac{1}{s-\rho} + \frac{1}{\rho}\right], \tag{4.1}$$

with

$$b(\pi) = -\sum_{\rho\in Z(\pi)}{}' \frac{1}{\rho}.$$

Hence,

$$\frac{d^{n-1}}{ds^{n-1}}\left[\frac{\xi'(s,\pi)}{\xi(s,\pi)}\right] = -\sum_{\rho\in Z(\pi)}{}' \frac{(n-1)!}{(\rho-s)^n}.$$

Since $\tau \geq 1$ is not in $Z(\pi)$, we obtain

$$\alpha_n(\pi,\tau) = \frac{1}{(n-1)!}\left[\sum_{k=0}^{n-1}\binom{k}{n}\frac{(n-1)!}{(n-1-k)!}s^{n-1-k}\left(\frac{d^{n-k}}{ds^{n-k}}\log\xi(s,\pi)\right)\right]_{s=\tau}$$

$$= -\frac{1}{(n-1)!}\left[\sum_{k=0}^{n-1}\binom{k}{n}\frac{(n-1)!}{(n-1-k)!}s^{n-1-k}\sum_{\rho\in Z(\pi)}{}' \frac{(n-1-k)!}{(\rho-s)^{n-k}}\right]_{s=\tau}$$
$$= -\sum_{k=0}^{n-1}\binom{k}{n}\tau^{n-k-1}\sum_{\rho\in Z(\pi)}{}' \frac{1}{(\rho-\tau)^{n-k}}.$$

Hence,

$$\begin{aligned}\alpha_n(\pi,\tau) &= -\frac{1}{\tau}\sum_{\rho\in Z(\pi)}{}' \sum_{k=0}^{n-1}\binom{k}{n}\left(\frac{\tau}{\rho-\tau}\right)^{n-k} \\ &= -\frac{1}{\tau}\sum_{\rho\in Z(\pi)}{}' \left[\sum_{k=0}^{n}\binom{k}{n}\left(\frac{\tau}{\rho-\tau}\right)^{n-k} - 1\right] \\ &= -\frac{1}{\tau}\sum_{\rho\in Z(\pi)}{}' \left[\left(1+\frac{\tau}{\rho-\tau}\right)^{n} - 1\right] \\ &= \frac{1}{\tau}\sum_{\rho\in Z(\pi)}{}' \left[1-\left(\frac{\rho}{\rho-\tau}\right)^{n}\right] \\ &= \frac{1}{\tau}\lambda_n(\pi,\tau).\end{aligned} \tag{4.2}$$

The same argument as used above yields[3]

$$\alpha_n(\pi,\tau) = \frac{1}{\tau^{n+1}}d_{n-1}(1-1/\tau,\pi). \tag{4.3}$$

Now, we are ready to state our first main theorem which extends Freitas's result [**8,** Theorem 1] to a general class of automorphic L-functions.

Theorem 4.2. *Let $\tau \geq 1$ be a real number. Then, we have*

$$1-\frac{\tau}{2} \leq \mathrm{Re}(\rho) \leq \frac{\tau}{2} \Longleftrightarrow \mathrm{Re}(\lambda_n(\pi,\tau)) \geq 0 \quad \textit{for all } n \in \mathbb{N},$$

where ρ denotes the zeros of $\xi(s,\pi)$.

The main remark here is that, when $\tau = 1$, we recover the Li coefficients for automorphic L-functions as defined by Lagarias [**11**]. Positivity of $\mathrm{Re}(\lambda_n(\pi,\tau))$ for other values of τ is equivalent to the non-existence of zeros in the half-plane $\mathrm{Re}(s) > \tau/2$. In particular, when

$\tau \le 1$, we obtain a region that contains the critical line. In the latter case, some of the coefficients $\mathrm{Re}(\lambda_n(\pi,\tau))$ must take negative values. On the other hand, for $\tau \ge 2$, we are outside of the critical strip, and hence, the $\mathrm{Re}(\lambda_n(\pi,\tau))$ must all be nonnegative.

For the proof of Theorem 4.2, we need the next lemma which is an adaptation of Bombieri and Lagarias's theorem [**2,** Theorem 1].

Lemma 4.3. *Let $\tau > 0$ be a real number and R a multi-set of complex numbers such that $\tau \notin R$ and*

$$\sum_R \frac{1+|\mathrm{Re}(\rho)|}{(1+|\rho|)^2} < \infty.$$

The following assertions are equivalent.

- (a) $\mathrm{Re}(\rho) \le \tau/2$ *for all* $\rho \in R$.
- (b) $\sum_R \mathrm{Re}[1-(\rho/(\rho-\tau))^n] \ge 0$ *for all* $n \in \mathbb{N}$.
- (c) *For all* $\epsilon > 0$*, there exists a constant* $c(\epsilon)$*, such that*

$$\sum_R \mathrm{Re}\left[1-\left(\frac{\rho}{\rho-\tau}\right)^n\right] \ge -c(\epsilon)e^{\epsilon n}$$

for all $n \ge 1$.

The proof is similar to that stated by Bombieri and Lagarias [**2,** Theorem 1] with $R_\tau = \{(\rho/\tau)/\rho \in R\}$, where ρ/τ has multiplicity in R_τ, equivalent to ρ in R.

Proof of Theorem 4.2. Since $\tau \notin Z(\pi)$, by Lemma 4.1 (c), the sum

$$\sum_{\rho\in Z(\pi)} \frac{1+|\mathrm{Re}(\rho)|}{(1+|\rho|)^2} < \infty.$$

Then, we apply Lemma 4.3 with $R = Z(\pi)$. Therefore,

$$\begin{aligned}&\mathrm{Re}(\rho) \le \frac{\tau}{2} \quad \text{for all } \rho \in Z(\pi)\\ &\quad\Longleftrightarrow \sum_{\rho\in Z(\pi)} \mathrm{Re}\left[1-\left(\frac{\rho}{\rho-\tau}\right)^n\right] \ge 0 \quad \text{for all } n \in \mathbb{N}\end{aligned}$$

$$\iff \text{for all } \epsilon>0, \text{ there exists } c(\epsilon) \text{ such that } \sum_{\rho\in Z(\pi)} \mathrm{Re}\left[1-\left(\frac{\rho}{\rho-\tau}\right)^n\right]$$
$$\geq -c(\epsilon)e^{\epsilon n} \quad \text{for all } n\geq 1.$$

Using the fact that

$$\rho\in Z(\pi) \iff 1-\overline{\rho}\in Z(\pi),$$

we conclude that $\mathrm{Re}(1-\overline{\rho}) \leq \tau/2$ for all $\rho \in Z(\pi)$. Then, $1-\tau/2 \leq \mathrm{Re}(\rho)$ for all $\rho\in Z(\pi)$. Therefore, Theorem 4.2 follows. □

5. Arithmetical formula for $\lambda_n(\pi,\tau)$. In this section, we derive an arithmetical formula for the coefficients $\lambda_n(\pi,\tau)$. This formula is stated in the next theorem.

Theorem 5.1. *Let π be an irreducible cuspidal automorphic representation of* $\mathrm{GL}(N)$ *over* $\mathbb{Q}$. *Then, for all $n\geq 1$, we have*

(5.1)

$$\begin{aligned}\lambda_n(\pi,\tau) &= \delta(\pi)\left[2+\frac{(-1)^{n+1}}{(\tau-1)^n}\right] + \frac{n\tau}{2}\log Q(\pi) - n\tau N\left(\gamma+\frac{1}{2}\log\pi\right)\\ &\quad+\frac{n\tau}{2}\sum_{j=1}^{N}\left(\frac{-2}{\tau+k_j(\pi)}+\sum_{l=1}^{\infty}\frac{\tau+k_j(\pi)}{l(2l+\tau+k_j(\pi))}\right)\\ &\quad+\sum_{j=1}^{N}\sum_{k=2}^{n}\binom{k}{n}(-\tau)^k\sum_{m=0}^{\infty}\frac{1}{(\tau+k_j(\pi)+2m)^k}\\ &\quad+\sum_{k=1}^{n}\binom{k}{n}\frac{(-\tau)^k}{(k-1)!}\sum_{m=1}^{\infty}\frac{\Lambda_\pi(m)(\log m)^{k-1}}{n^\tau},\end{aligned}$$

where

$$\delta(\pi)=\begin{cases}1 & \textit{if } \pi=\pi_{\mathrm{triv}},\\ 0 & \textit{otherwise.}\end{cases}$$

Proof. From equation (4.2), we have

(5.2)

$$\lambda_n(\pi,\tau) = \tau\alpha_n(\pi,\tau)$$

$$= \frac{\tau}{(n-1)!}\left[\sum_{k=0}^{n}\binom{k}{n}\left(\frac{d^{n-k}}{ds^{n-k}}s^{n-1}\right)\left(\frac{d^k}{ds^k}\log\xi(s,\pi)\right)\right]_{s=\tau}$$

$$= \frac{\tau}{(n-1)!}\sum_{k=1}^{n}\binom{k}{n}\frac{(n-1)!}{(k-1)!}\tau^{k-1}\left[\frac{d^k}{ds^k}\log\xi(s,\pi)\right]_{s=\tau}$$

$$= \sum_{k=1}^{n}\binom{k}{n}\frac{\tau^k}{(k-1)!}\left[\frac{d^k}{ds^k}\log\xi(s,\pi)\right]_{s=\tau}.$$

Now, we write

$$\frac{\xi'}{\xi}(s,\pi) = \frac{1}{2}\log Q(\pi) + \sum_{j=1}^{N}\frac{\Gamma'_{\mathbb{R}}}{\Gamma_{\mathbb{R}}}(s+k_j(\pi)) + \frac{L'}{L}(s,\pi) - \frac{e(0,\pi)}{s} - \frac{e(1,\pi)}{s-1}. \tag{5.3}$$

Therefore,

$$\begin{aligned}\lambda_n(\pi,\tau) = {} & \sum_{k=1}^{n}\binom{k}{n}\frac{\tau^k}{(k-1)!}\left[\frac{d^{k-1}}{ds^{k-1}}\left(\frac{L'}{L}(s,\pi) - \frac{e(1,\pi)}{s-1}\right)\right]_{s=\tau} \\ & - e(0,\pi)\sum_{k=1}^{n}\binom{k}{n}\frac{\tau^k}{(k-1)!}\frac{d^{k-1}}{ds^{k-1}}\left[\frac{1}{s}\right]_{s=\tau} \\ & + \frac{1}{2}\log Q(\pi)\sum_{k=1}^{n}\binom{k}{n}\frac{\tau^k}{(k-1)!}\delta_{k,1} \\ & + \sum_{k=1}^{n}\binom{k}{n}\frac{\tau^k}{(k-1)!}\sum_{j=1}^{N}\left[\frac{d^{k-1}}{ds^{k-1}}\frac{\Gamma'_{\mathbb{R}}}{\Gamma_{\mathbb{R}}}(s+k_j(\pi))\right]_{s=\tau},\end{aligned} \tag{5.4}$$

where $\delta_{k,1}$ is the Kronecker delta function (defined by $\delta_{k,1} = 1$ if $k = 1$, and $\delta_{k,1} = 0$, otherwise). Hence, we obtain

$$\begin{aligned}\lambda_n(\pi,\tau) = {} & -e(0,\pi) + \frac{n\tau}{2}\log Q(\pi) \\ & + \sum_{k=1}^{n}\binom{k}{n}\tau^k\eta_{k-1}(\pi,\tau) \\ & + \sum_{k=1}^{n}\binom{k}{n}\tau^k\vartheta_{k-1}(\pi,\tau),\end{aligned} \tag{5.5}$$

where

$$\eta_k(\pi,\tau) = \frac{1}{k!}\left[\frac{d^k}{ds^k}\left(\frac{L'}{L}(s,\pi) - \frac{e(1,\pi)}{s-1}\right)\right]_{s=\tau},$$
$$\vartheta_0(\pi,\tau) = -\frac{N}{2}\log\pi + \frac{1}{2}\sum_{j=1}^{N}\psi\left(\frac{\tau+k_j(\pi)}{2}\right)$$

and

$$\vartheta_k(\pi,\tau) = \frac{1}{k!}\sum_{j=1}^{N}\left(\frac{1}{2}\right)^{k+1}\psi^{(k)}\left(\frac{\tau+k_j(\pi)}{2}\right).$$

If $\tau > 1$, we obtain

$$\begin{aligned}(5.6)\quad \sum_{k=1}^{n}\binom{k}{n}\tau^k\eta_{k-1}(\pi,\tau) &= \sum_{k=1}^{n}\binom{k}{n}\frac{\tau^k}{(k-1)!}\left[\frac{d^{k-1}}{ds^{k-1}}\frac{L'}{L}(s,\pi)\right]_{s=\tau}\\ &\quad + e(1,\pi)\sum_{k=1}^{n}\binom{k}{n}\left(\frac{\tau}{1-\tau}\right)^k\\ &= e(1,\pi)\left[\left(1+\frac{\tau}{1-\tau}\right)^n - 1\right]\\ &\quad + \sum_{k=1}^{n}\binom{k}{n}\tau^k l_{k-1}(\pi,\tau)\\ &= -e(1,\pi)\left[1+(-1)^{n+1}\left(\frac{1}{\tau-1}\right)^n\right]\\ &\quad + \sum_{k=1}^{n}\binom{k}{n}\tau^k l_{k-1}(\pi,\tau),\end{aligned}$$

with

$$l_k(\pi,\tau) = \frac{1}{k!}\left[\frac{d^k}{ds^k}\frac{L'}{L}(s,\pi)\right]_{s=\tau}.$$

Hence, from equations (5.5) and (5.6) and using the fact that $e(0,\pi) = e(1,\pi)$, we deduce

$$\begin{aligned}(5.7)\qquad \lambda_n(\pi,\tau) = &-e(0,\pi)\left[2+(-1)^{n+1}\left(\frac{1}{\tau-1}\right)^n\right]\\ &+ \frac{n\tau}{2}\log Q(\pi)\end{aligned}$$

$$+\sum_{k=1}^{n}\binom{k}{n}\tau^{k}l_{k-1}(\pi,\tau)+\sum_{k=1}^{n}\binom{k}{n}\tau^{k}\vartheta_{k-1}(\pi,\tau).$$

By logarithmically differentiating the Euler product of $L(s,\pi)$ in the region $\mathrm{Re}(s)>1$, we obtain

$$\frac{L'}{L}(s,\pi)=\sum_{n=1}^{\infty}\frac{\Lambda_{\pi}(n)}{n^{s}},$$

in which, for $n=p^{m}$ a prime power,

$$\Lambda_{\pi}(n)=\frac{1}{m}\sum_{k=1}^{N}(\alpha_{k,p}(\pi))^{m},$$

and $\Lambda_{\pi}(n)=0$ otherwise. We formally obtain

$$l_{k}(\pi,\tau)=\frac{(-1)^{k+1}}{k!}\sum_{m=1}^{\infty}\frac{\Lambda_{\pi}(m)(\log m)^{k}}{n^{\tau}}. \tag{5.8}$$

Now, recall the following formula of the digamma function $\psi(z)=\Gamma'(z)/\Gamma$:

$$\psi(z)=-\gamma-\frac{1}{z}+\sum_{l=1}^{\infty}\frac{z}{l(l+z)},$$

in which $\gamma\approx 0.5771$ is the Euler constant. Then, for $k\geq 1$, we obtain

$$\begin{aligned}\vartheta_{k}(\pi,\tau)&=\frac{1}{k!}\sum_{j=1}^{N}\left(\frac{1}{2}\right)^{k}\psi^{(k)}\left(\frac{\tau+k_{j}(\pi)}{2}\right)\\&=\sum_{j=1}^{N}(-1)^{k+1}\sum_{m=0}^{\infty}\frac{1}{(\tau+k_{j}(\pi)+2m)^{k+1}}\end{aligned} \tag{5.9}$$

and

$$\begin{aligned}\vartheta_{0}(\pi,\tau)&=-\frac{N}{2}\log\pi+\frac{1}{2}\sum_{j=1}^{N}\psi\left(\frac{\tau+k_{j}(\pi)}{2}\right)\\&=-\frac{N}{2}\log\pi-N\gamma-\sum_{j=1}^{N}\frac{1}{\tau+k_{j}(\pi)}\end{aligned} \tag{5.10}$$

$$+\sum_{j=1}^{N}\sum_{l=1}^{\infty}\frac{\tau+k_j(\pi)}{l(2l+\tau+k_j(\pi))}.$$

Therefore, from (5.7), (5.8), (5.9) and (5.10), we deduce

$$\begin{aligned}\lambda_n(\pi,\tau) &= -e(0,\pi)\left[2+(-1)^{n+1}\left(\frac{1}{\tau-1}\right)^n\right] \\ &+ \frac{n\tau}{2}\log Q(\pi) - n\tau N\left(\gamma+\frac{1}{2}\log\pi\right) \\ &+ \frac{n\tau}{2}\sum_{j=1}^{N}\left(\frac{-2}{\tau+k_j(\pi)}+\sum_{l=1}^{\infty}\frac{\tau+k_j(\pi)}{l(2l+\tau+k_j(\pi))}\right) \\ &+ \sum_{j=1}^{N}\sum_{k=2}^{n}\binom{k}{n}(-\tau)^k\sum_{m=0}^{\infty}\frac{1}{(\tau+k_j(\pi)+2m)^k} \\ &+ \sum_{k=1}^{n}\binom{k}{n}\frac{(-\tau)^k}{(k-1)!}\sum_{m=1}^{\infty}\frac{\Lambda_\pi(m)(\log m)^{k-1}}{n^\tau}.\end{aligned} \tag{5.11}$$

We note that

$$-e(0,\pi)=\delta(\pi)=\begin{cases}1 & \text{if } \pi=\pi_{\text{triv}},\\ 0 & \text{otherwise.}\end{cases}$$

Then, Theorem 5.1 follows. $\square$

6. Asymptotic formula for $\lambda_n(\pi,\tau)$. In this section, we apply the saddle-point method in conjunction with the theory of the Nörlund-Rice integrals to derive an asymptotic formula for the coefficients $\lambda_n(\pi,\tau)$. This argument was used by the author **[14]** to derive an asymptotic formula for the Li coefficients for L-functions in the Selberg class.

Let $H_n(m,k)$ be the sum defined by

$$H_n(m,k)=\sum_{l=2}^{n}(-1)^l\binom{n}{l}\frac{\zeta(l,m/k)}{k^l},$$

where $\zeta(s,q)$ is the Hurwitz zeta function given by

$$\zeta(s,q) = \sum_{n=0}^{+\infty} \frac{1}{(n+q)^s}.$$

Using the saddle-point method in conjunction with the theory of the Nörlund-Rice integrals (see [**14,** Proposition 4.3]), the sums $H_n(m,k)$ satisfy the estimate

$$H_n(m,k) = \left(\frac{m}{k}-\frac{1}{2}\right)-\frac{n}{k}\left(\psi\left(\frac{m}{k}\right)+\log k+1-h_{n-1}\right)+a_n(m,k), \tag{6.1}$$

where the $a_n(m,k)$ are exponentially small:

$$\begin{aligned} a_n(m,k) = \frac{1}{k}\left(\frac{2n}{\pi k}\right)^{1/4} \exp\left(-\sqrt{\frac{4\pi n}{k}}\right) \cos\left(\sqrt{\frac{4\pi n}{k}} - \frac{5\pi}{8} - \frac{2\pi m}{k}\right) \\ + O(n^{-1/4}e^{-2\sqrt{\pi n/k}}). \end{aligned}$$

Here, $h_n = 1 + 1/2 + \cdots + 1/n$ is a harmonic number, and $\psi(x)$ is the logarithmic derivative of the Gamma function.

Write the arithmetical formula of $\lambda_n(\pi,\tau)$ (equation (5.1)) as

$$\begin{aligned} \lambda_n(\pi,\tau) &= \delta(\pi)\left[2 + \frac{(-1)^{n+1}}{(\tau-1)^n}\right] \\ &+ \frac{n\tau}{2}\log Q(\pi) - n\tau N\left(\gamma + \frac{1}{2}\log\pi\right) \\ &+ \frac{n\tau}{2}\sum_{j=1}^{N}\left(\frac{-2}{\tau+k_j(\pi)} + \sum_{l=1}^{\infty}\frac{\tau+k_j(\pi)}{l(2l+\tau+k_j(\pi))}\right) \\ &+ \sum_{j=1}^{N} I_j + \sum_{k=1}^{n}\binom{k}{n}(\tau)^k l_{k-1}(\pi,\tau), \end{aligned} \tag{6.2}$$

where

$$I_j = \sum_{k=2}^{n}\binom{k}{n}(-\tau)^k \sum_{m=0}^{\infty}\frac{1}{(\tau+k_j(\pi)+2m)^k}$$

and

$$l_k(\pi,\tau) = \frac{(-1)^{k+1}}{k!}\sum_{m=1}^{\infty}\frac{\Lambda_\pi(m)(\log m)^k}{n^\tau}.$$

Note that

$$I_j = \sum_{k=2}^{n} \binom{k}{n} (-1)^k \sum_{m=0}^{\infty} \left(\frac{\tau/2}{((\tau + k_j(\pi))/2) + m} \right)^k ,$$

which, with the above notation of $H_n(m,k)$, is equal to

$$I_j = H_n \left(1 + \frac{k_j(\pi)}{\tau}, \frac{2}{\tau} \right).$$

By applying the above estimate given by equation (6.1) with $m = 1 + (k_j(\pi))/\tau$ and $k = 2/\tau$, we deduce

(6.3)

$$I_j = \left(\frac{\tau}{2} + \frac{k_j(\pi)}{2} - \frac{1}{2} \right) - n\frac{\tau}{2}\left(\psi\left(\frac{\tau + k_j(\pi)}{2} \right) + \log\left(\frac{2}{\tau} \right) + 1 - h_n \right) + a_n\left(1 + \frac{k_j(\pi)}{\tau}, \frac{2}{\tau} \right).$$

If the a_n are exponentially small, then

$$a_n\left(1 + \frac{k_j(\pi)}{\tau}, \frac{2}{\tau} \right) = O(1)$$

and

$$I_j = \left(\frac{\tau}{2} + \frac{k_j(\pi)}{2} - \frac{1}{2} \right) - n\frac{\tau}{2}\left(\psi\left(\frac{\tau + k_j(\pi)}{2} \right) + \log\left(\frac{2}{\tau} \right) + 1 - h_n \right) + O(1).$$

Summing the last equation over j, we obtain

(6.4)

$$\sum_{j=1}^{N} I_j = \sum_{j=1}^{N} \left(\frac{\tau}{2} + \frac{k_j(\pi)}{2} - \frac{1}{2} \right) - n \sum_{j=1}^{N} \frac{\tau}{2} \left[\psi\left(\frac{\tau + k_j(\pi)}{2} \right) + \log\left(\frac{2}{\tau} \right) + 1 - h_n \right] + O(N).$$

Using the expression

$$\psi(z) = -\gamma - \frac{1}{z} + \sum_{l=1}^{\infty} \frac{z}{l(l+z)}$$

and the estimate

$$h_n = \log n - \gamma + \frac{1}{2n} + O\left(\frac{1}{2n^2}\right),$$

we obtain, from equations (6.2) and (6.4) that

(6.5)

$$\begin{aligned}\lambda_n(\pi,\tau) = {} & \frac{N\tau}{2} n \log n \\ & + \left\{ \frac{N\tau}{2}(\gamma - 1 - \log\pi) + \frac{\tau}{2}\log Q(\pi) + N\frac{\tau}{2}\log\left(\frac{\tau}{2}\right)\right\} n \\ & + O(N) + S_L(\pi,\tau),\end{aligned}$$

where

$$S_L(\pi,\tau) := \sum_{k=1}^{n} \binom{k}{n} \tau^k l_{k-1}(\pi,\tau).$$

This concludes the proof of the following theorem.

Theorem 6.1. *For any irreducible cuspidal (unitary) automorphic representation π on $\mathrm{GL}(N)$, we have*

(6.6)

$$\begin{aligned}\lambda_n(\pi,\tau) = {} & \frac{N\tau}{2} n \log n \\ & + \left\{ \frac{N\tau}{2}\left(\gamma - 1 - \log\pi\right) + \frac{\tau}{2}\log Q(\pi) + N\frac{\tau}{2}\log\left(\frac{\tau}{2}\right)\right\} n \\ & + O(N) + S_L(\pi,\tau),\end{aligned}$$

where

$$S_L(\pi,\tau) := \sum_{k=1}^{n} \binom{k}{n} \tau^k l_{k-1}(\pi,\tau),$$

γ is the Euler constant and the implied constant in the O-notation is absolute.

7. An interpolation function of the Archimedean contribution. The τ-Li coefficients (5.7) may be written as

$$\lambda_n(\pi,\tau) = S_\infty(n,\pi,\tau) + S_{NA}(n,\pi,\tau). \tag{7.1}$$

where

$$\begin{aligned} S_\infty(n,\pi,\tau) &= \delta(\pi)\left[2 + (-1)^{n+1}\left(\frac{1}{\tau-1}\right)^n\right] + \frac{n\tau}{2}\log Q(\pi) \\ &\quad + n\tau\left[-\frac{N}{2}\log\pi + \frac{1}{2}\sum_{j=1}^{N}\psi\left(\frac{\tau+k_j(\pi)}{2}\right)\right] \\ &\quad + \sum_{j=1}^{N}\sum_{k=2}^{n}\binom{k}{n}(-\tau)^k\sum_{m=0}^{\infty}\frac{1}{(\tau+k_j(\pi)+2m)^k} \end{aligned}$$

represents the Archimedean contribution, and

$$S_{NA}(n,\pi,\tau) = \sum_{k=1}^{n}\binom{k}{n}\tau^k l_{k-1}(\pi,\tau)$$

is the finite (non-Archimedean) term. The Archimedean contribution to the nth τ-Li coefficients may be explicitly given as in Theorem 5.1.

The same argument as used by Odzăk and Smajlovič in [**17,** Theorem 4.1, Remark 3.2] shows that there exists an entire function of order one and finite exponential type that interpolates the Archimedian contribution and the finite contribution to the τ-Li coefficients $\lambda_n(\pi,\tau)$ attached to the automorphic L-functions $L(s,\pi)$ at positive integers. Furthermore, the interpolation function may be obtained as $\lim_{m\to\infty}\varphi_{\infty,m}(z,\tau,F_\pi)$, with the same notation as in [**17**], where

(7.2)

$$\begin{aligned} \varphi_{\infty,m}(z,\tau,F_\pi) &= \sum_{j\in A_\pi} e^{i\pi z}\left(-\frac{k_j(\pi)}{\tau+k_j(\pi)}\right)^z + \sum_{j\in B_\pi|A_\pi}\left(\frac{k_j(\pi)}{\tau+k_j(\pi)}\right)^z \\ &\quad + \delta(\pi)\left[2 + \frac{(-1)^{n+1}}{(\tau-1)^n}\right] + z\left[\frac{\tau}{2}\log Q(\pi) - \frac{N\tau}{2}\log\pi\right] \\ &\quad + z\frac{\tau}{2}\sum_{j=1}^{N}\frac{\Gamma'}{\Gamma}\left(\frac{\tau+k_j(\pi)}{2}+1\right) \end{aligned}$$

$$+\sum_{j=1}^{N}\sum_{k=1}^{m-1}\left[\left(1-\frac{2}{\tau+k_j(\pi)+2k}\right)^{z}\right.$$
$$\left.-1+z\frac{2}{\tau+k_j(\pi)+2k}\right]$$
$$+\sum_{j=1}^{N}\sum_{l=2}^{\infty}\binom{z}{l}(-\tau)^k\zeta(l,\tau+k_j(\pi)+2m),$$

with

$$A_\pi=\{j\in\{1,\dots,N\}:\operatorname{Im}(k_j(\pi))=0\quad\text{and}\quad\operatorname{Re}(k_j(\pi))<0\}$$

and

$$B_\pi=\{j\in\{1,\dots,N\}:k_j(\pi))\neq 0\}.$$

Letting $m\mapsto\infty$, we obtain

(7.3)

$$\varphi_\infty(z,\tau,F_\pi)=\sum_{j\in A_\pi}e^{i\pi z}\left(-\frac{k_j(\pi)}{\tau+k_j(\pi)}\right)^{z}$$
$$+\sum_{j\in B_\pi|A_\pi}\left(\frac{k_j(\pi)}{\tau+k_j(\pi)}\right)^{z}+\delta(\pi)\left[2+\frac{(-1)^{n+1}}{(\tau-1)^n}\right]$$
$$+z\left[\frac{\tau}{2}\log Q(\pi)-\frac{N\tau}{2}\log\pi+\frac{\tau}{2}\sum_{j=1}^{N}\frac{\Gamma'}{\Gamma}\left(\frac{\tau+k_j(\pi)}{2}+1\right)\right]$$
$$+\sum_{j=1}^{N}\sum_{k=1}^{m-1}\left[\left(1-\frac{2}{\tau+k_j(\pi)+2k}\right)^{z}-1+z\frac{2}{\tau+k_j(\pi)+2k}\right].$$

Since, for a fixed z,

$$\lim_{m\longmapsto\infty}\sum_{j=1}^{N}\sum_{l=2}^{\infty}\binom{z}{l}(-\tau)^k\zeta(l,\tau+k_j(\pi)+2m)=0,$$

the interpolating function $F_{\pi,\infty}(z,\tau)$ of the Archimedean contribution, using the same notation as in [**11**] or in [**17,** Remark 3.2], is given by

(7.4)

$$F_{\pi,\infty}(z,\tau)=C_3(\pi)+C_4(\pi)z$$

$$-\sum_{j=1}^{N}\sum_{k=1}^{m-1}\left[\left(1-\frac{2}{\tau+k_j(\pi)+2k}\right)^{z}-1+z\frac{2}{\tau+k_j(\pi)+2k}\right],$$

for some constants $C_3(\pi)$ and $C_4(\pi)$. (In the notation of Lagarias, $F_{\pi,\infty}(z,1)=F_{\pi,\infty}(z)$ corresponds to our $-\varphi_\infty(z,1,F_\pi)=-\varphi_\infty(z,F_\pi)$ in the notation of Odzăk and Smajlovič.) Then, for $|z|$ large and $\mathrm{Re}(k_j(\pi))>-1/2$, we have

$$\left|\frac{k_j(\pi)}{\tau+k_j(\pi)}\right|<1.$$

Therefore,

(7.5)

$$\begin{aligned}\varphi_\infty(z,\tau,F_\pi)&=C_1(\pi)+C_2(\pi)z+o(1)\\&\quad+\sum_{j=1}^{N}\sum_{k=1}^{m-1}\left[\left(1-\frac{2}{\tau+k_j(\pi)+2k}\right)^{z}-1+z\frac{2}{\tau+k_j(\pi)+2k}\right],\end{aligned}$$

with

$$C_1(\pi)=\delta(\pi)\left[2+\frac{(-1)^{n+1}}{(\tau-1)^n}\right]$$

and

$$C_2(\pi)=\frac{\tau}{2}\left[\log Q(\pi)-N\log\pi+\sum_{j=1}^{N}\frac{\Gamma'}{\Gamma}\left(\frac{\tau+k_j(\pi)}{2}+1\right)\right].$$

Since

$$\left|1-\frac{2}{\tau+k_j(\pi)+2k}\right|<1\quad\text{for all } k\geq 1,$$

then the function $\varphi_\infty(z,\tau,F_\pi)$ is entire and of exponential type.

8. Concluding remarks. The coefficients $\alpha_n(\pi,\tau)$ satisfy an infinite system of the linear differential equation

$$\frac{\tau}{n}\alpha_n'(\pi,\tau)+\frac{(n+1)}{n}\alpha_n(\pi,\tau)=\alpha_{n+1}(\pi,\tau),\quad n\geq 1. \tag{8.1}$$

From the arithmetical formula, $1/\tau$ is the homogeneous solution of this set of equations. We have

$$\alpha_{n+1}(\pi,0)=\alpha_n(\pi,0)-\sigma_1$$

and, by the functional equation of $\xi(s,\pi)$,

$$\alpha_n(\pi,1)=-\lambda_1(\pi).$$

Then, $\alpha_n(\pi,0)=-n\lambda_1(\pi)<0$. Furthermore, from equation (8.1), we have

$$\alpha_n'(\pi,0)=n\lambda_{n+1}(\pi)-(n+1)\lambda_n(\pi).$$

Taking the $(j-1)$ derivative of (8.1), we obtain

$$\frac{\tau}{n}\alpha_n^{(j)}(\pi,\tau)+\frac{(n+j)}{n}\alpha_n^{(j-1)}(\pi,\tau)=\alpha_{n+1}^{(j-1)}(\pi,\tau),\quad n\geq 1.$$

This implies that all derivatives of $\alpha_n(\pi,\tau)$ can be calculated (recursively) at $\tau=1$ only in terms of the Li coefficients $\lambda_n(\pi)$. Therefore, we have

$$\begin{aligned}\alpha_n''(\pi,1)=(n+1)\lambda_{n+2}(\pi)-(n+1)(n+2)\lambda_{n+1}(\pi)\\ -(n+1)(n+2)\lambda_n(\pi).\end{aligned}$$

Hence, since the order of the Li coefficients is known, then all derivatives of $\alpha_{F,\tau}(n)$ at $\tau=1$ are known. Recall that $\lambda_n(\pi)=O(n\log n)$. Then, $\alpha_n'(\pi,1)$ is positive and linearly increasing with n and $\alpha_n'(\pi,1)\sim(n+1)>0$.

An interesting question on the Li criterion for a various class of L-functions is whether the positivity of a few Li coefficients would be enough to give us some information on the L-functions. The author, along with Omar and Ouni, gave a partial answer to the question. Indeed, it was proven in [**23, 24**] that there is a relation between the positivity of the first Li coefficients and the partial Li criterion PRH(T), i.e., the Riemann hypothesis holds up to a certain $T>0$, and showed that, if PRH(T) is true, then the Li coefficients of some L-functions are positive up to T^2. Brown [**3**] proved that the positivity of the real parts of the first N Li coefficients implies the existence of a zero-free region of the corresponding L-function. Furthermore, he showed that the positivity of the real part of the second Li coefficients yields the non existence of a Siegel zero (this result was generalized to the Selberg class with further conditions by the author in a work in progress and by a probabilistic argument in a special case of the Dirichlet L-functions in [**15**]).

Future research includes:

- extending the work of Brown on the Siegel zero for the $\tau > 1$ case as well as to consider the idea of the author [**15**] for giving a probabilistic interpretation for Li coefficients corresponding to automorphic L-functions or to other classes of L-functions for the $\tau > 1$ case;
- studying the distribution of the τ-Li coefficients $\lambda_n(\pi, \tau)$.

The author in [**15**], or with Omar and Ouni in [**23, 24**], conjectured that the Li coefficients for some L-functions are increasing. A natural question, then, is to see whether it remains true for the coefficients $\lambda_n(\pi, \tau)$. Furthermore, it might be interesting to numerically analyze the situation where τ is close to one but less than one and see how the negative part of the sequence $\mathrm{Re}(\lambda_n(\pi, \tau))$ behaves as τ approaches one.

These problems will be considered in a sequel to this article.

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ENDNOTES

1. The symbol $'$ indicates that the (conditionally convergent) sum is to be interpreted as $\lim_{T\to} \sum_{\rho,\ |\Im(\rho)|<T}$; we term this $*$-convergence.

2. The incomplete Li coefficient at height T is defined by

$$\lambda_n(T, \pi) = \sum_{\rho\in Z(\pi);|\mathrm{Im}(\rho)|<T} 1 - \left(1 - \frac{1}{\rho}\right)^n.$$

3. To prove (4.3), we begin with equation (4.1). If $1/(1-s) \notin Z(\pi)$ and $s \neq 1$, we write

$$\frac{\varphi'}{\varphi}(s, \pi) = \frac{1}{ds} \log\left(\xi\left(\frac{1}{1-s}, \pi\right)\right)$$

$$= \frac{b(\pi)}{(1-s)^2} + \sum_{Z(\pi)}{}' \left[\frac{1}{1-s} \frac{1}{1-\rho+s\rho} + \frac{1}{\rho(1-s)^2} \right].$$

Let $s_0 = 1 - 1/\tau \neq 1$. Since $1/(1-s_0) = \tau \notin Z(\pi)$, then s_0 is not a zero of $\varphi(s,\pi)$. Using Freitas's argument [**8,** Lemma 3.1], for any $\rho \in Z(\pi)$, we have

$$\frac{1}{1-\rho+s\rho} = \left(\frac{-\tau}{\rho-\tau} \right) \left(\frac{1}{1-(\rho\tau(s-s_0)/(\rho-\tau))} \right)$$

and

$$\frac{1}{1-s} = \tau \left(\frac{1}{1-\tau(s-s_0)} \right).$$

For s sufficiently close to s_0, we use geometric series and the fact that $1/2 \sum_{n\geq 0} 2^{-n} = 1$. We find, after simple computation,

$$\frac{\varphi'}{\varphi}(s,\pi) = \sum_{n=0}^{\infty} \tau^{n+1} \lambda_{n+1}(\pi,\tau)(s-s_0)^n = \sum_{n=0}^{\infty} d_n(s_0,\pi)(s-s_0)^n.$$

Therefore, formula (4.3) easily follows from the last equation and equation (4.2).

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