ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 47, Number 5, 2017

APPLICATIONS OF VARIATIONAL METHODS TO AN ANTI-PERIODIC BOUNDARY VALUE PROBLEM OF A SECOND-ORDER DIFFERENTIAL SYSTEM

YU TIAN AND YAJING ZHANG

ABSTRACT. In this paper, we discuss the existence of multiple solutions to a second order anti-periodic boundary value problem

$$\begin{cases} \ddot{x}(t) + Mx(t) + \nabla F(t, x(t)) = 0 & \text{almost every } t \in [0, T], \\ x(0) = -x(T) & \dot{x}(0) = -\dot{x}(T) \end{cases}$$

by using variational methods and critical point theory. Furthermore, we obtain the existence of periodic solutions for corresponding second-order differential systems.

1. Introduction. The study of anti-periodic solutions for nonlinear evolution equations was initiated by Okochi **[26]**. Okochi studied the nonlinear parabolic equation in a real Hilbert space H, which is of the form

$$\frac{du(t)}{dt} + \partial\varphi(u(t)) \ni f(t),$$

where $f \in L^2_{\text{loc}}(R; H)$, φ is a proper lower semi-continuous (lsc) convex functional on H and $\partial\varphi$ is the subdifferential of φ. By using fixed point theory, the existence of anti-periodic solutions was obtained in the case where $\partial\varphi$ is odd and f is T-anti-periodic. Inspired by **[26]**, anti-periodic problems for second- and higher-order differential equations have been extensively studied, see **[2, 3, 5, 6, 8, 9, 10, 11, 16, 17, 18, 19, 20, 22, 23, 25, 31]** and the references therein. It is very important to study anti-periodic boundary value problems since they can be applied

2010 AMS *Mathematics subject classification.* Primary 34B15, 58E30.

Keywords and phrases. Anti-periodic boundary value problem, variational methods, Mountain pass theorem.

Supported by the National Science Foundation for Young Scholars, grant No. 11001028, the National Natural Science Foundation, grant Nos. 61377067 and 11375033 and the Beijing University of Posts and Telecommunications teaching reform project, grant No. 500517183. This research was carried out while the first author was a visiting scholar at Baylor University.

Received by the editors on April 13, 2012, and in revised form on January 6, 2016.

DOI:10.1216/RMJ-2017-47-5-1721

to interpolation problems [**13, 15**], anti-periodic wavelets [**7**], the Hill differential operator [**14**] and physics [**1, 4, 21, 27**].

Many results have been obtained by using tools such as topological degree theory, lower and upper solution methods [**2, 3, 5, 6, 10, 16, 17, 18, 23, 31, 32**], the maximal monotone or m-accretive operator [**2, 6**].

In [**6**], Aizicovici and Pavel established the existence, uniqueness and continuous dependence upon data of anti-periodic solutions to some first- and second-order evolution equations. In [**31**], Wang and Li studied the existence of solutions of the following antiperiodic boundary value problem for a second-order conservative system:

$$\begin{cases} q'' = u(t,q) & t \in [0,T] \\ q(0) = -q(T) & q'(0) = -q'(T). \end{cases} \tag{1.1}$$

By using fixed point theory together with Green's function, the existence result is as follows: assume that there exist constants $0 \le c < 8$ and $M > 0$, such that

$$|u(t,q)| \le \frac{c}{T^2}|q| + M$$

for all $t \in [0,T], q \in R^1$. Problem (1.2) has at least one solution.

In [**32**], Wang and Shen studied the antiperiodic boundary value problem as follows

$$\begin{cases} x'' + f(t,x(t)) = 0 & t \in [0,T] \\ x(0) = -x(T) & x'(0) = -x'(T). \end{cases} \tag{1.2}$$

By using Schauder's fixed point theorem and the lower and upper solutions method, some sufficient conditions for the existence of solutions are obtained: assume that there exist constants $0 < r < 2$, $l > 0$, and functions $p, q, h \in C[0,T]$ such that

$$uf(t,u) \le p(t)u^2 + q(t)|u|^r + h(t)$$

for $t \in [0,T], |u| > 1$. Further, suppose that

$$\int_0^T p^+(s)\,ds < 4,$$

where $p^+(t) = \max\{p(t), 0\}$. Then (1.2) has at least one solution. To the best of our knowledge, few authors have studied the existence of solutions for anti-periodic boundary value problems by using variational methods and critical point theory. As a result, the motivation of this paper is to fill the gaps in this area. We study the existence of multiple solutions to anti-periodic boundary value problems for the second-order differential system

$$\begin{cases} \ddot{x}(t) + Mx + \nabla F(t,x) = 0 & \text{almost every } t \in [0,T] \\ x(0) = -x(T) & \dot{x}(0) = -\dot{x}(T), \end{cases} \tag{1.3}$$

$x = (x_1, x_2, \ldots, x_n)^T \in R^n, M$ is an $n \times n$ real symmetric matrix,

$$F : [0,T] \times R^n \longrightarrow R, (t,x) \longmapsto F(t,x)$$

is measurable in t for each $x \in R^n$, and continuously differentiable in x for almost every $t \in [0,T]$,

$$\nabla F(t,x) = \left(\frac{\partial F(t,x)}{x_1}, \frac{\partial F(t,x)}{x_2}, \ldots, \frac{\partial F(t,x)}{x_n} \right)^T.$$

In particular, our aim of this paper is to apply critical point theory to problem (1.3) and prove the existence of at least two solutions when the eigenvalues of M are less than π^2/T^2. In addition, we obtain the existence of $2T$-periodic solutions for $\ddot{x} + Mx + \nabla F(t,x) = 0$. The constraint conditions on F are new.

The main difficulties in the above problem are as follows:

(1) the construction of a suitable Banach space X;
(2) the construction of a functional φ on the space X;
(3) how to prove the critical point of the functional φ is only the solution of BVP (1.3).

In order to overcome these difficulties, especially (3), we prove an important fundamental lemma, which plays an important role in the proof of Lemma 2.4. The idea of this paper comes from **[24, 28, 29]**.

The following lemmas will be needed in our argument, which may be found in **[12, 24, 29, 33]**.

Lemma 1.1. (**[33,** Theorem 38A]). *For the functional*

$$F : S \subseteq X \longrightarrow [-\infty, +\infty] \quad \textit{with } S \neq \emptyset,$$

$\min_{u\in S} F(u) = \alpha$ *has a solution in the following cases*:

(i) X *is a real reflexive Banach space*;
(ii) S *is bounded and weak sequentially closed*;
(iii) F *is weak sequentially lower semi-continuous on* S, *i.e., by definition, for each sequence* (u_n) *in* S *such that* $u_n \rightharpoonup u$ *as* $n \to \infty$, *we have that* $F(u) \le \underline{\lim}_{n\to\infty} F(u_n)$ *holds.*

Lemma 1.2 (**[12, 24, 29]**). *Let* E *be a Banach space and* $\varphi \in C^1(E,R)$ *satisfy the Palais-Smale condition, i.e., every sequence* $\{x_n\}$ *in* E *satisfying* $\varphi(x_n)$ *is bounded, and* $\varphi'(x_n) \to 0$ *has a convergent subsequence. Assume that there exist* $x_0, x_1 \in E$ *and a bounded open neighborhood* Ω *of* x_0 *such that* $x_1 \in E \setminus \overline{\Omega}$ *and*

$$\max\{\varphi(x_0), \varphi(x_1)\} < \inf_{x\in\partial\Omega} \varphi(x).$$

Let

$$\Gamma = \{h \mid h : [0,1] \longrightarrow E \text{ is continuous and } h(0) = x_0,\ h(1) = x_1\}$$

and

$$c = \inf_{h\in\Gamma} \max_{s\in[0,1]} \varphi(h(s)).$$

Then, c *is a critical value of* φ, *that is, there exists an* $x^* \in E$ *such that* $\varphi'(x^*) = \Theta$ *and* $\varphi(x^*) = c$, *where* $c > \max\{\varphi(x_0), \varphi(x_1)\}$.

The remainder of the paper is organized as follows. In Section 2, some preliminary results will be given. In Section 3, we will state and prove the main results of the paper, as well as present some applications to (1.3).

2. Related lemmas. In order to begin, we introduce some notation. Here, and in the sequel, we assume that $T > 0$ is the limit. For $x \in R^n$, $x = 0$ means

$$x = (x_1, x_2, \ldots, x_n)^T = (0, 0, \ldots, 0)^T$$

and $x > 0$ means $x_i > 0$ for all $i = 1, 2, \ldots, n$. For $x = (x_1, x_2, \ldots, x_n)^T$, $y = (y_1, y_2, \ldots, y_n)^T \in R^n$,

$$x \cdot y = \sum_{i=1}^{n} x_i y_i.$$

For $x = (x_1, x_2, \ldots, x_n)^T \in R^n$,

$$|x| = \left(\sum_{i=1}^{n} x_i^2 \right)^{1/2}.$$

We define the space $X = \{x \in H^1([0,T], R^n) : x(0) = -x(T)\}$ equipped with the norm

$$\|x\|_X = \left(\int_0^T |\dot{x}(t)|^2 + |x(t)|^2 dt \right)^{1/2}.$$

We claim that $(X, \|\cdot\|_X)$ is a reflexive Banach space. In fact, $(H^1([0,T], R^n), \|\cdot\|_X)$ is a reflexive Banach space.

Now, we shall show that $(X, \|\cdot\|_X)$ is a closed subspace of $(H^1([0,T], R^n), \|\cdot\|_X)$. For any $(x_n) \in X, x_n \to x^*$ in X, i.e., $\|x_n - x^*\|_X \to 0$ as $n \to \infty$, which yields

$$x_n \longrightarrow x^* \quad \text{in } C([0,T], R^n) \quad \text{and} \quad x^* \in H^1([0,T], R^n).$$

Since $x_n(0) = -x_n(T)$, we have $x^*(0) = -x^*(T)$. Thus, $(X, \|\cdot\|_X)$ is a closed subspace of $(H^1([0,T], R^n), \|\cdot\|_X)$. By fundamental analysis, $(X, \|\cdot\|_X)$ is a reflexive Banach space.

For each $x \in X$, put

$$\varphi(x) := \int_0^T \frac{1}{2}|\dot{x}(t)|^2 - \frac{1}{2}Mx(t) \cdot x(t) - F(t, x(t))\, dt. \tag{2.1}$$

Clearly, φ is a Gâteaux differentiable functional whose Gâteaux derivative at the point $x \in X$ is the functional $\varphi'(x) \in X^*$, given by

$$\langle \varphi'(x), v \rangle = \int_0^T \dot{x}(t) \cdot \dot{v}(t) - Mx(t) \cdot v(t) - \nabla F(t, x(t)) \cdot v(t)\, dt, \tag{2.2}$$

for every $v \in X$. Obviously, $\varphi' : X \to X^*$ is continuous.

Now, we shall cite an important fundamental lemma, which is useful in proving Lemma 2.4, i.e., the critical point of φ is only the solution of BVP (1.3).

Let $Z = \{g \in C^\infty([0,T], R^n) : g(0) = -g(T)\}$.

Lemma 2.1. ([**30,** Lemma 3.4]). *Let $u, v \in L^1([0,T], R^n)$. If, for every $f \in Z$,*

$$\int_0^T (u(t), \dot{f}(t))\, dt = -\int_0^T (v(t), f(t))\, dt, \tag{2.3}$$

where $(\cdot,\cdot)$ denotes the inner product and $\dot{f}$ is the derivative of the vector function f, then

$$\frac{2}{T}\int_0^T u(t) + tv(t)\, dt = \int_0^T v(t)\, dt \tag{2.4}$$

and

$$u(t) = \int_0^t v(s)\, ds + a_0 \quad \textit{for almost every } t \in [0,T],\ a_0 \in R^n.$$

Remark 2.2. A function v satisfying (2.3) is called a *weak derivative* of u. By a Fourier series argument, the weak derivative, if it exists, is unique. The weak derivative of u will be denoted by $\dot{u}$.

Definition 2.3. A function $x \in X$ is said to be a *classical solution* of problem (1.3) if x satisfies equation in (1.3) for almost every $t \in [0,T]$ and the boundary conditions of (1.3).

Lemma 2.4. *If the function $x \in X$ is a critical point of the functional φ, then x is a classical solution of problem* (1.3).

Proof. Let $x \in X$ be a critical point of the functional φ. Then $\langle \varphi'(x), v\rangle = 0$ for every $v \in X$, i.e.,

$$\int_0^T \dot{x}(t)\cdot\dot{v}(t) - Mx(t)\cdot v(t) - \nabla F(t, x(t))\cdot v(t)\, dt = 0,$$

for all $v \in X$, and hence, for all $v \in Z$. By Lemma 2.1,

$$\dot{x}(t) = -\int_0^t Mx(s) + \nabla F(s, x(s))\, ds + a_0$$

for almost every $t \in [0, T]$, $a_0 \in R^n$. By Remark 2.2, $\dot{x}$ has a weak derivative $\ddot{x}$ and

$$\ddot{x} = -Mx - \nabla F(t, x(t)).$$

Since $x \in X$ and $\nabla F(t, x)$ are continuous in x for almost every $t \in [0, T]$, $\ddot{x}$ exists and is continuous for almost every $t \in [0, T]$. Thus, $\ddot{x}$ is a classical derivative of $\dot{x}$ for almost every $t \in [0, T]$. Therefore, x satisfies the equation in (1.3) for almost every $t \in [0, T]$.

Moreover, since the derivative of $\dot{x}$ is $\ddot{x}$, by (2.4) in Lemma 2.1, we have

$$\frac{2}{T}\int_0^T \dot{x}(t) + t\ddot{x}(t)\, dt = \int_0^T \ddot{x}(t)\, dt.$$

Now,

$$\frac{2}{T}\int_0^T \dot{x}(t) + t\ddot{x}(t)\, dt = \frac{2}{T}\left[\int_0^T \dot{x}(t)\, dt + T\dot{x}(T) - \int_0^T \dot{x}(t)\, dt\right] = 2\dot{x}(T),$$

and

$$\int_0^T \ddot{x}(t)\, dt = \dot{x}(T) - \dot{x}(0).$$

Thus, $\dot{x}(0) = -\dot{x}(T)$. Furthermore, $x \in X$ implies $x(0) = -x(T)$. Therefore, x is a classical solution of (1.3). $\square$

Next, we shall show some properties between the norms which are useful in estimating the norm $\|\cdot\|_X$.

Lemma 2.5. *If $x \in X$, then*

$$\|x\|_{L^2} \le \frac{1}{\sqrt{\lambda_0}}\|\dot{x}\|_{L^2} \quad \textit{and} \quad \|x\|_\infty \le \frac{\sqrt{T}}{2}\|\dot{x}\|_{L^2},$$

where

$$\|x\|_\infty = \max_{t\in[0,T]} |x(t)|,$$

$\lambda_0 = \pi^2/T^2$ *is the first nonzero eigenvalue of the eigenvalue problem*

$$\begin{cases} \ddot{x}(t) + \lambda x(t) = 0 & t \in [0,T],\ \lambda \in R \\ x(0) = -x(T) & \dot{x}(0) = -\dot{x}(T). \end{cases} \tag{2.5}$$

Proof. As is well known, (2.5) possesses a sequence of eigenvalues (λ_k), $\lambda_k = (2k+1)^2\pi^2/T^2$, with

$$0 < \lambda_0 < \lambda_1 < \cdots < \lambda_j < \cdots .$$

For each $\lambda_k, k = 0, 1, 2, \ldots$, (2.5) possesses eigenfunctions

$$\varphi_k(t) = a_k \cos\sqrt{\lambda_k}t + b_k \sin\sqrt{\lambda_k}t, \quad a_k, b_k \in R^n.$$

Let

$$\overline{x}(t) = \begin{cases} x(t-2kT) & t \in [2kT, (2k+1)T], \\ -x(t-(2k+1)T) & t \in [(2k+1)T, (2k+2)T]. \end{cases}$$

Then $\overline{x}$ is a $2T$-periodic function on R^n satisfying $\overline{x}(t) = -\overline{x}(t+T)$. By the expression of Fourier expansion, we have

$$\overline{x}(t) = \sum_{k=0}^{\infty}(a_k \cos\sqrt{\lambda_k}t + b_k \sin\sqrt{\lambda_k}t).$$

Thus,

$$\begin{aligned} \int_0^T |x(t)|^2 dt &= \frac{1}{2}\int_0^{2T} |\overline{x}(t)|^2 dt \\ &= \frac{1}{2}\int_0^{2T} \left|\sum_{k=0}^{\infty}(a_k \cos\sqrt{\lambda_k}t + b_k \sin\sqrt{\lambda_k}t)\right|^2 dt \\ &= \frac{1}{2}\int_0^{2T} \sum_{k=0}^{\infty} |a_k|^2 \cos^2\sqrt{\lambda_k}t + |b_k|^2 \sin^2\sqrt{\lambda_k}t\, dt \\ &= \frac{1}{2}\sum_{k=0}^{\infty}\left[|a_k|^2 \int_0^{2T} \cos^2\sqrt{\lambda_k}t\, dt + |b_k|^2 \int_0^{2T} \sin^2\sqrt{\lambda_k}t\, dt\right] \\ &= \frac{T}{2}\sum_{k=0}^{\infty}(|a_k|^2 + |b_k|^2). \end{aligned}$$

By the Parseval equality,

$$\begin{aligned}
\int_0^T |\dot{x}(t)|^2 dt &= \frac{1}{2}\int_0^{2T} |\dot{\bar{x}}(t)|^2 dt \\
&= \frac{1}{2}\int_0^{2T} \left|\sum_{k=0}^{\infty}(-a_k\sqrt{\lambda_k}\sin\sqrt{\lambda_k}t + b_k\sqrt{\lambda_k}\cos\sqrt{\lambda_k}t)\right|^2 dt \\
&= \frac{1}{2}\int_0^{2T} \sum_{k=0}^{\infty} |a_k|^2\lambda_k \sin^2\sqrt{\lambda_k}t + |b_k|^2\lambda_k\cos^2\sqrt{\lambda_k}t\, dt \\
&= \frac{T}{2}\sum_{k=0}^{\infty}(|a_k|^2 + |b_k|^2)\lambda_k.
\end{aligned}$$

Hence, we have $\|x\|_{L^2} \le 1/\sqrt{\lambda_0}\,\|\dot{x}\|_{L^2}$.

For the last inequality, we cite it from **[11,** Lemma 2.3]. For convenience, we prove it as follows. For $x \in X$, we have

$$x(t) = x(0) + \int_0^t \dot{x}(s)\,ds \quad \text{and} \quad x(t) = x(T) - \int_t^T \dot{x}(s)\,ds.$$

Thus, we have

$$\begin{aligned}
2x(t) &= \int_0^t \dot{x}(s)\,ds - \int_t^T \dot{x}(s)\,ds \\
&\le \int_0^t |\dot{x}(s)|\,ds + \int_t^T |\dot{x}(s)|\,ds \\
&= \int_0^T |\dot{x}(s)|\,ds.
\end{aligned}$$

It follows from Hölder's inequality that the result follows. □

Since M is a real symmetric matrix, M has n real eigenvalues, we denote $\lambda_1^M, \lambda_2^M, \dots, \lambda_n^M$.

Lemma 2.6. *For* $\max_{1\le i\le n}\{\lambda_i^M\} < \lambda_0 = \pi^2/T^2$, *we have*

$$\begin{aligned}
\theta_1\|\dot{x}\|_{L^2} &\le \left(\int_0^T |\dot{x}(t)|^2 - Mx(t)\cdot x(t)\,dt\right)^{1/2} \\
&\le \theta_2\|\dot{x}\|_{L^2} \quad \textit{for any } x \in X,
\end{aligned} \tag{2.6}$$

where

$$\theta_1 = \begin{cases} \sqrt{1 - (\max_{1\le i\le n}\{\lambda_i^M\}/\lambda_0)} & 0 \le \max_{1\le i\le n}\{\lambda_i^M\} < \lambda_0, \\ 1 & \max_{1\le i\le n}\{\lambda_i^M\} < 0, \end{cases}$$

and

$$\theta_2 = \begin{cases} 1 & 0 \le \min_{1\le i\le n}\{\lambda_i^M\} \\ & \quad < \max_{1\le i\le n}\{\lambda_i^M\} < \lambda_0, \\ \sqrt{1 - (\min_{1\le i\le n}\{\lambda_i^M\}/\lambda_0)} & \min_{1\le i\le n}\{\lambda_i^M\} < 0 \\ & \quad < \max_{1\le i\le n}\{\lambda_i^M\} < \lambda_0, \\ \sqrt{1 - (\min_{1\le i\le n}\{\lambda_i^M\}/\lambda_0)} & \max_{1\le i\le n}\{\lambda_i^M\} < 0, \end{cases}$$

that is to say,

$$\left(\int_0^T |\dot{x}(t)|^2 - Mx(t) \cdot x(t)\, dt \right)^{1/2}$$

is equivalent to $\|\dot{x}\|_{L^2}$.

Proof. If $0 \le \max_{1\le i\le n}\{\lambda_i^M\} < \lambda_0$, we have by Lemma 2.5

$$\begin{aligned} &\left(\int_0^T |\dot{x}(t)|^2 - Mx(t) \cdot x(t)\, dt \right)^{1/2} \\ &\quad \ge \left(\int_0^T |\dot{x}(t)|^2 - \max_{1\le i\le n}\{\lambda_i^M\}|x(t)|^2 dt \right)^{1/2} \\ &\quad \ge \left(\int_0^T |\dot{x}(t)|^2 - \frac{\max_{1\le i\le n}\{\lambda_i^M\}}{\lambda_0}|\dot{x}(t)|^2 dt \right)^{1/2} \\ &\quad = \theta_1 \|\dot{x}\|_{L^2}. \end{aligned}$$

If $\min_{1\le i\le n}\{\lambda_i^M\} < 0 < \max_{1\le i\le n}\{\lambda_i^M\} < \lambda_0$,

$$\begin{aligned} &\left(\int_0^T |\dot{x}(t)|^2 - Mx(t) \cdot x(t)\, dt \right)^{1/2} \\ &\quad \le \left(\int_0^T |\dot{x}(t)|^2 - \min_{1\le i\le n}\{\lambda_i^M\}|x(t)|^2 dt \right)^{1/2} \\ &\quad \le \left(\int_0^T |\dot{x}(t)|^2 - \frac{\min_{1\le i\le n}\{\lambda_i^M\}}{\lambda_0}|\dot{x}(t)|^2 dt \right)^{1/2} \le \theta_2 \|\dot{x}\|_{L^2}. \end{aligned}$$

If $0 < \min_{1\le i\le n}\{\lambda_i^M\} < \max_{1\le i\le n}\{\lambda_i^M\} < \lambda_0$,

$$\left(\int_0^T |\dot{x}(t)|^2 - Mx(t)\cdot x(t)\,dt\right)^{1/2} \le \left(\int_0^T |\dot{x}(t)|^2 - \min_{1\le i\le n}\{\lambda_i^M\}|x(t)|^2 dt\right)^{1/2} \le \|\dot{x}\|_{L^2}.$$

If $\max_{1\le i\le n}\{\lambda_i^M\} < 0$,

$$\begin{aligned}\|\dot{x}\|_{L^2} &\le \left(\int_0^T |\dot{x}(t)|^2 - Mx(t)\cdot x(t)\,dt\right)^{1/2}\\ &\le \left(\int_0^T |\dot{x}(t)|^2 - \frac{\min_{1\le i\le n}\{\lambda_i^M\}}{\lambda_0}|\dot{x}(t)|^2 dt\right)^{1/2}\\ &\le \theta_2\|\dot{x}\|_{L^2}.\end{aligned}$$

The result follows. □

Remark 2.7. $\|\cdot\|_X$ is equivalent to $\|\dot{x}\|_{L^2}$. In fact, by Lemma 2.5,

$$\|\dot{x}\|_{L^2} \le \|x\|_X = \left(\int_0^T |\dot{x}|^2 + |x|^2 dt\right)^{1/2} \le \sqrt{1+\frac{1}{\lambda_0}}\|\dot{x}\|_{L^2}. \tag{2.7}$$

Remark 2.8. If $\max_{1\le i\le n}\{\lambda_i^M\} < \lambda_0 = \pi^2/T^2$, $\|x\|_X$, $\|\dot{x}\|_{L^2}$ and

$$\left(\int_0^T |\dot{x}(t)|^2 - Mx(t)\cdot x(t)\,dt\right)^{1/2}$$

are equivalent.

Lemma 2.9. *Suppose that* $\max_{1\le i\le n}\{\lambda_i^M\} < \lambda_0 = \pi^2/T^2$, *where* λ_i^M, $i = 1, 2, \dots, n$, *are* n *real eigenvalue of matrix* M, *and*

(C1) *there exist* $\mu > 2$, l_1, $l_2 > 0$, $r \in L^1([0,T], R^+)$,

$$H(t,x) : [0,T] \times R^n \longrightarrow R$$

continuous in x *for almost every* $t \in [0,T]$, $H(t,x) > 0$ *for* $x > 0$, $t \in [0,T]$, *such that*

$$F(t,x) = \frac{r(t)|x|^\mu}{\mu} + H(t,x),$$

where

$$\limsup_{|x|\to+\infty} \frac{\mu H(t,x) - \nabla H(t,x)\cdot x}{|x|^2} = l_0. \tag{2.8}$$

(C2) $l_0/\lambda_0 < (\mu/2-1)\theta_1^2$, *where* θ_1 *is defined in Lemma* 2.6*. Then,* φ *satisfies the Palais-Smale condition.*

Proof. Let $(x_n) \subset X$ satisfy $\varphi(x_n)$ is bounded and

$$\varphi'(x_n) \longrightarrow 0 \quad \text{as } n \to \infty.$$

First, we shall show that (x_n) is a bounded sequence in X. By (2.1)–(2.2),

$$\begin{aligned}\left(\frac{\mu}{2}-1\right)\left[\|\dot{x}_n\|_{L^2}^2 - \int_0^T Mx_n \cdot x_n(t)\,dt\right] &= \mu\varphi(x_n) - \langle\varphi'(x_n), x_n\rangle + \mu\int_0^T F(t,x_n)\,dt \\ &\quad - \int_0^T \nabla F(t,x_n)\cdot x_n\,dt.\end{aligned} \tag{2.9}$$

By (C1) and Lemma 2.5,

$$\begin{aligned}&\mu\int_0^T F(t,x_n)\,dt - \int_0^T \nabla F(t,x_n)\cdot x_n\,dt \\ &\quad = \int_0^T r(t)|x_n|^\mu + \mu H(t,x_n) - [r(t)\phi_\mu(x_n) + \nabla H(t,x_n)]\cdot x_n\,dt \\ &\quad = \int_0^T r(t)|x_n|^\mu + \mu H(t,x_n) - \Big[r(t)|x_n|^\mu + \nabla H(t,x_n)\cdot x_n\Big]\,dt \\ &\quad = \int_0^T \mu H(t,x_n) - \nabla H(t,x_n)\cdot x_n\,dt,\end{aligned} \tag{2.10}$$

where $\phi_\mu(x) = |x|^{\mu-2}x$.

By (2.8), for any

$$\varepsilon \in \left(0, \lambda_0\left(\frac{\mu}{2}-1\right)\theta_1^2 - l_0\right), \tag{2.11}$$

there exists a positive constant $C > 0$ satisfying

$$\frac{\mu H(t,x) - \nabla H(t,x)\cdot x}{|x|^2} < l_0 + \varepsilon \quad \text{for all } |x| > C.$$

Thus, there exists a sufficiently large number $C' > 0$ satisfying

$$\mu H(t,x) - \nabla H(t,x)\cdot x < (l_0+\varepsilon)|x|^2 + C' \tag{2.12}$$

for all $x \in R^n$.

By (2.10)–(2.12) and Lemma 2.5, we have

$$\begin{aligned}\mu\int_0^T F(t,x_n)\,dt - \int_0^T \nabla F(t,x_n)\cdot x_n\,dt &\le (l_0+\varepsilon)\|x_n\|_{L^2}^2 + C'T\\ &\le \frac{l_0+\varepsilon}{\lambda_0}\|\dot{x}_n\|_{L^2}^2 + C'T.\end{aligned} \tag{2.13}$$

By (2.6) and (2.9), we have

$$\left(\frac{\mu}{2}-1\right)\left[\|\dot{x}_n\|_{L^2}^2 - \int_0^T Mx_n(t)\cdot x_n(t)\,dt\right] \ge \left(\frac{\mu}{2}-1\right)\theta_1^2\|\dot{x}_n\|_{L^2}^2. \tag{2.14}$$

Substituting (2.13) and (2.14) into (2.9), we have

$$\begin{aligned}\left(\frac{\mu}{2}-1\right)\theta_1^2\|\dot{x}_n\|_{L^2}^2 \le \mu\varphi(x_n) - \langle\varphi'(x_n),x_n\rangle\\ + \frac{l_0+\varepsilon}{\lambda_0}\|\dot{x}_n\|_{L^2}^2 + C'T.\end{aligned} \tag{2.15}$$

Suppose that $(\dot{x}_n)$ is unbounded in $L^2[0,T]$. Passing to a subsequence, we may assume, if necessary, that $\|\dot{x}_n\|_{L^2} \to \infty$ as $n \to \infty$. Dividing both sides of (2.15) by $\|\dot{x}_n\|_{L^2}^2$, we have

$$\left(\frac{\mu}{2}-1\right)\theta_1^2 \le \frac{\mu\varphi(x_n)}{\|\dot{x}_n\|_{L^2}^2} - \frac{\langle\varphi'(x_n),x_n\rangle}{\|\dot{x}_n\|_{L^2}^2} + \frac{l_0+\varepsilon}{\lambda_0} + \frac{C'T}{\|\dot{x}_n\|_{L^2}^2}.$$

Since $\varphi(x_n)$ is bounded and $\varphi'(x_n) \to 0$, let $n \to \infty$ in the above inequality. Then, we have, by (2.11),

$$\left(\frac{\mu}{2}-1\right)\theta_1^2 \le \frac{l_0+\varepsilon}{\lambda_0} < \left(\frac{\mu}{2}-1\right)\theta_1^2,$$

which is a contradiction. Therefore, $\|\dot{x}_n\|_{L^2} < \infty$, which, together with Remark 2.8, gives $\|x_n\|_X < \infty$.

From the reflexive property of X, we may extract a weakly convergent subsequence, which we call, for simplicity, $(x_n), x_n \rightharpoonup x$. In the following we shall show that (x_n) converges strongly to x, i.e., $\|x_n - x\|_X \to 0$ as $n \to \infty$. By (2.2),

$$\langle \varphi'(x_n) - \varphi'(x), x_n - x\rangle = \int_0^T |\dot{x}_n - \dot{x}|^2 - M(x_n - x) \cdot (x_n - x) - (\nabla F(t, x_n) - \nabla F(t, x), x_n - x)\, dt.$$

By $x_n \rightharpoonup x$ in X, we see that (x_n) converges uniformly to x in $C([0,T])$, i.e., $\|x_n - x\|_\infty \to 0$ as $n \to \infty$. Hence,

$$\int_0^T (\nabla F(t, x_n) - \nabla F(t, x)) \cdot (x_n - x)\, dt \longrightarrow 0 \quad \text{as } n \to \infty.$$

By $\varphi'(x_n) \to 0$ and $x_n \rightharpoonup x$, we have

$$\langle \varphi'(x_n) - \varphi'(x),\ x_n - x\rangle \longrightarrow 0.$$

Therefore,

$$\int_0^T |\dot{x}_n - \dot{x}|^2 - M(x_n - x) \cdot (x_n - x)\, dt \longrightarrow 0 \quad \text{as } n \to \infty.$$

By Remark 2.8, $\|x_n - x\|_X \to 0$ as $n \to \infty$. □

3. Main results.

Theorem 3.1. *Suppose that* $\max_{1\le i\le n}\{\lambda_i^M\} < \lambda_0 = \pi^2/T^2$ ($\lambda_i^M, i = 1, 2, \ldots, n$, *are* n *real eigenvalues of real symmetric matrix* M), *and* (C1)–(C2) *hold. Furthermore, we assume that*:

(C3) *there exists* $K_0 > 0$ *such that*

$$\frac{\theta_1^2 \lambda_0}{2(1+\lambda_0)} K_0^2 - \frac{\|r\|_{L^1} T^{\mu/2}}{\mu 2^\mu} K_0^\mu - \max_{|x|\le(\sqrt{T}K_0/2)} \int_0^T H(t,x)\, dt + \int_0^T H(t,0)\, dt > 0.$$

Then BVP (1.3) *has at least two solutions.*

Proof. We complete the proof with the following three steps.

Step 1. By Lemma 2.9, the functional φ satisfies the Palais-Smale condition.

Step 2. We shall show that there exists a $K_0 > 0$ such that the functional φ has a local minimum $x_0 \in B_{K_0} := \{x \in X : \|x\|_X < K_0\}$.

Let $K > 0$, which will be determined later. First, we claim that $\overline{B}_K$ is bounded and weakly sequentially closed. In fact, let $(u_n) \subseteq \overline{B}_K$ and $(u_n) \rightharpoonup u$ as $n \to \infty$. By the Mazur theorem **[24]**, there exists a sequence of convex combinations

$$v_n = \sum_{j=1}^{n} \alpha_{n_j} u_j, \qquad \sum_{j=1}^{n} \alpha_{n_j} = 1, \quad \alpha_{n_j} \geq 0,\ j \in N,$$

such that $v_n \to u$ in X. Since $\overline{B}_K$ is a closed convex set, $(v_n) \subset \overline{B}_K$ and $u \in \overline{B}_K$. Now, we claim that φ has a minimum $x_0 \in \overline{B}_K$. We will show that φ is weakly sequentially lower semi-continuous on $\overline{B}_K$. For this, let

$$\varphi^1(x) = \frac{1}{2}\int_0^T |\dot{x}|^2 - Mx \cdot x\, dt, \qquad \varphi^2(x) = \int_0^T -F(t,x)\, dt.$$

Then, $\varphi(x) = \varphi^1(x) + \varphi^2(x)$. By $x_n \rightharpoonup x$ on X, we have (x_n) uniformly converges to x in $C([0,T])$. Thus, φ^2 is weakly sequentially continuous. By Remark 2.8, φ^1 is continuous, which, together with the convexity of φ^1, gives that φ^1 is weakly sequentially lower semi-continuous. Therefore, φ is weakly sequentially lower semi-continuous on $\overline{B}_K$. Further, X is a reflexive Banach space, $\overline{B}_K$ is a bounded and weak sequentially closed set; thus, our claim follows from Lemma 1.1. Without loss of generality, we assume that $\varphi(x_0) = \min_{x \in \overline{B}_K} \varphi(x)$. Now we will show that

$$\varphi(x_0) < \inf_{x \in \partial B_K} \varphi(x). \tag{3.1}$$

If this is true, the result of Step 2 holds.

In fact, for any $x \in \partial B_K$, by Lemma 2.5, we have,

$$\|x\|_\infty \leq \frac{\sqrt{T}}{2}\|x\|_X = \frac{\sqrt{T}}{2}K.$$

By (C1), (2.6), (2.7) and Lemma 2.5, we have

$$
\begin{aligned}
\varphi(x) &\geq \frac{\theta_1^2}{2}\|\dot{x}\|_{L^2}^2 - \int_0^T r(t)\frac{|x|^\mu}{\mu} + H(t,x)\,dt \\
&\geq \frac{\theta_1^2\lambda_0}{2(1+\lambda_0)}K^2 - \frac{\|r\|_{L^1}}{\mu}\|x\|_\infty^\mu - \int_0^T H(t,x)\,dt \\
&\geq \frac{\theta_1^2\lambda_0}{2(1+\lambda_0)}K^2 - \frac{\|r\|_{L^1}T^{\mu/2}}{\mu 2^\mu}\|\dot{x}\|_{L^2}^\mu - \int_0^T H(t,x)\,dt \\
&\geq \frac{\theta_1^2\lambda_0}{2(1+\lambda_0)}K^2 - \frac{\|r\|_{L^1}T^{\mu/2}}{\mu 2^\mu}\|x\|_X^\mu - \int_0^T H(t,x)\,dt \\
&\geq \frac{\theta_1^2\lambda_0}{2(1+\lambda_0)}K^2 - \frac{\|r\|_{L^1}T^{\mu/2}}{\mu 2^\mu}K^\mu \\
&\quad - \max_{|x|\leq(\sqrt{T}/2)K}\int_0^T H(t,x)\,dt.
\end{aligned} \tag{3.2}
$$

Noting

$$
\varphi(0) = -\int_0^T F(t,0)\,dt = -\int_0^T H(t,0)\,dt,
$$

and $\varphi(x_0) \leq \varphi(0)$, let $K = K_0$. By (C3), we have $\inf_{x\in\partial B_{K_0}}\varphi(x) > \varphi(0) \geq \min_{x\in\overline{B}_{K_0}}\varphi(x)$. Thus, (3.1) holds and $x_0 \in B_{K_0}$.

Step 3. We shall show that there exists an $x_1 \in X$ with $\|x_1\|_X > K_0$ such that $\varphi(x_1) < \inf_{x\in\partial B_{K_0}}\varphi(x)$.

Let $\widetilde{e}(t) = \sin(\pi/T)t(1,0,\ldots,0)^T$, $\overline{\lambda} > 0$. Then, by (C1),

$$
\begin{aligned}
\varphi(\overline{\lambda}\widetilde{e}) &= \int_0^T \frac{1}{2}|\overline{\lambda}\dot{\widetilde{e}}|^2 - \frac{1}{2}M\overline{\lambda}\widetilde{e}\cdot\overline{\lambda}\widetilde{e} - F(t,\overline{\lambda}\widetilde{e})\,dt \\
&\leq \int_0^T \frac{1}{2}\left|\overline{\lambda}\frac{\pi}{T}\cos\frac{\pi t}{T}\right|^2 + \frac{\max_{1\leq i\leq n}\{|\lambda_i^M|\}}{2}\left|\overline{\lambda}\sin\frac{\pi t}{T}\right|^2 \\
&\quad - F\left(t,\overline{\lambda}\sin\frac{\pi t}{T}e_1\right)dt \\
&= \left(\overline{\lambda}\frac{\pi}{T}\right)^2\frac{T}{4} + \max_{1\leq i\leq n}\{|\lambda_i^M|\}|\overline{\lambda}|^2\frac{T}{4}
\end{aligned} \tag{3.3}
$$

$$- \int_0^T \frac{r(t)|\overline{\lambda}|^\mu \sin^\mu(\pi t/T)}{\mu}\,dt - \int_0^T H\left(t, \overline{\lambda}\sin\frac{\pi t}{T}e_1\right) dt$$
$$\leq \left(\overline{\lambda}\frac{\pi}{T}\right)^2 \frac{T}{4} + \max_{1\leq i\leq n}\{|\lambda_i^M|\}|\overline{\lambda}|^2\frac{T}{4} - |\overline{\lambda}|^\mu \int_0^T \frac{r(t)\sin^\mu(\pi t/T)}{\mu}\,dt.$$

Since

$$\int_0^T r(t)\sin^\mu\frac{\pi t}{T}\,dt > 0$$

and $\mu > 2$, we have

$$\lim_{\overline{\lambda}\longrightarrow+\infty} \varphi(\overline{\lambda}\widetilde{e}) = -\infty.$$

Thus, there exists a sufficiently large $\overline{\lambda}_0 > 0$ with $\|\overline{\lambda}_0\widetilde{e}\| > K_0$ such that

$$\varphi(\overline{\lambda}_0\widetilde{e}) < \inf_{x\in\partial B_{K_0}} \varphi(x).$$

Therefore, let $x_1 = \overline{\lambda}_0\widetilde{e}$ and

$$\varphi(x_1) < \inf_{x\in\partial B_{K_0}} \varphi(x).$$

Lemma 1.2 now gives the critical value

$$c = \inf_{h\in\gamma}\max_{t\in[0,1]} \varphi(h(t)),$$

where

$$\gamma = \{h \mid h : [0,1] \longrightarrow X \text{ is continuous and } h(0) = x_0,\, h(1) = x_1\},$$

that is, there exists an $x^* \in X$ such that $\varphi'(x^*) = 0$. Therefore, x_0 and x^* are two critical points of φ, $\|x_0\|_X < K_0$. By Lemma 2.4 x_0 and x^* are two classical solutions of (1.3). $\square$

By Lemma 2.9 and Theorem 3.1, we have the following corollary.

Corollary 3.2. *Suppose that* $\max_{1\leq i\leq n}\{\lambda_i^M\} < \lambda_0$, ($\lambda_i^M$, $i = 1, 2, \ldots, n$, *are* n *real eigenvalues of real symmetric matrix* M), (C3) *and*

(C4) *there exist* $\mu > 2$, l_1, $l_2 > 0$, $r \in L^1([0,T], R^+)$,

$$H(t,x) : [0,T] \times R^n \longrightarrow R$$

continuous in x for almost every $t \in [0,T]$, $H(t,x) > 0$ for $x > 0$, $t \in [0,T]$, such that

$$F(t,x) = \frac{r(t)|x|^\mu}{\mu} + H(t,x),$$

and $\mu H(t,x) < \nabla H(t,x) \cdot x$ for $|x|$ sufficiently large. Then, BVP (1.3) *has at least one solution.*

Proof. Condition (C4) implies (C1) and (C2). Therefore, the result follows. □

Finally, we consider the system

$$\ddot{x} + Mx + \nabla F(t,x) = 0, \quad \text{almost everywhere } t \in R, \tag{3.4}$$

where

$$F : R \times R^n \longrightarrow R \quad \text{and} \quad (t,x) \longmapsto F(t,x)$$

is measurable in t for each $x \in R^n$, and continuously differentiable in x for almost every $t \in R$,

$$\nabla F(t,x) = \left(\frac{\partial F(t,x)}{\partial x_1}, \frac{\partial F(t,x)}{\partial x_2}, \dots, \frac{\partial F(t,x)}{\partial x_n}\right)^T.$$

Theorem 3.3. *Assume that* $\max_{1\le i\le n}\{\lambda_i^M\} < \lambda_0 = (\pi^2/T^2)$, *($\lambda_i^M$, $i = 1, 2, \dots, n$, are eigenvalues of real symmetric matrix M),* (C1), (C2), (C3), *and*

(C5) *$F(t+T,x) = F(t,x)$, $\nabla F(t,-x) = -\nabla F(t,x)$ hold.*
Then, (3.4) *has at least two $2T$-periodic solutions.*

Proof. Assume that $x(t)$ is a solution of BVP (1.3). Let $x_1(t) = -x(t-T)$ for $t \in [T, 2T]$. Then, $\ddot{x}_1(t) = -\ddot{x}(t-T)$. By (C5), we have

$$\nabla F(t, x_1(t)) = \nabla F(t-T, -x(t-T)) = -\nabla F(t-T, x(t-T)).$$

Thus,

$$\begin{aligned}\ddot{x}_1(t) + Mx_1(t) + \nabla F(t, x_1(t)) = &- \ddot{x}(t-T) - Mx(t-T)\\ &- \nabla F(t-T, x(t-T)) = 0\end{aligned}$$

and

$$x_1(T) = -x(0), \qquad x_1(2T) = -x(T),$$
$$\dot{x}_1(T) = -\dot{x}_1(0), \qquad \dot{x}_1(2T) = -\dot{x}_1(T).$$

Let

$$\overline{x}(t) = \begin{cases} x(t) & t \in [0,T], \\ x_1(t) & t \in [T,2T]. \end{cases}$$

Then, $\overline{x}(t)$ is a solution of (3.4) with $x(0) = x(2T)$, $\dot{x}(0) = \dot{x}(2T)$. Since $F(t,x)$ is 2T-periodic in t, $\overline{x}$ can be extended by $2T$-periodic over R to give a $2T$-periodic solution of (3.4), and Theorem 3.1 gives the existence of two solutions. The result follows. □

By Corollary 3.2 and Theorem 3.3 we have the following corollary.

Corollary 3.4. *Suppose that* $\max_{1\le i\le n}\{\lambda_i^M\} < \lambda_0 = (\pi^2/T^2)$, *(*$\lambda_i^M$, $i = 1,2,\ldots,n$, *are* n *real eigenvalues of real symmetric matrix* M*),* (C3), (C4) *and* (C5) *hold. Then,* (3.4) *has at least one* $2T$*-periodic solution.*

Acknowledgments. The first author expresses her gratitude to Professor Johnny Henderson for his helpful advice during the preparation of this paper.

REFERENCES

1. A. Abdurahman, F. Anton and J. Bordes, *Half-string oscillator approach to string field theory*, Nuclear Phys. **397** (1993), 260–282.

2. A.R. Aftabizadeh, S. Aizicovici and N.H. Pavel, *Anti-periodic boundary value problems for higher order differential equations in Hilbert spaces*, Nonlin. Anal. **18** (1992), 253–267.

3. B. Ahmad and J.J. Nieto, *Existence of solutions for anti-periodic boundary value problems involving fractional differential equations via Leray-Schauder degree*, Topol. Meth. Nonlin. Anal. **35** (2010), 295–304.

4. C. Ahn and C. Rim, *Boundary flows in general coset theories*, J. Physica **32** (1999), 2509–2525.

5. S. Aizicovici, M. McKibben and S. Reich, *Anti-periodic solutions to non-monotone evolution equations with discontinuous nonlinearities*, Nonlin. Anal. **43** (2001), 233–251.

6. S. Aizicovici and N.H. Pavel, *Anti-periodic solutions to a class of nonlinear differential equations in Hilbert space*, J. Funct. Anal. **99** (1991), 387–408.

7. H.L. Chen, *Antiperiodic wavelets*, J. Comp. Math. **14** (1996), 32–39.

8. Y.Q. Chen, *Note on Massera's theorem on anti-periodic solution*, Adv. Math. Sci. Appl. **9** (1999), 125–128.

9. Y.Q. Chen, Y.J. Cho and D. O'Regan, *Anti-periodic solutions for evolution equations with mappings in the class* (S_+), Math. Nachr. **278** (2005), 356–362.

10. Y.Q. Chen, J.J. Nieto and Donal O'Regan, *Anti-periodic solutions for fully nonlinear first-order differential equations*, Math. Comp. Model. **46** (2007), 1183–1190.

11. Y.Q. Chen, X.D. Wang and H.X. Xu, *Anti-periodic solutions for semilinear evolution equations*, J. Math. Anal. Appl. **273** (2002), 627–636.

12. Guo Dajun, *Nonlinear functional analysis*, Shandong Science and Technology Press, Shandong, 1985.

13. F.J. Delvos and L. Knoche, *Lacunary interpolation by antiperiodic trigonometric polynomials*, BIT **39** (1999), 439–450.

14. P. Djiakov and B. Mityagin, *Simple and double eigenvalues of the Hill operator with a two-term potential*, J. Approx. Theory **135** (2005), 70–104.

15. J.Y. Du, H.L. Han and G.X. Jin, *On trigonometric and paratrigonometric Hermite interpolation*, J. Approx. Theory **131** (2004), 74–99.

16. D. Franco and J.J. Nieto, *First order impulsive ordinary differential equations with anti-periodic and nonlinear boundary conditions*, Nonlin. Anal. **42** (2000), 163–173.

17. D. Franco, J.J. Nieto and D. O'Regan, *Anti-periodic boundary value problem for nonlinear first order ordinary differential equations*, Math. Inequal. Appl. **6** (2003), 477–485.

18. ———, *Existence of solutions for first order ordinary differential equations with nonlinear boundary conditions*, Appl. Math. Comp. **153** (2004), 793–802.

19. A. Haraux, *Nonlinear evolution equations, Global behavior of solutions*, Lect. Notes Math. **841**, Springer-Verlag, Berlin, 1981.

20. ———, *Anti-periodic solutions of some nonlinear evolution equations*, Manuscr. Math. **63** (1989), 479–505.

21. H. Kleinert and A. Chervyakov, *Functional determinants from Wronski Green function*, J. Math. Phys. **40** (1999), 6044–6051.

22. B. Liu, *An anti-periodic LaSalle oscillation theorem for a class of functional differential equations*, J. Comp. Appl. Math. **223** (2009), 1081–1086.

23. Z.G. Luo, J.H. Shen and J.J. Nieto, *Antiperiodic boundary value problem for first-order impulsive ordinary differential equations*, Comp. Math. Appl. **49** (2005), 253–261.

24. J. Mawhin and M. Willem, *Critical point theory and Hamiltonian systems*, Springer-Verlag, Berlin, 1989.

25. M. Nakao, *Existence of anti-periodic solution for the quasilinear wave equation with viscosity*, J. Math. Anal. Appl. **204** (1996), 754–764.

26. H. Okochi, *On the existence of periodic solutions to nonlinear abstract parabolic equations*, J. Math. Soc. Japan **40** (1988), 541–553.

27. S. Pinsky and U. Tritman, *Antiperiodic boundary conditions to supersymmetric discrete light cone quantization*, Physica **62** (2000), 087701.

28. Y. Tian and W.G. Ge, *Periodic solutions of non-autonomous second-order systems with a p-Laplacian*, Nonlin. Anal. **66** (2007), 192–203.

29. ______, *Applications of variational methods to boundary value problem for impulsive differential equations*, Proc. Edinburgh Math. Soc. **51** (2008), 509–527.

30. Yu Tian and Johnny Henderson, *Three anti-periodic solutions for second-order impulsive differential inclusions via nonsmooth critical point theory*, Nonlin. Anal. **75** (2012), 6496–6505.

31. K. Wang and Y. Li, *A note on existence of anti-periodic and heteroclinic solutions for a class of second-order odes*, Nonlin. Anal. **70** (2009), 1711–1724.

32. W. Wang and J. Shen, *Existence of solutions for anti-periodic boundary value problems*, Nonlin. Anal. **70** (2009), 598–605.

33. E. Zeidler, *Nonlinear functional analysis and its applications*, Volume 3, Springer, Berlin, 1985.

Beijing University of Posts and Telecommunications, School of Science, Beijing, 100876 P.R. China
Email address: **tianyu2992@163.com**

Beijing University of Posts and Telecommunications, School of Science, Beijing, 100876 P.R. China
Email address: **zhangyajing1977@163.com**