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SOME CONSTRUCTIONS OF K-FRAMES AND THEIR DUALS

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ABSTRACT. K-frames, as a new generalization of frames, have important applications, especially in sampling theory, to help us to reconstruct elements from a range of a bounded linear operator K in a separable Hilbert space. In this paper, we focus on the reconstruction formulae to characterize all K-duals of a given K-frame. Also, we give several approaches for constructing K-frames.

1. Introduction and preliminaries. A family of local atoms for a closed subspace $\mathcal{H}_0$ of a Hilbert space $\mathcal{H}$ was introduced in [**11**] with frame-like properties. In contrast to frames, local atoms do not necessarily belong to $\mathcal{H}_0$. This property is especially worthwhile in some problems arising in sampling theory, see [**10, 13, 14**]. K-frames were recently introduced by Găvruţa to study atomic systems with respect to a bounded operator $K \in B(\mathcal{H})$ [**12**]. It was shown that atomic systems for K are equivalent with K-frames; in addition, every family of local atoms for a closed subspace $\mathcal{H}_0$ of $\mathcal{H}$ is a $\pi_{\mathcal{H}_0}$-frame, where $\pi_{\mathcal{H}_0}$ is the orthogonal projection of $\mathcal{H}$ onto $\mathcal{H}_0$.

Let $\mathcal{H}$ be a separable Hilbert space, and recall that a sequence $F := \{f_i\}_{i\in I} \subseteq \mathcal{H}$ is called a K-*frame* for $\mathcal{H}$, if there exist constants $A, B > 0$ such that

$$A\|K^*f\|^2 \leq \sum_{i\in I} |\langle f, f_i\rangle|^2 \leq B\|f\|^2, \quad f \in \mathcal{H}. \tag{1.1}$$

Clearly, if $K = I_{\mathcal{H}}$, then F is an ordinary frame; hence, K-frames arise naturally as a generalization of ordinary frames. For more details and applications of ordinary frames, see [**2**]–[**7**]. The constants A and B in (1.1) are called the *lower* and the *upper bounds* of F, respectively.

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Also, the supremum of all lower bounds is called the *optimal lower bound*, and likewise, the *optimal upper bound* is defined as the infimum of all upper frame bounds of K-frame F. If $A = B = 1$, we call F a Parseval K-frame. Obviously, every K-frame is a Bessel sequence; hence, similar to ordinary frames, the synthesis operator can be defined as

$$T_F : l^2 \longrightarrow \mathcal{H}; \qquad T_F(\{c_i\}_{i\in I}) = \sum_{i\in I} c_i f_i.$$

It is a bounded operator, and its adjoint, called the *analysis operator*, is given by $T_F^*(f) = \{\langle f, f_i\rangle\}_{i\in I}$. Finally, the frame operator is given by

$$S_F : \mathcal{H} \longrightarrow \mathcal{H}; \qquad S_F f = T_F T_F^* f = \sum_{i\in I} \langle f, f_i\rangle f_i.$$

Many properties of ordinary frames do not hold for K-frames; for example, the frame operator of a K-frame is not invertible in general. It is worthwhile to mention that, if K has close range, then S_F from $R(K)$ onto $S_F(R(K))$ is an invertible operator [**15**]. In particular,

$$B^{-1}\|f\| \le \|S_F^{-1} f\| \le A^{-1}\|K^\dagger\|^2\|f\|, \quad f \in S_F(R(K)), \tag{1.2}$$

where $K^\dagger$ is the pseudo inverse of K, see [**7**]. For further information on K-frames refer to [**15, 16**].

Every Bessel sequence $\{f_i\}_{i\in I}$ can be considered as a K-frame. Define the operator $K : \mathcal{H} \to \mathcal{H}$ by $Ke_i = f_i$ for all $i \in I$, where $\{e_i\}_{i\in I}$ is an orthonormal basis of $\mathcal{H}$. Then, by [**7,** Lemma 3.3.6], K is a bounded operator and has a unique extension to a bounded operator on $\mathcal{H}$; thus, $\{f_i\}_{i\in I}$ is a K-frame for $\mathcal{H}$, by [**15,** Corollary 3.7].

In addition, every frame sequence $\{f_i\}_{i\in I}$ can be considered as a π_{H_0}-frame, where $H_0 = \overline{\operatorname{span}}_{i\in I}\{f_i\}$. In fact, let $\{f_i\}_{i\in I}$ be a frame sequence with bounds A and B, respectively, and $K = \pi_{\mathcal{H}_0}$. Then, for every $f \in \mathcal{H}$,

$$A\|K^* f\|^2 \le \sum_{i\in I} |\langle K^* f, f_i\rangle|^2 = \sum_{i\in I} |\langle f, f_i\rangle|^2 \le B\|K^*\|^2\|f\|^2.$$

However, the converse does not hold in general. In order to see this, note that the sequence $F = \{e_i + e_{i+1}\}_{i\in I}$ is a complete and Bessel sequence; however, it is not a frame for $\mathcal{H}$, see [**7,** Example 5.1.10]. On the other hand, consider the mapping $K : \mathcal{H} \to \mathcal{H}$ defined by

$K(e_i) = e_i + e_{i+1}$. It is a bounded operator, and therefore, F is a K-frame for $\mathcal{H}$ by [**15,** Corollary 3.7].

Throughout this paper, we suppose that $\mathcal{H}$ is a separable Hilbert space, I a countable index set and $I_{\mathcal{H}}$ is the identity operator on $\mathcal{H}$. For two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ we denote the collection of all bounded linear operators between $\mathcal{H}_1$ and $\mathcal{H}_2$ by $B(\mathcal{H}_1, \mathcal{H}_2)$, and we abbreviate $B(\mathcal{H}, \mathcal{H})$ by $B(\mathcal{H})$. Also, we denote the range of $K \in B(\mathcal{H})$ by $R(K)$, and the orthogonal projection of $\mathcal{H}$ onto a closed subspace $V \subseteq \mathcal{H}$ is denoted by π_V.

The main results of this paper are presented in two sections. In Section 2, we describe the notion of K-duals and investigate an explicit K-dual, the so-called canonical K-dual. As a result, some characterizations of K-duals are presented; moreover, it is proven that a K-dual is the canonical K-dual if and only if their optimal upper bounds are equal. Section 3 is devoted to introducing the notion of approximate K-duals and presenting some methods of constructing K-frames and their duals.

2. K-duals. In this section, we introduce the notion of K-duals for K-frames and characterize such duals. Moreover, by using a K-frame and its K-dual, we can construct a frame for $R(K^*)$ and $R(K)$. In addition, the relation between the optimal bounds of a K-frame and its K-dual is established.

Definition 2.1. Let $\{f_i\}_{i\in I}$ be a K-frame. A Bessel sequence $\{g_i\}_{i\in I} \subseteq \mathcal{H}$ is called a *K-dual* of $\{f_i\}_{i\in I}$ if

$$Kf = \sum_{i\in I} \langle f, g_i\rangle f_i, \quad f \in \mathcal{H}. \tag{2.1}$$

In [**12**], it was shown that, for every K-frame of $\mathcal{H}$, there exists at least a Bessel sequence $\{g_i\}_{i\in I}$ which satisfies (2.1). The sequences $\{f_i\}_{i\in I}$ and $\{g_i\}_{i\in I}$ in (2.1) are not interchangeable in general, see [**15**]. More precisely, from (2.1), it follows that

$$K^*f = \sum_{i\in I} \langle f, f_i\rangle g_i, \quad f \in \mathcal{H}. \tag{2.2}$$

Hence, $\{f_i\}_{i\in I}$ and $\{g_i\}_{i\in I}$ in (2.1) are interchangeable if and only if K is self adjoint.

Lemma 2.2. *Let $\{f_i\}_{i\in I}$ and $\{g_i\}_{i\in I}$ be two Bessel sequences as in* (2.1)*. Then, $\{f_i\}_{i\in I}$ and $\{g_i\}_{i\in I}$ are a K-frame and a K^*-frame, respectively.*

Proof. Using (2.1) for any $f \in \mathcal{H}$, we have

$$\begin{aligned}\|Kf\|^4 = |\langle Kf, Kf\rangle|^2 &= \left|\left\langle \sum_{i\in I}\langle f, g_i\rangle f_i, Kf\right\rangle\right|^2\\ &\leq \sum_{i\in I}|\langle f, g_i\rangle|^2 \sum_{i\in I}|\langle Kf, f_i\rangle|^2\\ &\leq \sum_{i\in I}|\langle f, g_i\rangle|^2 B\|Kf\|^2,\end{aligned}$$

where B is an upper bound of $\{f_i\}_{i\in I}$. This shows that $\{g_i\}_{i\in I}$ is a K^*-frame. In order to obtain a lower bound for $\{f_i\}_{i\in I}$ apply (2.2) and repeat the above argument for K^* instead of K. $\square$

Now, we are ready to introduce an explicit K-dual for every K-frame. This helps to characterize all K-duals of a K-frame.

Proposition 2.3. *Let K be a bounded operator on $\mathcal{H}$ with closed range, and let $F = \{f_i\}_{i\in I}$ be a K-frame with A and B bounds, respectively. Then, $\{K^* S_F^{-1}\pi_{S_F(R(K))} f_i\}_{i\in I}$ is a K-dual of $\pi_{R(K)}F$ with B^{-1} and $A^{-1}\|K\|^2\|K^\dagger\|^2$ bounds, respectively.*

Proof. First, note that S_F is a bounded operator. In addition, by using (1.2), the mapping

$$S_F : R(K) \longrightarrow S_F(R(K))$$

is invertible. It follows that $\{K^* S_F^{-1}\pi_{S(R(K))} f_i\}_{i\in I}$ is a Bessel sequence. Moreover, $S_F = T_F T_F^*$ is self-adjoint on $\mathcal{H}$ and $S_F^{-1} S_F|_{R(K)} = I_{R(K)}$. Hence,

$$\begin{aligned}Kf &= (S_F^{-1} S_F)^* Kf = S_F^* (S_F^{-1})^* Kf\\ &= S_F^* \pi_{S_F(R(K))}(S_F^{-1})^* Kf\end{aligned}$$

$$= \sum_{i\in I} \langle \pi_{S_F(R(K))}(S_F^{-1})^* Kf, f_i\rangle \pi_{R(K)} f_i$$
$$= \sum_{i\in I} \langle f, K^* S_F^{-1} \pi_{S_F(R(K))} f_i\rangle \pi_{R(K)} f_i,$$

for all $f \in \mathcal{H}$. Therefore, $\{K^* S_F^{-1} \pi_{S_F(R(K))} f_i\}_{i\in I}$ is a K-dual of $\pi_{R(K)}F$ with the lower bound B^{-1} obtained by Lemma 2.2. Furthermore, by (1.2), for each $f \in R(K)$, we obtain

$$\|(S_F^{-1})^* f\|^2 = \langle S_F^{-1}(S_F^{-1})^* f, f\rangle \le A^{-1}\|K^\dagger\|^2 \|(S_F^{-1})^* f\|\|f\|.$$

Therefore,

$$\begin{aligned}\sum_{i\in I} |\langle f, K^* S_F^{-1} \pi_{S_F(R(K))} f_i\rangle|^2 &= \sum_{i\in I} |\langle (S_F^{-1})^* Kf, f_i\rangle|^2\\ &= \langle S_F(S_F^{-1})^* Kf, (S_F^{-1})^* Kf\rangle\\ &= \langle Kf, (S_F^{-1})^* Kf\rangle\\ &\le A^{-1}\|K\|^2\|K^\dagger\|^2\|f\|^2,\end{aligned}$$

for all $f \in \mathcal{H}$; thus, the result follows. □

Remark 2.4. We call the K-dual introduced in Proposition 2.3 the *canonical K-dual* of $\pi_{R(K)}F = \{\pi_{R(K)} f_i\}_{i\in I}$ and use $\mathfrak{F}_i := K^* S_F^{-1} \pi_{S_F(R(K))} f_i$, for convenience. Obviously, this coincides with the canonical dual if $\{f_i\}_{i\in I}$ is an ordinary frame.

In the next theorem, we characterize all K-duals of $\pi_{R(K)}F$ by using the canonical K-dual.

Theorem 2.5. *Let K be a bounded operator on $\mathcal{H}$ with closed range, and let $F = \{f_i\}_{i\in I}$ be a K-frame of $\mathcal{H}$. Then, $\{g_i\}_{i\in I}$ is a K-dual of $\pi_{R(K)}F$ if and only if*

$$g_i = \mathfrak{F}_i + \varphi^* \delta_i, \quad i \in I,$$

where $\{\delta_i\}_{i\in I}$ is the standard orthonormal basis of l^2 and $\varphi \in B(\mathcal{H}, l^2)$ such that $\pi_{R(K)} T_F \varphi = 0$.

Proof. Suppose that $\varphi \in B(\mathcal{H}, l^2)$ and $\pi_{R(K)}T_F\varphi = 0$. Then $\{g_i\}_{i\in I} = \{\mathfrak{F}_i + \varphi^*\delta_i\}_{i\in I}$ is a Bessel sequence; in fact,

$$\sum_{i\in I} |\langle f, g_i\rangle|^2 \leq 2(A^{-1}\|K\|^2\|K^\dagger\|^2 + \|\varphi\|^2)\|f\|^2,$$

for all $f \in \mathcal{H}$, where A is a lower bound for $\{f_i\}_{i\in I}$. Moreover,

$$\sum_{i\in I}\langle f, g_i\rangle\pi_{R(K)}f_i = \sum_{i\in I}\langle f, \mathfrak{F}_i + \varphi^*\delta_i\rangle\pi_{R(K)}f_i = Kf + \pi_{R(K)}T_F\varphi f = Kf.$$

Therefore, $\{g_i\}_{i\in I}$ is a K-dual of $\pi_{R(K)}F$. Conversely, let $\{g_i\}_{i\in I}$ be a K-dual of $\pi_{R(K)}F$. Define

$$\varphi = T_G^* - T_F^*\pi_{S_F(R(K))}(S_F^{-1})^*K.$$

Then, $\varphi \in B(\mathcal{H}, l^2)$ and

$$\begin{aligned}\pi_{R(K)}T_F\varphi f &= \pi_{R(K)}T_FT_G^*f - S_F^*(S_F^{-1})^*Kf\\ &= \sum_{i\in I}\langle f, g_i\rangle\pi_{R(K)}f_i - (S_F^{-1}S_F)^*Kf = 0,\end{aligned}$$

for every $f \in \mathcal{H}$. Moreover,

$$\begin{aligned}\mathfrak{F}_i + \varphi^*\delta_i &= K^*S_F^{-1}\pi_{S_F(R(K))}f_i + \varphi^*\delta_i\\ &= K^*S_F^{-1}\pi_{S_F(R(K))}f_i + T_G\delta_i - K^*S_F^{-1}\pi_{S_F(R(K))}f_i = g_i,\end{aligned}$$

for all $i \in I$. This completes the proof. □

In the next theorem, we characterize all K-duals of a K-frame when $K \in B(\mathcal{H})$ is not necessarily a closed range operator.

Theorem 2.6. *Let $\{f_i\}_{i\in I}$ be a K-frame. Then, $\{g_i\}_{i\in I}$ is a K-dual of $\{f_i\}_{i\in I}$ if and only if $\{g_i\}_{i\in I} = \{V\delta_i\}_{i\in I}$ where $\{\delta_i\}_{i\in I}$ is the standard orthonormal basis of l^2 and $V : l^2 \to \mathcal{H}$ is a bounded operator such that $T_FV^* = K$. In this case, $\{g_i\}_{i\in I}$ is in fact a Parseval V-frame.*

Proof. Suppose that $\{g_i\}_{i\in I}$ is a K-dual of $\{f_i\}_{i\in I}$ with the synthesis operator T_G. Take $V = T_G$. Then $T_FV^* = K$ by (2.1). Conversely, if $V \in B(l^2, \mathcal{H})$ such that $T_FV^* = K$, take $g_i = V\delta_i$. Then $\{g_i\}_{i\in I}$ is a

Bessel sequence and, for each $f \in \mathcal{H}$, we have

$$\sum_{i\in I}\langle f, g_i\rangle f_i = \sum_{i\in I}\langle V^* f, \delta_i\rangle f_i = T_F V^* f = Kf.$$

This implies that $\{g_i\}_{i\in I}$ is a K-dual of $\{f_i\}_{i\in I}$, and thus, $\{g_i\}_{i\in I}$ is a K^*-frame. Moreover,

$$\sum_{i\in I}|\langle f, g_i\rangle|^2 = \sum_{i\in I}|\langle V^* f, \delta_i\rangle|^2 = \|V^* f\|^2, \quad f \in \mathcal{H}.$$

Thus, $\{g_i\}_{i\in I}$ is a Parseval V-frame. $\square$

In **[15]**, it was shown that, if $\{f_i\}_{i\in I}$ and $\{g_i\}_{i\in I}$ are as in (2.1) and K is a closed range operator on $\mathcal{H}$, then there exists a sequence $\{h_i\}_{i\in I} = \{(K^\dagger|_{R(K)})^* g_i\}_{i\in I}$ such that

$$f = \sum_{i\in I}\langle f, h_i\rangle f_i, \quad f \in R(K). \tag{2.3}$$

Moreover, $\{h_i\}_{i\in I}$ and $\{f_i\}_{i\in I}$ are interchangeable for every $f \in R(K)$. In the following proposition, we give a characterization of sequences $\{h_i\}_{i\in I}$ in (2.3). Its proof is similar to that of Theorem 2.6.

Proposition 2.7. *Let $K \in B(\mathcal{H})$ be a closed range operator and $\{f_i\}_{i\in I}$ a K-frame for $\mathcal{H}$. Then $\{h_i\}_{i\in I}$ satisfies* (2.3) *if and only if $\{h_i\}_{i\in I} = \{V\delta_i\}_{i\in I}$, where $\{\delta_i\}_{i\in I}$ is the standard orthonormal basis of l^2 and $V : l^2 \to R(K)$ is a left inverse of $T_F^*|_{R(K)}$. Moreover, $\{h_i\}_{i\in I}$ is a Parseval V-frame.*

Now, we will obtain the relation between optimal frame bounds of a K-frame and its K-duals.

Theorem 2.8. *Let K be a closed range operator, and let $F = \{f_i\}_{i\in I}$ be a K-frame of $\mathcal{H}$ with the optimal bounds A and B, respectively. Also, let $G = \{g_i\}_{i\in I}$ be a K-dual of F with the optimal bounds C and D, respectively. Then,*

(i) *$C \geq 1/B$ and $D \geq 1/A$.*
(ii) *G is the canonical K-dual of $F \subset R(K)$ if and only if $S_G = S_{\mathfrak{F}}$, where $\mathfrak{F} = \{\mathfrak{F}_i\}_{i\in I}$ is the canonical K-dual.*

Proof. Applying Lemma 2.2 yields

$$\sum_{i\in I}|\langle f, g_i\rangle|^2 \geq \frac{1}{B}\|Kf\|^2,$$

for each $f \in \mathcal{H}$; therefore, $C \geq 1/B$. Similarly,

$$\sum_{i\in I}|\langle f, f_i\rangle|^2 \geq \frac{1}{D}\|K^* f\|^2.$$

Thus, $D \geq 1/A$. In order to show (ii), first note that $K^* S_F^{-1}\pi_{S_F(R(K))}K$ is the frame operator of the canonical K-dual. Indeed,

$$\begin{aligned}
S_{\mathfrak{F}} f &= \sum_{i\in I}\langle f, \mathfrak{F}_i\rangle \mathfrak{F}_i \\
&= K^* S_F^{-1}\pi_{S_F(R(K))}\sum_{i\in I}\langle (S_F^{-1})^* Kf, f_i\rangle f_i \\
&= K^* S_F^{-1}\pi_{S_F(R(K))} S_F (S_F^{-1})^* Kf \\
&= K^* S_F^{-1}\pi_{S_F(R(K))} Kf.
\end{aligned}$$

Now, by using Theorem 2.5,

$$g_i = \mathfrak{F}_i + \varphi^*\delta_i, \quad i \in I,$$

for some $\varphi \in B(\mathcal{H}, l^2)$ with $T_F\varphi = 0$. Hence, for any $f, g \in \mathcal{H}$, we have

$$\begin{aligned}
\langle S_G f, f\rangle &= \left\langle \sum_{i\in I}\langle f, \mathfrak{F}_i + \varphi^*\delta_i\rangle(\mathfrak{F}_i + \varphi^*\delta_i), f\right\rangle \\
&= \left\langle \sum_{i\in I}\langle f, \mathfrak{F}_i\rangle \mathfrak{F}_i, f\right\rangle + \langle \varphi^* T_F^* \pi_{S_F(R(K))} S_F^{-1} Kf, f\rangle \\
&\quad + \left\langle \sum_{i\in I}\langle \varphi f, \delta_i\rangle \varphi^*\delta_i, f\right\rangle + \langle K^* S_F^{-1}\pi_{S_F(R(K))} T_F \varphi, f\rangle \\
&= \left\langle \sum_{i\in I}\langle f, \mathfrak{F}_i\rangle \mathfrak{F}_i, f\right\rangle + \left\langle \sum_{i\in I}\langle \varphi f, \delta_i\rangle \varphi^*\delta_i, f\right\rangle \\
&= \langle S_{\mathfrak{F}} f, f\rangle + \|\varphi f\|^2.
\end{aligned}$$

On the other hand, $K^* S_F^{-1}\pi_{S_F(R(K))}K$, the frame operator of K^*-frame

$\{\mathfrak{F}_i\}_{i\in I}$, is a positive operator. Hence,

$$D = \sup_{\|f\|\leq 1} \langle S_G f, f\rangle \leq \sup_{\|f\|\leq 1} \langle S_{\mathfrak{F}} f, f\rangle + \|\varphi\|^2 = \|S_{\mathfrak{F}}\| + \|\varphi\|^2.$$

Therefore, $S_G = S_{\mathfrak{F}}$ if and only if G is the canonical K-dual. $\square$

A K-frame $F = \{f_i\}_{i\in I}$ of $\mathcal{H}$ is called a *K-exact frame* if, for every $j \in I$, the sequence $\{f_i\}_{i\neq j}$ is not a K-frame for $\mathcal{H}$. Also, we call F a *K-minimal frame* whenever, for each $\{c_i\}_{i\in I} \in l^2$ such that $\sum_{i\in I} c_i f_i = 0$, then $c_i = 0$ for all $i \in I$. By Lemma 2.2, every K-dual of a K-minimal frame is a K^*-frame. However, the next example shows that it is not a K^*-minimal frame.

Example 2.9. Let $\mathcal{H} = \mathbb{C}^4$ and $\{e_i\}_{i=1}^4$ be the standard orthonormal basis of $\mathcal{H}$. Define $K : \mathcal{H} \to \mathcal{H}$ by

$$K \sum_{i=1}^{4} c_i e_i = c_1 e_1 + c_1 e_2 + c_2 e_3.$$

Then, $K \in B(\mathcal{H})$, and the sequence $F = \{e_1, e_2, e_3\}$ is a K-minimal frame with bounds $A = 1/8$ and $B = 1$, respectively. On the other hand, if $\{g_i\}_{i=1}^3$ is a K-dual of F, then

$$\langle f, e_1\rangle e_1 + \langle f, e_1\rangle e_2 + \langle f, e_2\rangle e_3 = \sum_{i=1}^{3} \langle f, g_i\rangle f_i, \quad f \in \mathcal{H}.$$

Hence, $g_1 = g_2 = e_1$ and $g_3 = e_2$. This shows that $\{g_i\}_{i=1}^3$ is not a K^*-minimal frame.

Lemma 2.10. *Every K-exact frame is a K-minimal frame. The converse does not hold in general.*

Proof. Assume that $F = \{f_i\}_{i\in I}$ is not a K-minimal frame. Without loss of generality, we let $f_i \neq 0$ for each $i \in I$. Then, there exists a $\{c_i\}_{i\in I} \in l^2$ with $c_m \neq 0$ such that $f_m = (-1/c_m) \sum_{i\neq m} c_i f_i$ for some $m \in I$. Now, by an argument similar to the proof of [**7,** Theorem 5.2.3], we can see that $\{f_i\}_{i\neq m}$ is a K-frame. This shows that F is not a K-exact frame.

For the converse, consider $\mathcal{H} = \mathbb{C}^3$, and assume that $F = \{e_i\}_{i=1}^3$ is the orthonormal basis of $\mathcal{H}$. Define $K : \mathcal{H} \to \mathcal{H}$ by

$$K \sum_{i \in I} c_i e_i = c_1 e_1 + c_2 e_1 + c_3 e_2.$$

Then, $K \in B(\mathcal{H})$, and F is a K-minimal frame. We can easily see that $\{e_1, e_2\}$ is also a K-frame with bounds $A = 1/8$ and $B = 1$, respectively. $\square$

Theorem 2.11. *A K-frame $F = \{f_i\}_{i \in I}$ has a unique K-dual if and only if it is a K-minimal frame.*

Proof. First, note that, by the definition of the K-minimal frame, the sufficiency is clear; we only prove the necessity. Assume that $\{g_i\}_{i \in I}$ is a unique K-dual of F and F is not a K-minimal frame. Let $g_i \neq 0$ for each $i \in I$. Then, F is not a K-exact frame by Lemma 2.10. Hence, there exists an $m \in I$ such that $\{f_i\}_{i \neq m}$ is a K-frame, and thus, it has a K-dual as $\{h_i\}_{i \neq m}$. Take $h_m = 0$, so $\{h_i\}_{i \in I} \neq \{g_i\}_{i \in I}$ and

$$Kf = \sum_{i \in I} \langle f, h_i \rangle f_i = \sum_{i \in I} \langle f, g_i \rangle f_i, \quad f \in \mathcal{H}.$$

Thus, $\{h_i\}_{i \in I}$ is also a K-dual of F, which is a contradiction. Now, suppose that $g_i = 0$ except $i \in J$ for a set $J \subseteq I$. Then, $\{f_i\}_{i \in J}$ is a K-minimal frame. More precisely, since $g_i = 0$ for each $i \in I \setminus J$, we obtain $Kf = \sum_{i \in J} \langle f, g_i \rangle f_i$ for each $f \in \mathcal{H}$. Hence, by Lemma 2.2, $\{f_i\}_{i \in J}$ is a K-frame. We claim that $\{g_i\}_{i \in J}$ is a unique K-dual of $\{f_i\}_{i \in J}$. Suppose that $\{h_i\}_{i \in J}$ is a K-dual of $\{f_i\}_{i \in J}$ such that $h_j \neq g_j$ for some $j \in J$. Taking $h_i = 0$ for all $i \in I \setminus J$, we can easily see that $Kf = \sum_{i \in I} \langle f, h_i \rangle f_i$. Therefore, by Lemma 2.2, $\{h_i\}_{i \in J}$ is a K-dual of F distinguishable from $\{g_i\}_{i \in I}$, which is a contradiction. Therefore, $\{f_i\}_{i \in J}$ has a unique K-dual $\{g_i\}_{i \in J}$ such that $g_i \neq 0$ for all $i \in J$. Thus, $\{f_i\}_{i \in J}$ is a K-minimal frame by the first part of the proof. Also, by assumption, $\{f_i\}_{i \in I}$ is not a K-minimal frame. Thus, there exists an $m \in I \setminus J$ such that $f_m = (-1/c_m) \sum_{i \neq m} c_i f_i$. Choose $h_m \neq 0$, and take $h_i = g_i + (\overline{c_i}/\overline{c_m}) h_m$ for $i \neq m$. Therefore,

$$\sum_{i \in I} \langle f, h_i \rangle f_i = \langle f, h_m \rangle f_m + \sum_{i \neq m} \langle f, h_i \rangle f_i$$

$$= \sum_{i\neq m} \left\langle f, -\frac{\overline{c_i}}{\overline{c_m}} h_m \right\rangle f_i + \sum_{i\neq m} \langle f, h_i\rangle f_i$$
$$= \sum_{i\neq m} \langle f, g_i\rangle f_i = Kf,$$

for every $f \in \mathcal{H}$. Hence, $\{h_i\}_{i\in I}$ is also a K-dual of $\{f_i\}_{i\in I}$ distinguishable from $\{g_i\}_{i\in I}$, which is a contradiction. □

For each K-frame $\{f_i\}_{i\in I}$ of $\mathcal{H}$, we can construct a frame for $R(K^*)$. More precisely, let $K \in B(\mathcal{H})$ be a bounded operator with closed range, and let $\{f_i\}_{i\in I}$ be a K-frame of $\mathcal{H}$ with bounds A and B, respectively. Then $\{K^\dagger \pi_{R(K)} f_i\}_{i\in I}$ is a frame for $R(K^*)$. In fact, for every $f \in R(K^*)$, we have

$$A\|f\|^2 = A\|K^*(K^*)^\dagger f\|^2 \le \sum_{i\in I} |\langle \pi_{R(K)}(K^*)^\dagger f, f_i\rangle|^2$$
$$= \sum_{i\in I} |\langle f, K^\dagger \pi_{R(K)} f_i\rangle|^2 \le B\|(K^*)^\dagger\|^2\|f\|^2.$$

Similarly, for each K^*-frame $\{f_i\}_{i\in I}$ of $\mathcal{H}$, the sequence

$$\{(K^*)^\dagger \pi_{R(K^*)} f_i\}_{i\in I}$$

is a frame for $R(K)$. In particular, if $\{f_i\}_{i\in I}$ is a K-frame with a K-dual $\{g_i\}_{i\in I}$, then the sequence $\{(K^*)^\dagger \pi_{R(K^*)} g_i\}_{i\in I}$ is a frame for $R(K)$. We summarize the above discussion in the next corollary.

Corollary 2.12. *Let K be a bounded operator on $\mathcal{H}$ with closed range, and let $\{f_i\}_{i\in I}$ be a K-frame of $\mathcal{H}$. Then, $\{(K^*)^\dagger K^* S_F^{-1} \pi_{S_F(R(K))} f_i\}_{i\in I}$ is a frame for $R(K)$.*

3. Construction of K-frames. In this section, we present some approaches for construction of K-frames and their K-duals since more reconstructions for the elements of $R(K)$ were obtained. In this respect, we first introduce the notion of approximate K-dual, adopted from **[9]** for ordinary frames. Approximate K-duals are easier to construct and lead to perfect reconstructions for the elements of $R(K)$.

Definition 3.1. The Bessel sequence $\{g_i\}_{i\in I}$ is called an *approximate K-dual* of a K-frame $\{f_i\}_{i\in I}$ whenever

$$\|Kf - \sum_{i\in I}\langle f, g_i\rangle f_i\| < 1, \quad f \in \mathcal{H}.$$

Theorem 3.2. *Let K be a bounded operator on $\mathcal{H}$ with a closed range. In addition, let $G = \{g_i\}_{i\in I}$ be an approximate K-dual of $F = \{f_i\}_{i\in I}$. Then,*

(i) $\{U^{-1}\pi_{R(K)}f_i\}_{i\in I}$ *is a K-frame for $\mathcal{H}$ with the K-dual*

$$\{K^*(K^\dagger|_{R(K)})^* g_i\}_{i\in I},$$

in which $U = \pi_{R(K)}T_F T_G^* K^\dagger|_{R(K)}$.

(ii) $\{(K^\dagger|_{R(K)})^* g_i\}_{i\in I}$ *is a K-frame for $\mathcal{H}$ with the K-dual*

$$\{K^* U^{-1}\pi_{R(K)} f_i\}_{i\in I}.$$

Proof. By taking $h_i = (K^\dagger|_{R(K)})^* g_i$, for each $f \in R(K)$, we have

$$\left\| f - \sum_{i\in I}\langle f, h_i\rangle f_i \right\| = \left\| KK^\dagger f - \sum_{i\in I}\langle K^\dagger f, g_i\rangle f_i \right\| < 1.$$

Therefore, $\|I|_{R(K)} - T_F T_H^*|_{R(K)}\| < 1$, where $H = \{h_i\}_{i\in I}$. Thus,

$$\|I|_{R(K)} - \pi_{R(K)}T_F T_H^*|_{R(K)}\| < 1.$$

This implies that $U = \pi_{R(K)}T_F T_H^*|_{R(K)}$ is an invertible operator on $R(K)$, and thus, for each $f \in \mathcal{H}$, we have

$$\begin{aligned} Kf = U^{-1}UKf &= \sum_{i\in I}\langle Kf, h_i\rangle U^{-1}\pi_{R(K)} f_i \\ &= \sum_{i\in I}\langle f, K^*(K^\dagger|_{R(K)})^* g_i\rangle U^{-1}\pi_{R(K)} f_i. \end{aligned}$$

Hence, Lemma 2.2 follows (i). In order to show (ii), note that, for every $f \in R(K)$, we have

$$f = U^*(U^*)^{-1} f = \sum_{i\in I}\langle f, U^{-1}\pi_{R(K)} f_i\rangle h_i;$$

thus,

$$Kf = \sum_{i\in I} \langle f, K^* U^{-1} \pi_{R(K)} f_i \rangle (K^\dagger|_{R(K)})^* g_i,$$

for each $f \in \mathcal{H}$. This completes the proof. $\square$

Example 3.3. Let $\mathcal{H}$, F and K be as in Example 2.9. Consider $G = \{g_i\}_{i\in I}$ such that

$$g_1 = \frac{e_1}{2}, \qquad g_2 = \frac{e_1}{2}, \qquad g_3 = \frac{e_2}{3}.$$

Clearly, G is an approximate K-dual of F, and $R(K) = \overline{\text{span}}\{e_1 + e_2, e_3\}$. Hence, for each $f \in \mathcal{H}$ and $g \in R(K)$, we have

$$\begin{aligned}\langle (K^\dagger|_{R(K)})^* f, g\rangle &= \langle f, K^\dagger|_{R(K)} g\rangle \\ &= \langle f, \langle g, e_1+e_2\rangle K^\dagger|_{R(K)}(e_1+e_2) + \langle g, e_3\rangle K^\dagger|_{R(K)} e_3\rangle \\ &= \langle f, \langle g, e_1+e_2\rangle e_1 + \langle g, e_3\rangle e_2\rangle \\ &= \langle \langle f, e_1\rangle (e_1+e_2) + \langle f, e_2\rangle e_3, g\rangle.\end{aligned}$$

Moreover,

$$(K^\dagger|_{R(K)})^* f = \langle f, e_1\rangle (e_1 + e_2) + \langle f, e_2\rangle e_3.$$

Therefore,

$$(K^\dagger|_{R(K)})^* G = \left\{ \frac{e_1 + e_2}{2}, \frac{e_1 + e_2}{2}, \frac{e_3}{3} \right\}$$

is a K-frame for $\mathcal{H}$ with the K-dual $\{e_1, e_1, 2e_2\}$ by Theorem 3.2 (ii).

In the next theorem, we will construct K-frames from a given K-frame and characterize their K-duals.

Theorem 3.4. *Let $F = \{f_i\}_{i\in I}$ be a K-frame for $\mathcal{H}$ with bounds A and B, respectively. Also, let $\{\delta_i\}_{i\in I}$ be the standard orthonormal basis of l^2 and $\psi \in B(l^2)$ such that there exists an $\varepsilon > 0$ such that*

$$\|\psi a\|^2 \geq \varepsilon \|a\|^2, \quad a = \{a_i\}_{i\in I} \in R(T_F^*). \tag{3.1}$$

Then, the following assertions hold:

(i) $\Psi = \{T_F \psi^* \delta_j\}_{j\in I}$ *is also a K-frame for $\mathcal{H}$.*

(ii) *There exists a correspondence between K-duals of F and K-duals of Ψ.*

(iii) *The correspondence in* (ii) *is one-to-one if $\psi : l^2 \to l^2$ is an invertible operator.*

Proof. Since F is a K-frame and ψ is a bounded linear operator, we conclude that Ψ is well defined. In fact, it is a Bessel sequence and

$$\|T_\Psi\|^2 \le B\|\psi\|^2.$$

Also,

$$\begin{aligned} T_\Psi^* f &= \{\langle f, T_F \psi^* \delta_j\rangle\}_{j\in I} \\ &= \left\{ \sum_{i\in I} \langle f, f_i\rangle \langle \psi\delta_i, \delta_j\rangle \right\}_{j\in I} \\ &= \left\{ \left\langle \psi \sum_{i\in I} \langle f, f_i\rangle \delta_i, \delta_j \right\rangle \right\}_{j\in I} \\ &= \psi\{\langle f, f_i\rangle\}_{i\in I} = \psi T_F^* f, \end{aligned}$$

for every $f \in \mathcal{H}$. Moreover,

$$A\varepsilon \|K^* f\|^2 \le \varepsilon \|T_F^* f\|^2 \le \|\psi T_F^* f\|^2 = \|T_\Psi^* f\|^2 \le B\|\psi\|^2 \|f\|^2.$$

Therefore, Ψ is a K-frame for $\mathcal{H}$, which follows from (i). Suppose that $G = \{g_i\}_{i\in I}$ is a K-dual of F. Then (3.1) implies that $\psi|_{R(T_F^*)}$ has a left inverse such as $\theta \in B(l^2, R(T_F^*))$. Consider $\Theta = \{T_G \theta \delta_i\}_{i\in I}$. Then Θ is a K-dual of Ψ. Indeed,

$$T_\Psi T_\Theta^* = T_\Psi \theta^* T_G^* = T_F \psi^* \theta^* T_G^* = (\theta\psi T_F^*)^* T_G^* = T_F T_G^* = K.$$

On the other hand, let $\Gamma = \{\Gamma_i\}_{i\in I}$ be a K-dual of Ψ. Then $\{T_\Gamma \psi \delta_i\}_{i\in I}$ is a K-dual of F. More precisely,

$$\begin{aligned} \sum_{i\in I} \langle f, T_\Gamma \psi\delta_i\rangle f_i &= \sum_{i\in I} \left\langle f, \sum_{j\in I} \langle \psi\delta_i, \delta_j\rangle \Gamma_j \right\rangle f_i \\ &= \sum_{j\in I} \langle f, \Gamma_j\rangle T_F \{\langle \delta_j, \psi\delta_i\rangle\}_{i\in I} \\ &= \sum_{j\in I} \langle f, \Gamma_j\rangle T_F \psi^* \delta_j \\ &= T_\Psi T_\Gamma^* f = Kf, \end{aligned}$$

for each $f \in \mathcal{H}$. Thus, (ii) has been obtained. In order to show (iii), let Λ and Δ be the set of all K-duals F and Ψ, respectively. Also, let θ be the inverse of ψ. Define $\rho : \Lambda \to \Delta$ by

$$\rho\{g_i\}_{i\in I} = \{T_G\theta\delta_i\}_{i\in I}.$$

From (ii), we conclude that ρ is well defined. Moreover, if $G = \{g_i\}_{i\in I}$ and $H = \{h_i\}_{i\in I}$ are K-duals of F such that $\rho G = \rho H$, then

$$g_i = T_G\delta_i = T_G\theta\psi\delta_i = T_H\theta\psi\delta_i = T_H\delta_i = h_i.$$

Therefore, ρ is one-to-one. Also, if $\Gamma = \{\Gamma_i\}_{i\in I}$ is a K-dual of Ψ, then $H = \{T_\Gamma\psi\delta_i\}_{i\in I}$ is a K-dual of F by (ii). Furthermore,

$$T_H\theta\delta_i = T_\Gamma\psi\theta\delta_i = T_\Gamma\delta_i = \Gamma_i$$

It follows that $\rho H = \Gamma$. Therefore, ρ is onto. This completes the proof. $\square$

In the case of $K = I_{\mathcal{H}}$, Theorem 3.4 (i) reduces to a result due to Aldroubi, see [**1**]. Any pair of Bessel sequences in $\mathcal{H}$ can be extended to a pair of dual frames for $\mathcal{H}$, see [**8,** Proposition 2.1]. Now, let $K \in B(\mathcal{H})$. For every two Bessel sequences $\{f_i\}_{i\in I}$ and $\{g_i\}_{i\in I}$ we provide a new reconstruction of the elements $R(K)$. Take $U = K - T_F T_G^*$. Then $U \in B(\mathcal{H})$, and thus, by combining Theorem 2.5, and [**12,** Theorem 3], there exists a U-frame $\{\varphi_i\}_{i\in I}$ for $\mathcal{H}$. If $\{\psi_i\}_{i\in I}$ is a U-dual of $\{\varphi_i\}_{i\in I}$, then it is clear that the sequences $\{f_i\}_{i\in I} \cup \{\varphi_i\}_{i\in I}$ and $\{g_i\}_{i\in I} \cup \{\psi_i\}_{i\in I}$ are Bessel. Moreover,

$$Kf = Uf + T_F T_G^* f = \sum_{i\in I}\langle f, \psi_i\rangle\varphi_i + \sum_{i\in I}\langle f, g_i\rangle f_i, \quad f \in \mathcal{H}.$$

The result follows from Lemma 2.2.

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