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# CONSTRUCTION OF NEW MULTIPLE KNOT B-SPLINE WAVELETS

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ABSTRACT. This paper deals with construction of non-uniform multiple knot B-spline wavelet basis functions (with minimal support). These wavelets are semi-orthogonal on a bounded interval. A large family of multiple knot B-spline wavelets is presented that gives a variety of basis functions with explicit formulas and locally compact supports. Moreover, the structure of this wavelet is conceptually simple and easy to implement. Finally, some examples of multiple knot B-spline wavelets are also presented.

**1. Introduction.** Wavelets play a crucial role in many areas of mathematics and engineering such as speech, image and signal processing, approximation theory and numerical solution of partial differential equations. Enormous progress has been made in the construction and analysis of wavelet methods in recent years. Construction of wavelets is usually based on the determination of filter coefficients of the scaling functions.

The main goal is of course to find a possibly simple construction from a technical point of view. Therefore, in order to minimize computational effort, construction of compactly supported wavelets is prominent. A typical class of these wavelets are spline wavelets which are constructed by B-spline functions. Some decomposition and reconstruction algorithms for spline wavelet packets on a closed interval are given in **[12]**. B-splines are defined based on the knot points. An important class of B-splines are multiple knot B-spline functions which are defined based on multiple knot points. Construction of multiple knot B-spline wavelets (MKBSWs) which are notable in mathematics and engineering, has become a powerful mathematical tool. A generalized

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multiresolution of multiplicity $r$ generated by $r$ linearly independent spline functions with multiple knots is introduced in [**10**].

Chui and Quak [**2**] investigated scaling function spaces of polynomial splines of order $m$ which, for a given level $j \in \mathbb{N}$, are defined on a knot set $\mathbf{t}_j$ of equidistant knots of spacing $2^{-j}$ ($j$ denotes the level) and knots of multiplicity $m$ at both endpoints of the interval $[0, 1]$. For some applications, this wavelet or a modification of it can work well. A modification of Chui and Quak's multiple knot B-spline wavelet for solving (partially) Dirichlet boundary value problem is given in [**11**].

In [**2**], only the endpoint knots are multiple and the midpoint knots are single. However, in many applications, we need multiple knot B-spline wavelets (MKBSWs) whose knot points are multiple not only in the endpoints but also in the middle knots. For example, hydroelastic analysis of fully nonlinear water waves with the floating elastic plate is a difficult task. When the water wave encounters the plate, the wave function would not be smooth enough at the edge of the plate compared to the other points. Hence, to numerically analyze the behavior of the wave, the solution space should include basis functions that are not smooth enough at the edge of plate [**9**].

Lyche et al. [**6**] constructed the mutually orthogonal spline wavelet spaces on non-uniform partitions of a bounded interval with multiple knots on the interval. In this paper, we follow another methodology for constructing MKBSWs on a non-uniform partition of a bounded interval. Our structure has the following advantages:

- the wavelet is expressed explicitly in terms of polynomials that are conceptually simple and easy to implement;
- a large family of MKBSWs is presented that gives a variety of basis functions with explicit formulas and locally compact supports;
- MKBSWs may be found with minimal support;
- the algorithm is readily implemented with a variety in the order of B-splines;
- multiple knot wavelets are constructed on non-uniform partitions from a bounded interval.

In [**1**], a flexible and efficient single knot wavelet construction is proposed for non-uniform B-spline curves and surfaces. Because of multiplicity of the knots and presentation of an explicit formula for the wavelet, the wavelet introduced in this paper is more efficient than that of [**1**]. Hence, this paper fulfills the construction of wavelets on the single and multiple knot B-splines.

The structure of the paper is as follows. In Section 2, some preliminaries of multiresolution analysis (MRA) and wavelets are presented. The definition of multiple knot B-splines with their properties are given in detail in Section 3. The main body of the paper is presented in Section 4 where the construction of MKBSW is detailed. A variety of MKBSW examples that support our theoretical results are given in Section 5. The conclusion of the paper is presented in Section 6.

**2. Preliminaries.** It is well known that wavelets can be constructed from a multiresolution analysis (MRA). In this section, we first recall the definition of MRA that was already introduced by Mallat [**7**] and Meyer [**8**] as follows:

**Definition 2.1.** A multiresolution analysis (MRA) is a nested sequence of closed subspaces $V_j \in L^2(\mathbb{R})$ that satisfy:

(i) $\cdots \subset V_{-1} \subset V_0 \subset V_1 \subset \cdots$;
(ii) $\mathrm{clos}_{L^2}(\cup_{j\in\mathbb{Z}} V_j) = L^2(\mathbb{R})$;
(iii) $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$;
(iv) $f \in V_j \Leftrightarrow f(\cdot + 2^{-j}) \in V_j \Leftrightarrow f(2\cdot) \in V_{j+1}$, $j \in \mathbb{Z}$; and
(v) there exists a function $\phi \in V_0$ that $\{\phi(\cdot - k)\}_{k\in\mathbb{Z}}$ forms a Riesz basis for $V_0$, i.e., there are constants $A$ and $B$, with $0 < A \leq B < \infty$ such that

$$A\|\{c_k\}\|_{\ell^2}^2 \leq \left\| \sum_{k=-\infty}^{\infty} c_k \phi(\cdot - k) \right\|_2^2 \leq B\|\{c_k\}\|_{\ell^2}^2$$

for all sequences $\{c_k\}$ for which

$$\|\{c_k\}\|_{\ell^2}^2 = \sum_{k=-\infty}^{\infty} |c_k|^2 < \infty.$$

The function $\phi$ that satisfies the above conditions is called a *scaling function.*

Since $V_j \subset V_{j+1}$, one can write $V_{j+1} = V_j \bigoplus W_j$ that $W_j$ is a complementary orthogonal subspace for $V_j$ in $V_{j+1}$. Moreover, one can show that [**4**]

$$L^2(\mathbb{R}) = \overline{\bigoplus_{j\in\mathbb{Z}} W_j}.$$

Let $\psi \in V_1$ and $W_j = \operatorname{clos}_{L^2}\{\psi(2^j. - k) : j \in \mathbb{Z}, k \in \mathbb{Z}\}$. Then the function $\psi$ is called a *wavelet.*

Let us denote $V_j^{[0,1]}$ instead of the subspace $V_j$ when dealing with a bounded interval $[0,1]$ and take $V_j^{[0,1]}$ merely for $j \geq 0$ in the concept of a multi-resolution analysis. Then we have $V_0^{[0,1]} \subset V_1^{[0,1]} \subset \cdots$ and

$$\operatorname{clos}_{L^2[0,1]}(\cup_{j\in\mathbb{Z}_+} V_j) = L^2[0,1].$$

Furthermore, the complementary orthogonal subspaces $W_j^{[0,1]}$ satisfy the following relations,

$$V_{j+1}^{[0,1]} = V_j^{[0,1]} \bigoplus W_j^{[0,1]}, \quad j \in \mathbb{Z}_+,$$

and the orthogonal decomposition of $L^2[0,1]$, namely,

$$L^2[0,1] = V_0^{[0,1]} \bigoplus_{j\in\mathbb{Z}_+} W_j^{[0,1]}.$$

In this paper, we will construct the wavelet on the interval $[0,1]$.

## 3. Multiple knot B-splines.

In this section, we define multiple knot B-splines and prove some related properties.

**Definition 3.1.** Let $m > 1$ be a fixed integer number. For $j \in \mathbb{Z}_+$ and $N_j \in \mathbb{N}$, consider the set $\{n_k\}_{k=0}^{N_j}$ such that

$$\begin{cases} n_k = m & k = 0,\ N_j, \\ 1 \leq n_k \leq m-1 & k = 1, 2, \ldots, N_j - 1. \end{cases}$$

A general multiple knot set on $[0,1]$ is defined by $\mathbf{t}^{(j)} := \mathbf{t}_m^{(j)} = \{t_k\}_{k=-m+1}^{S_0}$, where

$$S_0 = \sum_{k=1}^{N_j} n_k, \quad 0 \leq t_k^j < t_{k+1}^j \leq 1 \quad \text{for } k = 0, 1, \ldots, N_j - 1,$$

with

$$
\begin{array}{lllllll}
t_{-m+1} & = & t_{-m+2} & = & \cdots & = & t_{-m+n_0} = 0 =: t_0^j, \\
t_1 & = & t_2 & = & \cdots & = & t_{n_1} =: t_1^j, \\
t_{n_1+1} & = & t_{n_1+2} & = & \cdots & = & t_{n_1+n_2} =: t_2^j, \\
 & \vdots & & \vdots & \cdots & & \vdots \\
t_{S_2+1} & = & t_{S_2+2} & = & \cdots & = & t_{S_2+n_{N_j-1}} =: t_{N_j-1}^j, \\
t_{S_1+1} & = & t_{S_1+2} & = & \cdots & = & t_{S_1+n_{N_j}} = 1 =: t_{N_j}^j,
\end{array}
\tag{3.1}
$$

where

$$S_2 = \sum_{k=1}^{N_j-2} n_k \quad \text{and} \quad S_1 = \sum_{k=1}^{N_j-1} n_k.$$

For $i = 0, 1 \ldots, S = \sum_{k=0}^{N_j-1} n_k$, the B-spline basis functions of degree $m-1$ (the order of $m$) are recursively defined starting with piecewise constants for $m = 1$:

$$B_{i,0,j}(x) = \begin{cases} 1 & t_i \le x < t_{i+1},\ t_i < t_{i+1}, \\ 0 & \textit{otherwise.} \end{cases}$$

For $m = 2, 3, \ldots$, they are defined by

$$
\begin{aligned}
&B_{i,m-1,j}(x) \\
&= \begin{cases} 0 & t_i = t_{i+m-1},\ t_{i+1} = t_{i+m}, \\ \alpha_i B_{i,m-2,j}(x) & t_i < t_{i+m-1},\ t_{i+1} = t_{i+m}, \\ \beta_i B_{i+1,m-2,j}(x) & t_i = t_{i+m-1},\ t_{i+1} < t_{i+m}, \\ \alpha_i B_{i,m-2,j}(x) + \beta_i B_{i+1,m-2,j}(x) & t_i < t_{i+m-1},\ t_{i+1} < t_{i+m}, \end{cases}
\end{aligned}
\tag{3.2}
$$

where $\alpha_i = (x - t_i)/(t_{i+m-1} - t_i)$ and $\beta_i = (t_{i+m} - x)/(t_{i+m} - t_{i+1})$.

It is well known that the number of B-splines $B_{i,m-1,j}$ of order $m$ associated with $\mathbf{t}_m^{(j)}$ is $S = \sum_{k=0}^{N_j-1} n_k$. Moreover, the number of B-splines $B_{i,2m-1,j}$ of order $2m$ on $\mathbf{t}_m^{(j)}$ is equal to $S - m$. For example, if $m = 3, j = 2$ and $n_0 = 3$, $n_1 = 1$, $n_2 = n_3 = 2$, $n_4 = 3$, then

$$\mathbf{t}_3^{(2)} = \{0,\ 0,\ 0,\ 1/4,\ 1/2,\ 1/2,\ 3/4,\ 3/4,\ 1,\ 1,\ 1\}.$$

The multiple knot B-splines of orders 3 and 6 associated with $\mathbf{t}_3^{(2)}$ are $\{B_{i,2,2}\}_{i=1}^{8}$ and $\{B_{i,5,2}\}_{i=1}^{5}$, respectively.

We denote by $V_{m,\mathbf{t}_m^j}$ the space generated by the multiple knot B-spline basis functions of order $m$ at level $j$ on $\mathbf{t}_m^{(j)}$, i.e., $V_{m,\mathbf{t}_m^j} := \operatorname{span}\{B_{i,m-1,j}\}_{i=1}^{S}$. In a particular case where

$$n_1 = n_2 = \cdots = n_{N_j-1} = n, \quad 1 \le n \le m-1,$$

the space is denoted by $V^n_{m,\mathbf{t}_m^j}$.

The next lemma gives an explicit rule of multiple knot B-splines in $V^{m-1}_{m,\mathbf{t}_m^j}$:

**Lemma 3.2.** *Let the space $V^{m-1}_{m,\mathbf{t}_m^j}$ be defined as above. The following two statements hold for* MKBS *basis functions in the space $V^{m-1}_{m,\mathbf{t}_m^j}$:*

(i) *If $t_i < t_{i+1} = \cdots = t_{i+m-1} < t_{i+m}$, then two real numbers $A$ and $B$ exist such that*

$$B_{i,m-1,j}(x) = \begin{cases} A(x-t_i)^{m-1} & t_i \le x < t_{i+1}, \\ B(t_{i+m}-x)^{m-1} & t_{i+1} \le x < t_{i+m}, \\ 0 & \textit{otherwise}. \end{cases} \tag{3.3}$$

(ii) *If $t_i = t_{i+1} = \cdots = t_{i+k} < t_{i+k+1} = t_{i+k+2} = \cdots = t_{i+m}$ for $k = 0, 1, \ldots, m-1$, then a real number $C$ exists such that*

$$B_{i,m-1,j}(x) = \begin{cases} C\binom{m-1}{k}(t_{i+m}-x)^k(x-t_i)^{m-k-1} & t_i \le x < t_{i+m}, \\ 0 & \textit{otherwise}, \end{cases} \tag{3.4}$$

*where $\binom{m-1}{k} := (m-1)!/k!(m-k-1)!$.*

*Proof.* First note that, for the knot vector $\{t_i, t_{i+1}, \ldots, t_{i+m}\}$, only two cases (i) and (ii) occur. The proof proceeds by induction with respect to $m$. For $m = 1$, there is nothing to prove.

Assume that the lemma is true for $m-1 \ge 2$. In order to prove (i), by the recursive relation (3.2), we have:

$$B_{i,m-1,j}(x) = \frac{x-t_i}{t_{i+m-1}-t_i} B_{i,m-2,j}(x) + \frac{t_{i+m}-x}{t_{i+m}-t_{i+1}} B_{i+1,m-2,j}(x). \tag{3.5}$$

On the other hand, by the induction assumption,

$$B_{i,m-2,j}(x) = \begin{cases} C_1(x-t_i)^{m-2} & t_i \le x < t_{i+m-1}, \\ 0 & otherwise. \end{cases} \tag{3.6}$$

and

$$B_{i+1,m-2,j}(x) = \begin{cases} C_2(t_{i+m}-x)^{m-2}, & t_{i+1} \le x < t_{i+m}, \\ 0 & otherwise, \end{cases} \tag{3.7}$$

hold. Now relation (i) is derived by substituting (3.6) and (3.7) in (3.5).

In order to prove (ii), by (3.2), we have:

$$\begin{aligned} &B_{i,m-1,j}(x) \\ &= \begin{cases} (x-t_i)/(t_{i+m-1}-t_i)B_{i,m-2,j}(x) & k=0, \\ (t_{i+m}-x)/(t_{i+m}-t_{i+1})B_{i+1,m-2,j}(x) & k=m-1, \\ (x-t_i)/(t_{i+m-1}-t_i)B_{i,m-2,j}(x) & \\ \quad +(t_{i+m}-x)/(t_{i+m}-t_{i+1})B_{i+1,m-2,j}(x) & 1 \le k \le m-2. \end{cases} \end{aligned}$$

For cases $k=0$ and $k=m-1$, by the induction assumption, we respectively have:

$$B_{i,m-2,j}(x) = \begin{cases} C_1(x-t_i)^{m-2} & t_i \le x < t_{i+m-1}, \\ 0 & otherwise, \end{cases} \tag{3.8}$$

and

$$B_{i+1,m-2,j}(x) = \begin{cases} C_2(t_{i+m}-x)^{m-2} & t_{i+1} \le x < t_{i+m}, \\ 0 & otherwise. \end{cases} \tag{3.9}$$

For $1 \le k \le m-2$, first note that $t_i = t_{i+1}$ and $t_{i+m-1} = t_{i+m}$. Second, by the induction assumption, we have:

$$\begin{aligned} &B_{i,m-2,j}(x) \\ &\quad = \begin{cases} C_1\binom{m-2}{k}(t_{i+m}-x)^k(x-t_i)^{m-k-2} & t_i \le x < t_{i+m}, \\ 0 & otherwise, \end{cases} \end{aligned} \tag{3.10}$$

and

$$B_{i+1,m-2,j}(x) = \begin{cases} C_2\binom{m-2}{k-1}(t_{i+m}-x)^{k-1}(x-t_i)^{m-k-1} & t_i \le x < t_{i+m}, \\ 0 & \textit{otherwise}. \end{cases} \tag{3.11}$$

Now, by substituting (3.10) and (3.11) in (3.5) and using Pascal's binomial theorem

$$\binom{m-2}{k-1} + \binom{m-2}{k} = \binom{m-1}{k},$$

statement (ii) is established. □

The next lemma identifies the properties of the space $V_{m,\mathbf{t}_m^j}$.

**Lemma 3.3.** *Let $S$ and $V_{m,\mathbf{t}_m^j}$ be given as before. Then,*

(i) $\{B_{i,m-1,j}\}_{i=1}^{S}$ *is linearly independent and*
(ii) $V_{m,\mathbf{t}_m^j} = \{f : [0,1] \to R | f|_{[t_k^j, t_{k+1}^j]} \in \Pi_{m-1},\ k = 0,1,\ldots,N_j-1$ *and* $f_-^{(L)}(t_k^j) = f_+^{(L)}(t_k^j)$ *for* $L = 0,1,\ldots,m-n_k-1$ *and* $k = 1,2,\ldots,N_j-1\}$, *in which* $f_-^{(L)}$ *and* $f_+^{(L)}$ *show the left and right* $L$*th derivatives of* $f$, *respectively.*

$L = 0$ *is defined by* $f_\pm^{(0)}(t_k^j) = \lim_{t\to t_k^{j\pm}} f(t)$. *Also,* $\Pi_{m-1}$ *denotes the set of polynomials with degree not exceeding* $m-1$.

A proof of this lemma may be found in some standard textbooks on spline theory, see for example, [**13,** Theorems 4.14, 4.18]. An immediate result of Lemma 3.3 is as follows:

**Corollary 3.4.** *Let $V_{m,\mathbf{t}_m^j}^{m-1}$ and $V_{m,\mathbf{t}_m^j}$ be given as before. Then,*

(i) *for any set* $\{n_k^j\}_{k=0}^{N_j}$ *where* $n_k^j \le m$, *the space* $V_{m,\mathbf{t}_m^j}$ *is a subspace of* $V_{m,\mathbf{t}_m^j}^{m-1}$, *i.e.,*

$$V_{m,\mathbf{t}_m^j} \subseteq V_{m,\mathbf{t}_m^j}^{m-1};$$

(ii) *for the knot sequences* $\mathbf{t}_m^j$ *and* $\mathbf{t}_m^{j+1}$,

$$V_{m,\mathbf{t}_m^j} \subset V_{m,\mathbf{t}_m^{j+1}},$$

*holds.*

Case (ii) follows from the fact that $N_j < N_{j+1}$ and $\mathbf{t}_m^j \subset \mathbf{t}_m^{j+1}$ with the multiplicities $\{n_k^j\}_{k=0}^{N_j}$ and $\{n_k^{j+1}\}_{k=0}^{N_{j+1}}$, such that $\{(t_k^j, n_k^j)\}_{k=0}^{N_j} \subset \{(t_k^{j+1}, n_k^{j+1})\}_{k=0}^{N_{j+1}}$. Since $V_{m,\mathbf{t}_m^j}$ and $V_{m,\mathbf{t}_m^{j+1}}$ are two closed subspaces, there exists a subspace $W_{m,\mathbf{t}_m^j}$ so that $V_{m,\mathbf{t}_m^{j+1}} = V_{m,\mathbf{t}_m^j} \oplus W_{m,\mathbf{t}_m^j}$. Hereafter, when there is no ambiguity, $V_{m,\mathbf{t}_m^j}$, $V_{m,\mathbf{t}_m^{j+1}}$ and $W_{m,\mathbf{t}_m^j}$ will be denoted by $V_j, V_{j+1}$ and $W_j$, respectively.

By Lemma 3.3 and Corollary 3.4, it turns out that

$$\dim V_{j+1} = \sum_{k=0}^{N_{j+1}-1} n_k^{j+1}, \qquad \dim V_j = \sum_{k=0}^{N_j-1} n_k^j.$$

Hence,

$$\dim W_j = \sum_{k=0}^{N_{j+1}-1} n_k^{j+1} - \sum_{k=0}^{N_j-1} n_k^j,$$

since $\dim W_j = \dim V_{j+1} - \dim V_j$. Also, it may be seen that a change on the knot points can be made to have the same dimension for $W_j$ and $V_j$.

**Remark 3.5.** Note that $\{V_j\}_{j\geq 0}$ does not satisfy concept (iv) of Definition 2.1. For instance, consider the knot set $\mathbf{t}_2^{(1)} = \{0, 0, 1/2, 1, 1\}$, $\mathbf{t}_2^{(2)} = \{0, 0, 1/4, 1/2, 1, 1\}$, and assume that the function $f$ is defined on $\mathbf{t}_2^{(1)}$ by

$$f(x) = \begin{cases} x - 1/2 & \text{if } 0 \leq x < 1/2, \\ 1/2 - x & \text{if } 1/2 \leq x < 1, \\ 0 & \textit{otherwise}. \end{cases} \tag{3.12}$$

By Lemma 3.3, it is clear that $f \in V^1_{2,\mathbf{t}_2^1}$; however, $f(2\cdot)$ does not belong

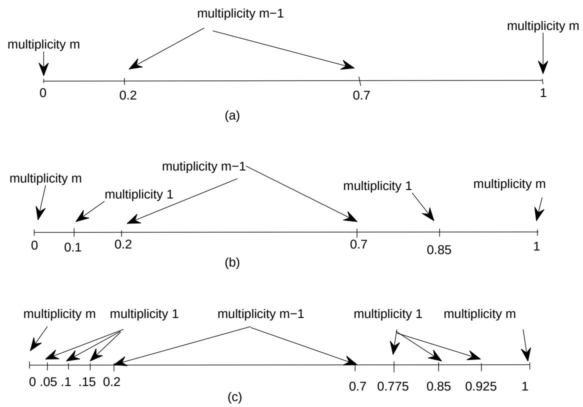


FIGURE 1. An illustration of knot vector on (a) level 1 or $j = 0$, (b) level 2 or $j = 1$ and (c) level 3 or $j = 2$.

to $V^1_{2,\mathbf{t}_2^2}$, since

$$f(2x) = \begin{cases} 2x - 1/2 & \text{if } 0 \le x < 1/4, \\ 1/2 - 2x & \text{if } 1/4 \le x < 1/2, \\ 0 & \textit{otherwise}, \end{cases} \tag{3.13}$$

that is not continuous in $1/2$.

Now we are ready to determine the nature of the space $W_j$. This is the subject of next section.

**4. Construction of MKBSW.** In this section, we present the construction of MKBSW. For this purpose, we introduce the following subspace:

$$S_{m,\mathbf{t}^{j+1}} = \{s \in V_{2m,\mathbf{t}_m^{j+1}} \mid s^{(L)}(t_k^j) = 0, \\ L = 0, 1, \ldots, n_k^j - 1,\ k = 1, \ldots, N_j - 1\},$$

in which $s^{(L)}$ shows the $L$th derivative of $s$. It is obvious that the zero function belongs to $S_{m,\mathbf{t}^{j+1}}$ which implies $S_{m,\mathbf{t}^{j+1}} \neq \emptyset$. An illustration of the knot vector from $j = 0$ to $j = 2$ is shown in Figure 1.

Now, we can state the following theorem:

**Theorem 4.1.** *Let $m = 2, 3, \ldots$ and $j \in \mathbb{N}$. Then the mth order differential operator $D^m$ maps the space $S_{m,\mathbf{t}^{j+1}}$ one-to-one onto the space $W_j$.*

*Proof.* Firstly, we prove that, for $s \in S_{m,\mathbf{t}^{j+1}}$,

$$D^m s = s^{(m)} \in W_j.$$

To this end, we show that

$$\text{(a)} \quad \mathrm{s}^{(\mathrm{m})} \in \mathrm{V}_{\mathrm{m},\mathbf{t}_{\mathrm{m}}^{\mathrm{j+1}}} \quad \text{and} \quad \text{(b)} \quad \mathrm{s}^{(\mathrm{m})} \perp \mathrm{V}_{\mathrm{m},\mathbf{t}_{\mathrm{m}}^{\mathrm{j}}}.$$

Next, $s^{(m)} \in V_{m,\mathbf{t}_m^{j+1}}$ is followed from a characterization of $V_{m,\mathbf{t}_m^{j+1}}$ given in Lemma 3.3 (b). In order to prove (b), we must show that for, each $B \in V_{m,\mathbf{t}_m^j}$,

$$\int_0^1 B(x) s^{(m)}(x)\, dx = 0.$$

If $1 \leq k \leq N_j - 1$, then $1 \leq n_k^j \leq m-1$ yields $0 \leq m - n_k^j - 1 \leq m-2$. Now, define $B^j := \{m - n_k^j - 1 \mid 1 \leq k \leq N_j - 1\}$. We can order the elements of $B^j$ as $B^j = \{b_i\}_{i=1}^r$ such that

$$b_1 < b_2 < \cdots < b_r.$$

Let $\Delta_c^j := \{t_k^j \mid 0 \leq k \leq N_j \text{ and } m - n_k^j - 1 \leq b_c\}$ where $c = 1, 2, \ldots, r$. Note that $\Delta_c^j \neq \emptyset$, for each $c$, since at least $0, 1$ and $\{t_k^j \mid 1 \leq k \leq N_j - 1 \text{ and } m - n_k^j - 1 = b_c\}$ belong to $\Delta_c^j$. Moreover, $\Delta_c^j$ can be denoted by $\Delta_c^j =: \{t_{\theta_i^c}\}_{i=1}^{\kappa_c}$ with

$$t_{\theta_1^c}^j < t_{\theta_2^c}^j < \cdots < t_{\theta_{\kappa_c}^c}^j,$$

where $\kappa_c$ denotes the cardinal number of $\Delta_c^j$. Also, note that $\Delta_c^j \subset \Delta_{c+1}^j$, $\Delta_r^j = \mathbf{t}_m^j$ and $\kappa_r = N_j + 1$. The most multiplicity of $t_k^{j+1}$ in $\mathbf{t}_m^{j+1}$ is $m$ due to endpoints 0 and 1. Therefore, by Lemma 3.3, it turns out that $s \in C^{m-1}(R)$, which implies $s^{(L)}(0) = s^{(L)}(1) = 0$ for $L = 0, 1, \ldots, m-1$. Furthermore, we have $B \in C^{b_1}([0, 1])$. By $(b_1 + 1)$

times integration by parts, it follows that

$$\int_0^1 B(x)s^{(m)}(x)\,dx = (-1)^{b_1+1}\int_0^1 B^{(b_1+1)}(x)s^{(m-b_1-1)}(x)\,dx \tag{4.1}$$
$$= (-1)^{b_1+1}\sum_{i=1}^{\kappa_1-1}\int_{t^j_{\theta^1_i}}^{t^j_{\theta^1_{i+1}}} B^{(b_1+1)}(x)s^{(m-b_1-1)}(x)\,dx.$$

By Lemma 3.3 and the definition of $\Delta^j_c$, we have $B \in C^{b_2}([t^j_{\theta^1_i}, t^j_{\theta^1_{i+1}}])$ for $i = 1, 2, \ldots, \kappa_1 - 1$. On the other hand, $m - n^j_k - 1 \le b_1$ for $t^j_k \in \Delta^j_1$, and thus, $m - b_1 - 2 \le n^j_k - 1$ implies

$$s^{(L)}(t^j_{\theta^1_i}) = 0, \quad L = 0, 1, \ldots, m - b_1 - 2,\ i = 1, 2, \ldots, \kappa_1,$$

by the definition of $S_{m,\mathbf{t}^{j+1}}$. Thus, $b_2 - b_1$ times integration by parts gives

$$\int_{t^j_{\theta^1_i}}^{t^j_{\theta^1_{i+1}}} B^{(b_1+1)}(x)s^{(m-b_1-1)}(x)\,dx \tag{4.2}$$
$$= (-1)^{b_2-b_1}\int_{t^j_{\theta^1_i}}^{t^j_{\theta^1_{i+1}}} B^{(b_2+1)}(x)s^{(m-b_2-1)}(x)\,dx.$$

Continuing this process $b_2 - b_1, b_3 - b_2, \ldots$ and $b_r - b_{r-1}$ times integration we have

$$\int_0^1 B(x)s^{(m)}(x)\,dx \tag{4.3}$$
$$= (-1)^{b_r+1}\sum_{i=1}^{N_j}\int_{t^j_{\theta^r_i}}^{t^j_{\theta^r_{i+1}}} B^{(b_r+1)}(x)s^{(m-b_r-1)}(x)\,dx.$$

In the case where $b_r = m - 2$, by (4.3), we have:

$$\int_0^1 B(x)s^{(m)}(x)\,dx = (-1)^{m-1}\sum_{i=1}^{N_j}\int_{t^j_{\theta^r_i}}^{t^j_{\theta^r_{i+1}}} B^{(m-1)}(x)s'(x)\,dx. \tag{4.4}$$

If $b_r < m-2$, applying $m-b_r-2$ times integration by parts,

(4.5)
$$\begin{aligned}\int_0^1 B(x)s^{(m)}(x)\,dx &= (-1)^{b_r+1}\sum_{i=1}^{N_j}\int_{t^j_{\theta^r_i}}^{t^j_{\theta^r_{i+1}}} B^{(b_r+1)}(x)s^{(m-b_r-1)}(x)\,dx\\ &= (-1)^{m-1}\sum_{i=1}^{N_j}\int_{t^j_{\theta^r_i}}^{t^j_{\theta^r_{i+1}}} B^{(m-1)}(x)s'(x)\,dx,\end{aligned}$$

which is the same as (4.4). In addition, $B^{(m-1)}$ is a piecewise constant polynomial, i.e.,

$$B^{(m-1)} \equiv c_i \quad \text{in } [t^j_{\theta^r_i},\ t^j_{\theta^r_{i+1}}],$$

for $i = 1, 2, \ldots, N_j$ where $c_i$ is a real constant. Also, $s$ vanishes at the endpoints of $[t^j_{\theta^r_i},\ t^j_{\theta^r_{i+1}}]$. Then, for $i = 1, 2, \ldots, N_j$, it follows that

$$(4.6)\qquad \begin{aligned}\int_{t^j_{\theta^r_i}}^{t^j_{\theta^r_{i+1}}} B^{(m-1)}(x)s'(x)\,dx &= c_i\int_{t^j_{\theta^r_i}}^{t^j_{\theta^r_{i+1}}} s'(x)\,dx\\ &= c_i[s(t^j_{\theta^r_i}) - s(t^j_{\theta^r_{i+1}})] = 0.\end{aligned}$$

Consequently,

$$\int_0^1 B(x)s^{(m)}(x)\,dx = 0$$

establishes (b).

Next, we show that $D^m$ is an injection operator. Let $s^{(m)} = 0$. Then

$$s|_{[t^{j+1}_k, t^{j+1}_{k+1}]} \in \Pi_{m-1}, \quad k = 0, 1, \ldots, N_j - 1.$$

We know that $s^{(L)}(0) = 0$ for $L = 0, 1, \ldots, m-1$. Thus,

$$s|_{[0, t^{j+1}_1]} = 0.$$

Furthermore $s \in C^{m-1}(\mathbb{R})$ that gives $s^{(L)}(t^{j+1}_1) = 0$ for $L = 0, 1, \ldots, m-1$. Therefore,

$$s|_{[t^{j+1}_1, t^{j+1}_2]} = 0.$$

Continuing this process for $k = 2, 3, \ldots, N_j - 1$, we can conclude that $s = 0$. Thus, $D^m$ is an injective mapping.

Now, as previously mentioned,

$$\dim V_{2m,\mathbf{t}_m^{j+1}} = \sum_{k=0}^{N_{j+1}-1} n_k^{j+1}.$$

Also, the conditions

$$s^{(L)}(t_k^j) = 0, \quad L = 0, 1, \ldots, n_k^j - 1,\ k = 1, 2, \ldots, N_j - 1,$$

are independent, which yields

$$\dim S_{m,\mathbf{t}_m^{j+1}} = \sum_{k=0}^{N_{j+1}-1} n_k^{j+1} - \sum_{k=0}^{N_j-1} n_k^j. \tag{4.7}$$

On the other hand,

$$\dim W_j = \sum_{k=0}^{N_{j+1}-1} n_k^{j+1} - \sum_{k=0}^{N_j-1} n_k^j,$$

which yields the mapping is onto. This completes the proof. □

Now, the fundamental question is: by Theorem 4.1, for any $\psi$ belonging to $W_j$, there exists an $s \in S_{m,\mathbf{t}_m^{j+1}}$ such that $D^m s = \psi$. The question is then how to explicitly identify the members of $W_j$?

By the definition of $S_{m,\mathbf{t}_m^{j+1}}$ and Corollary 3.4 (i), we have $S_{m,\mathbf{t}_m^{j+1}} \subset V^{2m-1}_{2m,t_m^{j+1}}$. Then, by Lemma 3.3 (ii), we have

$$s\,|_{[t_k^{j+1}, t_{k+1}^{j+1}]} \in \Pi_{2m-1},$$

for $k = 0, 1, \ldots, N_{j+1} - 1$. We assume that

$$s\,|_{[t_{2k}^{j+1}, t_{2k+1}^{j+1}]} = P_k, \qquad s\,|_{[t_{2k+1}^{j+1}, t_{2k+2}^{j+1}]} = Q_k, \quad k = 0, 1, \ldots, N_j - 1.$$

We consider the particular case that, for any $t_k^j$, $k = 1, 2, \ldots, N_j - 1$, there exists a $t_{k'}^{j+1}$ such that $t_k^j < t_{k'}^{j+1} = (t_k^j + t_{k+1}^j)/2 < t_{k+1}^j$. Now, noting that $n_{2k}^{j+1} = n_k^j$ and $N_{j+1} = 2N_j$, we have:

$$s^{(L)}(t_{2k}^{j+1}) = 0, \quad L = 0, 1, \ldots, n_{2k} - 1,\ k = 1, 2, \ldots, N_j - 1.$$

Then, by Lemmas 3.2 and 3.3 (i) for $k = 0, 1, \ldots, N_j - 1$, there exist two sets $\{\alpha_{k,i}\}_{i=0}^{2m-n_{2k}-1}$ and $\{\beta_{k,i}\}_{i=0}^{2m-n_{2k+2}-1}$ such that

$$P_k(x) = \sum_{i=0}^{2m-n_{2k}-1} \alpha_{k,i}(x - t_{2k}^{j+1})^{2m-i-1}(t_{2k+1}^{j+1} - x)^i, \tag{4.8}$$

and

$$Q_k(x) = \sum_{i=0}^{2m-n_{2k+2}-1} \beta_{k,i}(t_{2k+2}^{j+1} - x)^{2m-i-1}(x - t_{2k+1}^{j+1})^i \tag{4.9}$$

hold. Then, we have

$$\sum_{k=0}^{N_j-1} (4m - n_{2k} - n_{2k+2})$$

unknowns which should be identified. In order to form the corresponding equations, we first note that

$$s_-^{(L)}(t_k^{j+1}) = s_+^{(L)}(t_k^{j+1}) \quad \text{for } L = 0, 1, \ldots, 2m - n_k - 1,$$

and $k = 1, 2, \ldots, N_{j+1} - 1$. Thus, we have the following equations:

$$P_k^{(L)}(t_{2k+1}^{j+1}) = Q_k^{(L)}(t_{2k+1}^{j+1}), \quad L = 0, 1, \ldots, 2m - n_{2k+1} - 1, \quad k = 0, 1, \ldots, N_j - 1, \tag{4.10}$$

$$Q_k^{(L)}(t_{2k+2}^{j+1}) = P_{k+1}^{(L)}(t_{2k+2}^{j+1}), \quad L = 0, 1, \ldots, 2m - n_{2k+2} - 1, \quad k = 0, 1, \ldots, N_j - 2. \tag{4.11}$$

On the other hand, for $k = 0, 1, \ldots, N_j - 2$,

$$Q_k^{(L)}(t_{2k+2}^{j+1}) = P_{k+1}^{(L)}(t_{2k+2}^{j+1}) = 0, \quad L = 0, 1, \ldots, n_{2k+2} - 1.$$

Then, the second relation of (4.10) reduces as:

$$\begin{gathered} Q_k^{(L)}(t_{2k+2}^{j+1}) = P_{k+1}^{(L)}(t_{2k+2}^{j+1}) \\ L = n_{2k+2}, n_{2k+2} + 1, \ldots, 2m - n_{2k+2} - 1, \\ k = 0, 1, \ldots, N_j - 2. \end{gathered}$$

Consequently, we obtain a system with

$$\sum_{k=0}^{N_j-1}(4m - n_{2k} - n_{2k+2})$$

unknowns and

$$\sum_{k=0}^{N_j-1}(2m - n_{2k+1}) + \sum_{k=0}^{N_j-2}(2m - 2n_{2k+2})$$

equations. Since

$$\sum_{k=0}^{N_j-1}(n_{2k} + n_{2k+2}) = n_0 + 2\sum_{k=0}^{N_j-2} n_{2k+2} + n_{N_{j+1}} = 2m + 2\sum_{k=0}^{N_j-2} n_{2k+2},$$

then there exist $\sum_{k=0}^{N_j-1} n_{2k+1}$ degrees of freedom. We can form more explicitly the system of equations. For this, we note the following simple proposition:

**Proposition 4.2.** *Let $g(x) = ax + c$ and $h(x) = bx + d$ be two linear functions in which $a$ and $b$ are equal to $1$ or $-1$ and $c$ and $d$ are arbitrary real numbers. Setting $f = g^m h^n$ for $m, n \in \mathbb{Z}^+$, the $L$th derivative of $f$ satisfies the following*:

$$f^{(L)}(x) = (L!)\sum_{i=0}^{L} a^i b^{L-i}\binom{m}{i}\binom{n}{L-i} g^{m-i}(x) h^{n-L+i}(x), \tag{4.12}$$

*where $g^q = 0$ and $h^q = 0$ if $q < 0$.*

Now, by Proposition 4.2, for $k = 0, 1, \ldots, N_j - 1$, we have:

$$\begin{aligned} P_k^{(L)}(x) &= L! \sum_{i=0}^{2m-n_{2k}-1} \sum_{q=0}^{L} (-1)^{L-q}\binom{2m-i-1}{q}\binom{i}{L-q}\alpha_{k,i} p(x), \\ Q_k^{(L)}(x) &= L! \sum_{i=0}^{2m-n_{2k+2}-1} \sum_{q=0}^{L} (-1)^{q}\binom{2m-i-1}{q}\binom{i}{L-q}\beta_{k,i} q(x), \end{aligned} \tag{4.13}$$

where

$$p(x) = (x - t_{2k}^{j+1})^{2m-i-q-1}(t_{2k+1}^{j+1} - x)^{i+q-L},$$

and

$$q(x) = (t^{j+1}_{2k+2} - x)^{2m-i-q-1}(x - t^{j+1}_{2k+1})^{i+q-L}.$$

We define the index sets of the wavelet by

$$\mathcal{J}_j^{(1)} := \{(k,i)|\ 0 \le k \le N_j - 1, 0 \le i \le 2m - n_{2k} - 1\}$$

and

$$\mathcal{J}_j^{(2)} := \{(k,i)|\ 0 \le k \le N_j - 1, 0 \le i \le 2m - n_{2k+2} - 1\}.$$

Now by (4.10) and (4.13), we can write:

$$(\tau^j_{2k})^{2m-L-1}\sum_{i=0}^{\xi_k}(-1)^i\binom{2m-i-1}{L-i}\alpha_{k,i} = \sum_{i=0}^{\xi_{k+1}}(-1)^{L-i}\binom{2m-i-1}{L-i}\beta_{k,i}, \tag{4.14}$$

for $0 \le L \le 2m - n_{2k+1} - 1$, $0 \le k \le N_j - 1$, where $\xi_k = \min\{2m - n_{2k} - 1, L\}$, $\tau^j_k = (t^{j+1}_{k+1} - t^{j+1}_k)/(t^{j+1}_{k+2} - t^{j+1}_{k+1})$ and

$$\sum_{i=2m-L-1}^{2m-n_{2k+2}-1}\binom{i}{L-2m+i+1} \times ((-1)^i(\tau^j_{2k+1})^{2m-L-1}\beta_{k,i} + (-1)^{L-i}\alpha_{k+1,i}) = 0, \tag{4.15}$$

for $n_{2k+2} \le L \le 2m - n_{2k+2} - 1$, $0 \le k \le N_j - 2$, where $\{\alpha_{k,i}\}_{(k,i)\in\mathcal{J}_j^{(1)}}$ and $\{\beta_{k,i}\}_{(k,i)\in\mathcal{J}_j^{(2)}}$ are unknown values. We know that the maximum value of $n_{2k}$ is $m$ and $n_{2k+1} \le m - 1$ for $k = 0, 1, \ldots, N_j - 1$. Then $n_{2k+1} - 1 < 2m - n_{2k} - 1$.

On one hand, by Theorem 4.1, for any $\psi \in W_j$, there exists an $s \in S_{m,\mathbf{t}^{j+1}_m}$. Conversely, the coefficients $\alpha_{k,i}$ and $\beta_{k,i}$ are handled to form the function $s$. Therefore, the existence of the solution for the system comprised by (4.14) and (4.15) is guaranteed.

Now, we find a solution for the system given by (4.14) and (4.15) with a free selection of $\{\alpha_{k,i}\}_{k,i}$ for $k = 0, 1, \ldots, N_j - 1$ and $i = 0, 1, \ldots, n_{2k+1} - 1$. For $k^* = 0, 1, \ldots, N_j - 1$ and $i^* = 0, 1, \ldots, n_{2k+1} - 1$, we define the set $\Lambda_{k^*,i^*}$ as

$$\Lambda_{k^*,i^*} := \{\delta_{k,k^*}\delta_{i,i^*}\}_{0\le k\le N_j-1, 0\le i\le n_{2k+1}-1}$$

in which $\delta$ is the Kronecker delta defined by

$$\delta_{i,j} = \begin{cases} 1 & i=j, \\ 0 & i\neq j. \end{cases}$$

Let us fix $k^*$ and $i^*$. By defining

$$\mathcal{J}_j^* := \{(k,i) \mid 0\le k\le N_j-1, 0\le i\le n_{2k+1}-1\},$$

and setting $\{\alpha_{k,i}\}_{(k,i)\in\mathcal{J}_j^*} := \Lambda_{k^*,i^*}$, a full system in terms of

$$\{\alpha_{k,i}\}_{0\le k\le N_j-1, n_{2k+1}\le i\le 2m-n_{2k}-1} \quad \text{and} \quad \{\beta_{k,i}\}_{(k,i)\in\mathcal{J}_j^{(2)}}$$

is obtained from which a unique solution is derived. We represent this solution by the sets $\{\alpha_{k,i}^{\Lambda_{k^*,i^*}}\}_{(k,i)\in\mathcal{J}_j^{(1)}}$ and $\{\beta_{k,i}^{\Lambda_{k^*,i^*}}\}_{(k,i)\in\mathcal{J}_j^{(2)}}$. Hence, for each selection of $k^*$ and $i^*$, a solution is determined.

The set $\{s^{\Lambda_{k^*,i^*}}\}_{(k^*,i^*)\in\mathcal{J}_j^*}$ may be thought of as a set of polynomials specified by substituting $\{\alpha_{k,i}^{\Lambda_{k^*,i^*}}\}_{(k,i)\in\mathcal{J}_j^{(1)}}$ and $\{\beta_{k,i}^{\Lambda_{k^*,i^*}}\}_{(k,i)\in\mathcal{J}_j^{(2)}}$ in the relations (4.8) and (4.9), respectively. Now, we will show that the set of $\{s^{\Lambda_{k^*,i^*}}\}_{(k^*,i^*)\in\mathcal{J}_j^*}$ is independent. For this, assume that there exists an $\{A_{k^*,i^*}\}_{(k^*,i^*)\in\mathcal{J}_j^*}$ such that

$$S(x) = \sum_{(k^*,i^*)\in\mathcal{J}_j^*} A_{k^*,i^*} s^{\Lambda_{k^*,i^*}}(x) \equiv 0. \tag{4.16}$$

By (4.8) and (4.9), the polynomials of degree $2m-1$, $P_k^{\Lambda_{k^*,i^*}}$ and $Q_k^{\Lambda_{k^*,i^*}}$ can be considered such that

$$s^{\Lambda_{k^*,i^*}}\,|_{[t_{2k}^{j+1}, t_{2k+1}^{j+1}]} = P_k^{\Lambda_{k^*,i^*}},$$
$$s^{\Lambda_{k^*,i^*}}\,|_{[t_{2k+1}^{j+1}, t_{2k+2}^{j+1}]} = Q_k^{\Lambda_{k^*,i^*}}, \quad k=0,1,\dots,N_j-1.$$

Setting $\theta := t_{2k+1}^{j+1}$, for $k=0,1,\dots,N_j-1$, we have

$$\lim_{x\to\theta^-} S(x) = 0,$$

which yields

$$\lim_{x\to\theta^-} \sum_{(k^*,i^*)\in\mathcal{J}_j^*} A_{k^*,i^*} P_k^{\Lambda_{k^*,i^*}} = 0, \quad k = 0,1,\ldots,N_j-1.$$

Then,

$$\lim_{x\to\theta^-} \sum_{(k^*,i^*)\in\mathcal{J}_j^*} A_{k^*,i^*} \sum_{i=0}^{2m-n_{2k}-1} \alpha_{k,i}^{\Lambda_{k^*,i^*}} (x-t_{2k}^{j+1})^{2m-i-1}(t_{2k+1}^{j+1}-x)^i = 0,$$

for $k = 0,1,\ldots,N_j-1$. Thus,

$$\sum_{(k^*,i^*)\in\mathcal{J}_j^*} A_{k^*,i^*} \alpha_{k,0}^{\Lambda_{k^*,i^*}} = 0, \quad k = 0,1,\ldots,N_j-1. \tag{4.17}$$

Since $\alpha_{k,0}^{\Lambda_{k^*,i^*}} = \delta_{k,k^*}\delta_{0,i^*}$, relation (4.17) implies that $A_{k,0} = 0$ for $k = 0,1,\ldots,N_j-1$.

On the other hand, by differentiation of (4.16), we have

$$\frac{dS(x)}{dx} = \sum_{(k^*,i^*)\in\mathcal{J}_j^*} A_{k^*,i^*} \frac{ds^{\Lambda_{k^*,i^*}}(x)}{dx} \equiv 0. \tag{4.18}$$

Now, similar to the above operation, for $[ds^{\Lambda_{k^*,i^*}}(x)]/dx$, (4.18) clearly gives $A_{k,1} = 0$, for $k = 0,1,\ldots,N_j-1$, if $n_{2k+1} > 1$. Applying the consecutive derivatives and continuing this process for $i = 2,3,\ldots,n_{2k+1}-1$, implies that

$$A_{k^*,i^*} = 0, \quad (k^*,i^*) \in \mathcal{J}_j^*;$$

hence, $\{s^{\Lambda_{k^*,i^*}}\}_{(k^*,i^*)\in\mathcal{J}_j^*}$ is linearly independent. Thus, we proved the next theorem:

**Theorem 4.3.** *Let $j \geq 0$ be an integer number. For any $\{(k^*,i^*)\}_{(k^*,i^*)\in\mathcal{J}_j^*}$, there exist two sets $\{\alpha_{k,i}^{\Lambda_{k^*,i^*}}\}_{(k,i)\in\mathcal{J}_j^*}$ and $\{\beta_{k,i}^{\Lambda_{k^*,i^*}}\}_{(k,i)\in\mathcal{J}_j^*}$ such that*

$\{\psi_{(k^*,i^*),j}\}_{(k^*,i^*)\in\mathcal{J}_j^*}$, *defined by*

$$\psi_{(k^*,i^*),j}(x)=\begin{cases}\displaystyle\sum_{i=0}^{2m-n_{2k}-1}\sum_{q=0}^{m}\alpha_{k,i}^{\Lambda_{k^*,i^*}}F_{i,q}^{k,j}(x) & t_{2k}^{j+1}\le x<t_{2k+1}^{j+1},\\ \displaystyle\sum_{i=0}^{2m-n_{2k+2}-1}\sum_{q=0}^{m}\beta_{k,i}^{\Lambda_{k^*,i^*}}G_{i,q}^{k,j}(x) & t_{2k+1}^{j+1}\le x<t_{2k+2}^{j+1},\end{cases}$$

*for* $k=0,1,\ldots,N_j-1$, *constitutes a basis for the wavelet space* $W_j$, *where*

$$F_{i,q}^{k,j}(x)=(-1)^{m-q}\binom{2m-i-1}{q}\binom{i}{m-q}(x-t_{2k}^{j+1})^{2m-i-q-1}(t_{2k+1}^{j+1}-x)^{i+q-m},$$

*and*

$$G_{i,q}^{k,j}(x)=(-1)^{q}\binom{2m-i-1}{q}\binom{i}{m-q}(t_{2k+2}^{j+1}-x)^{2m-i-q-1}(x-t_{2k+1}^{j+1})^{i+q-m}.$$

Theorem 4.3 can be stated in a simpler form: *Let* $j\ge 0$ *and* $m\ge 1$ *be integer numbers, and let* $\{n_k\}_{k=0}^{N_j}$ *be given as before. For* $h=1,2,\ldots,N$ *such that*

$$N=\sum_{k=0}^{N_j-1}n_{2k+1},$$

*there exist two sets* $\{\alpha_{k,i}^h\}_{i=0}^{m-1}$ *and* $\{\beta_{k,i}^h\}_{i=0}^{m-1}$ *such that* $\{\psi_h\}_{h=1}^{N}$, *defined by*

(4.19)
$$\psi_h(x)=\sum_{k=0}^{N_j-1}\sum_{i=0}^{m-1}(\alpha_{k,i}^hF_{k,i}(x)\chi_{[t_{2k}^{j+1},t_{2k+1}^{j+1})}+\beta_{k,i}^hG_{k,i}(x)\chi_{[t_{2k+1}^{j+1},t_{2k+2}^{j+1})})$$

*constitutes a basis for the wavelet space* $W_j$, *where*

$$F_{k,i}(x)=(x-t_{2k}^{j+1})^{m-i-1}(t_{2k+1}^{j+1}-x)^{i}, \tag{4.20}$$

*and*

$$G_{k,i}(x) = (t^{j+1}_{2k+2} - x)^{m-1}(x - t^{j+1}_{2k+1})^i. \tag{4.21}$$

**4.1. Minimal support.** The MKBSW defined by Theorem 4.3 can be constructed with minimal support. In fact, in order to have the wavelet $\psi_h$ with minimal support, we solve the following optimization problem:

$$\begin{aligned} &\max\ p - q \\ &\text{s.t. } \text{System (4.14)}, \\ &\qquad \text{System (4.15)}, \\ &\qquad \int_0^p |\psi_h| + \int_q^1 |\psi_h| = 0, \\ &\qquad 0 \le p \le q \le 1. \end{aligned} \tag{4.22}$$

This problem yields the coefficients $p, q, \alpha^h_{k,i}$ and $\beta^h_{k,i}$ for $k = 0, \dots, 2^j - 1$ and $i = 0, \dots, m-1$. The interval $[p, q]$ would be the minimal support of $\psi_h$. Many methods exist for solving this optimization problem. For instance, an efficient heuristic method is the Particle Swarm Optimization (PSO) that can be rapidly converged ([**5**]).

Taking the optimization problem (4.22) into account, an algorithm for deriving the wavelet basis functions can be given as follows:

**Algorithm 1.** `Construction of Multiple Knot B-Spline Wavelet.`

(1) `INPUT: multiple knots, level` $j$ `and order of B-spline` $m$.
(2) `OUTPUT: The wavelet basis functions` $\psi_h$.
(3) `Solve` 4.22;
(4) `Compute` $\psi_h$ `by` (4.19).

**4.2. Description and discussion.** Theorem 4.3 presents new multiple knot B-spline wavelets on the interval $[0, 1]$ that contain the following properties:

(i) Let the degree of B-spline $m$, the level $j$ and the knot set $\mathbf{t}_m^{(j)}$ be fixed. Every selection of $\{\Lambda_{k^*,i^*}\}_{(k^*,i^*)\in\mathcal{J}_j^*}$ gives a new wavelet basis for $W_j$. For instance, we can take $\Lambda_{k^*,i^*} = \{\alpha_{k,i}\}_{(k,i)\in\mathcal{J}_j^*}$ with $\alpha_{k,i} = c_{k,i}\delta_{k,k^*}\delta_{i,i^*}$ in which $\{c_{k,i}\}_{k,i}$ are arbitrary real numbers.

(ii) The free selection of multiplicities for the knot points makes it the flexible and efficient wavelet in many applications.

(iii) The form of the basis functions is explicit and, in addition, their derivatives of any order would be easily available. Moreover, the structure of the wavelet is based upon polynomials; therefore, it is conceptually simple and easy to implement.

(iv) Although the basis functions of the space $W_j^{[0,1]}$ are not refinable, in the case of linear knots $k2^{-j}$, they can be represented as follows:

$$\psi_h = 2^{(-j-1)(m-1)} \sum_{k=0}^{2^j-1} \sum_{i=0}^{m-1} (\alpha_{k,i}^h \psi_{m,i}^1(2^j\cdot -k) + \beta_{k,i}^h \psi_{m,i}^2(2^j\cdot -k))$$

where

$$\psi_{m,i}^1 = (2\cdot)^{m-1}(1-2\cdot)^i \chi_{[0,1/2)}(\cdot),$$

and

$$\psi_{m,i}^2 = (2-2\cdot)^{m-1}(2\cdot -1)^i \chi_{[1/2,1)}(\cdot).$$

(v) As the level $j$ increases, the nonzero values of $\alpha_{k,i}^{\Lambda_{k^*,i^*}}$ and $\beta_{k,i}^{\Lambda_{k^*,i^*}}$ will decrease, see Section 5.

(vi) *Boundary adapted* and *boundary symmetric* are two important features of the basis functions of a wavelet space that are noteworthy. These are important, for example when we are solving a boundary value problem. We recall a system of functions $F := \{f_1, f_2, \ldots, f_M\}$, $f_i : [0,1] \to R$,

(a) boundary adapted, if

$$f_i(0) \neq 0 \quad \text{for } i = 1, 2, \ldots, k$$

and

$$f_i(1) \neq 0 \quad \text{for } i = M, M-1 \ldots, M-k+1;$$

(b) boundary symmetric, if

$$f_i(0) = f_{M-i+1}(1) \quad \text{for } i = 1, 2, \ldots, M.$$

For more details, see **[15]**. The number of wavelet basis functions which are nonzero on the endpoints 0 and 1 are at least $n_1$ and $n_{2^{j+1}-1}$, respectively. Furthermore, the wavelet bases generally are not boundary symmetric. Neither are they boundary adapted except when the multiplicities of the mid knots are equal. In addition, in general, the bases also do not have any axis of symmetry. Moreover, in Figures 2, 3 and 4, in which the multiplicities of the middle knots are fixed and equal to $n$, it appears that there exist $n$ bases which the other bases are a translation thereof, i.e., we have $2^j - 1$ translation from any $n$ basis functions.

**5. Numerical examples.** In this section, some numerical examples are presented to show the graph of the basis constructed in the preceding section. To this end, the different levels of $j$ with different orders of B-splines $m$ along with a variety of knot set $\{\mathbf{t}_m^{(j)}\}$ are considered.

Figure 2 shows $\psi_{(k^*,i^*),3}$ with $m = 6$ and the knot multiplicity of $\{n_0 = 6, n_1 = 3, n_2 = 4, n_3 = 2, n_4 = 5, n_5 = 3, n_6 = 5, n_7 = 2, n_8 = 6\}$, while Figures 3 and 4 show $\psi_{(k^*,i^*),4}$ with $m = 6$ and the knot multiplicity of

$$\{n_0 = 6, n_1 = 5, n_2 = 3, n_3 = 2, n_4 = 4, n_5 = 1, n_6 = 2, \\ n_7 = 3, n_8 = 5, n_9 = 1, n_{10} = 3, n_{11} = 3, \\ n_{12} = 5, n_{13} = 4, n_{14} = 2, n_{15} = 4, n_{16} = 6\}.$$

As expected, the lengths of the supports of $\psi_{(k,i),4}$ for $(k,i) \in \mathcal{J}_4^*$ are shorter than those of $\psi_{(k,i),3}$ for $(k,i) \in \mathcal{J}_3^*$. Also, Figures 5, 6 and 7 show the wavelet basis functions of $\psi_{(k,i),2}$ for $(k,i) \in \mathcal{J}_2^*$ on the spaces $V^1_{4,\mathbf{t}_4^2}$, $V^2_{4,\mathbf{t}_4^2}$ and $V^3_{4,\mathbf{t}_4^2}$, respectively. Figure 8 shows the bases $\{\psi_{(k,i),2}\}_{(k,i)\in\mathcal{J}_2^*}$ on the space $V^3_{4,\mathbf{t}_4^2}$ for the different selections of $\Lambda_{k^*,i^*} = \{\alpha_{k,i}\}_{(k,i)\in\mathcal{J}_2^*}$, i.e., $\alpha_{k,i} = c_{k^*,i^*}\delta_{k,k^*}\delta_{i,i^*}$ with $\{c_{k^*,i^*}\} = \{-1, 0.1, 2, 1, 5, -3, 2, 2.5, -0.75, 4, 9, -0.2\}$. Finally, Figure 9 shows

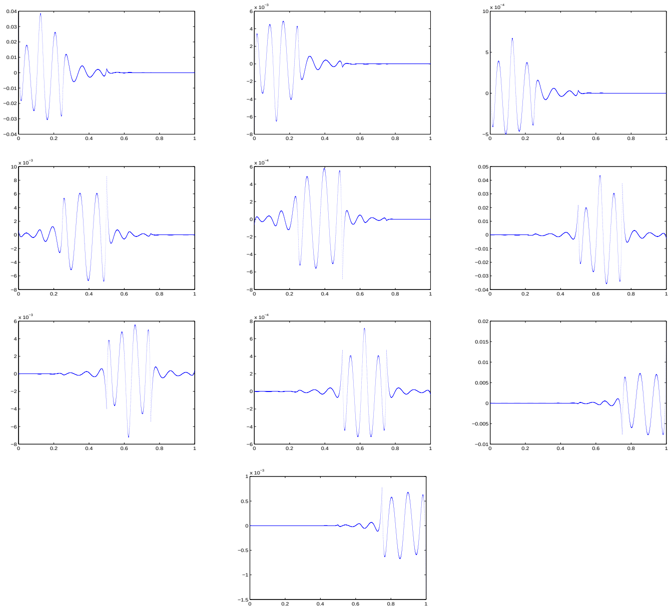

FIGURE 2. $\psi_{(k,i),3}$, $0 \le k \le 3$, $0 \le i \le n_{2k+1}-1$ and $m=6$.

$\psi_{(k,i),j}$ with $m=3$, $j=2$, first knots 0.4 and 0.87 the knot multiplicity of $\{n_0=3, n_1=1, n_2=1, n_3=1, n_4=2, n_5=1, n_6=1, n_7=1, n_8=2, n_9=1, n_{10}=1, n_{11}=1, n_{12}=3\}$, respectively. No symmetry is observed in Figures 2, 3, 4 and 8. Figures 5, 6 and 7 show the wavelet basis functions that are boundary adapted, but they do not have boundary nor axis symmetry. Note that, for any $f \in L^2([0,1])$, we have

$$f = \sum_k \langle f, \varphi_{j,k}\rangle \widetilde{\varphi}_{j,k} + \sum_{j \ge j_0} \sum_k \langle f, \psi_{j,k}\rangle \widetilde{\psi}_{j,k},$$

where $\widetilde{\varphi}_{j,k}$ and $\widetilde{\psi}_{j,k}$ are dual scaling functions and dual wavelets, respectively. Hence, when the wavelets $\psi_{j,k}$ are of small magnitude, the corresponding dual wavelet $\widetilde{\psi}_{j,k}$ should be of large s magnitude to approximate $f$ well.

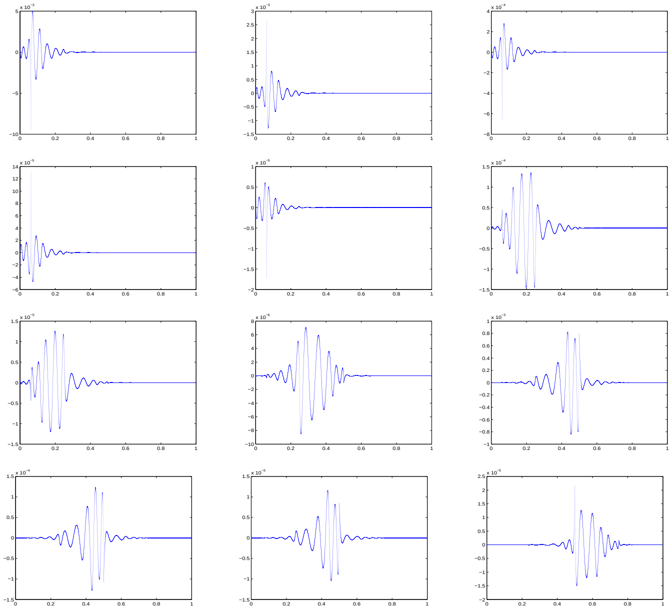

FIGURE 3. Some basis functions of $\psi_{(k,i),4}$, $0 \leq k \leq 7$, $0 \leq i \leq n_{2k+1} - 1$ and $m = 6$.

**6. Adaptive approximation.** We know that wavelet compression allows us to obtain a sparse representation of a function $f$ which is smooth, except at some isolated singularities, in the sense that most of the wavelet coefficients are small and hence can be neglected, see [**3**]. The level $j$ usually should be large enough to lay the functions such as $f$ in the multi-resolution space $V_j$. However, this is generally not the case for the multi-resolution space generated by MKBS basis functions, since it can contain the basis functions that are not smooth in the singularities points. Hence, a better approximation of $f$ is expected when we use MKBS basis functions. In addition, $f$ can be approximated by keeping only a small number of wavelet coefficients over the other wavelets and significantly reduce the complexity of the description of $f$ without affecting its accuracy.

In this section, we present two examples whose functions are smooth, except at some isolated singularities, and are approximated

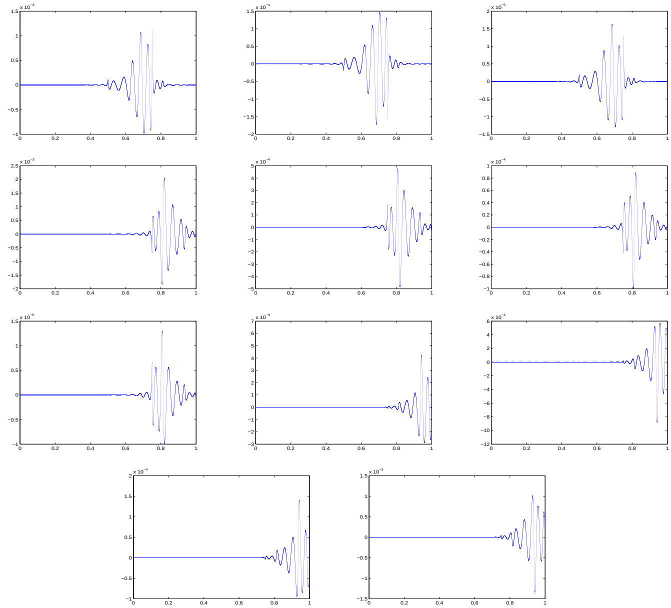

FIGURE 4. The rest basis functions of $\psi_{(k,i),4}$, $0 \leq k \leq 7$, $0 \leq i \leq n_{2k+1} - 1$ and $m = 6$.

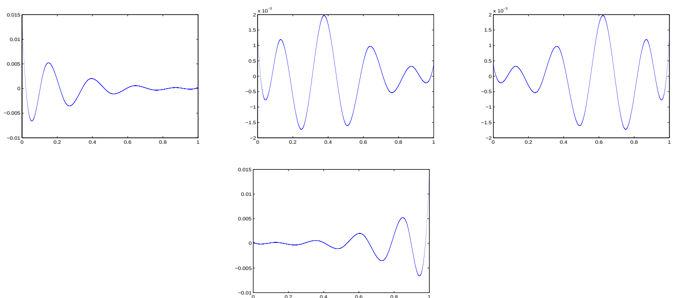

FIGURE 5. $\psi_{(0,0),2}$, $\psi_{(1,0),2}$, $\psi_{(2,0),2}$, $\psi_{(3,0),2}$ in the space $V^1_{4,2}$.

by MKBSW and some other wavelets. The functions of both examples are approximated at the level $j = 9$. For each example, a table is given that shows a comparison of errors ($\epsilon$) and the number of zeros of

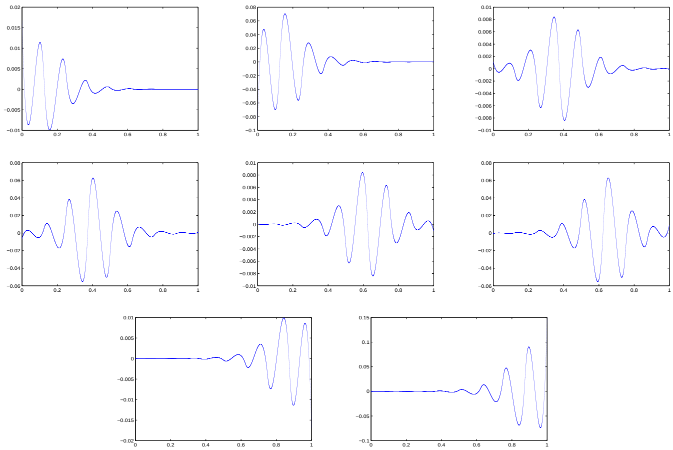

FIGURE 6. $\psi_{(k,i),j}$, $0 \le k \le 3$, $0 \le i \le 1$ in the space $V^2_{4,2}$.

detail coefficients in percent between MKBSW and the other wavelets up to level $j = 9$. In the first column of the tables, dbN denotes the Daubechies family wavelets, where $N$ is the order, and db the "surname" of the wavelet. Also, *coif* and *bior* show the coiflet wavelets and biorthogonal spline wavelets, respectively. The error is taken by $\|.\|_2$.

**Example 1.** For the first example, we consider the function $f : [0,1] \to \mathbb{R}$, defined by

$$f(x) = \begin{cases} x^7 - 3x^5 + 15x^2 & x \in [0, 1/2), \\ 5x^7 - 2x^5 + x + 3.1875 & x \in [1/2, 1]. \end{cases} \tag{6.1}$$

Figure 9 shows the function $f$ that contains a singularity in $x = 1/2$. Function $f$ is only continuous at this point and there is no more smoothness. Hence, the multiple knot function with multiplicity $n = 7$ is considered. Moreover, we use hard global thresholding thr $= 0.001$. A comparison of the errors and the number of zeros in Table 1 is given. As expected, the MKBSW has a lower error and more zeros for the detail coefficients. The number of zeros in percent is few for Haar and db2 while it is prolific for MKBSW and the other wavelets. Note

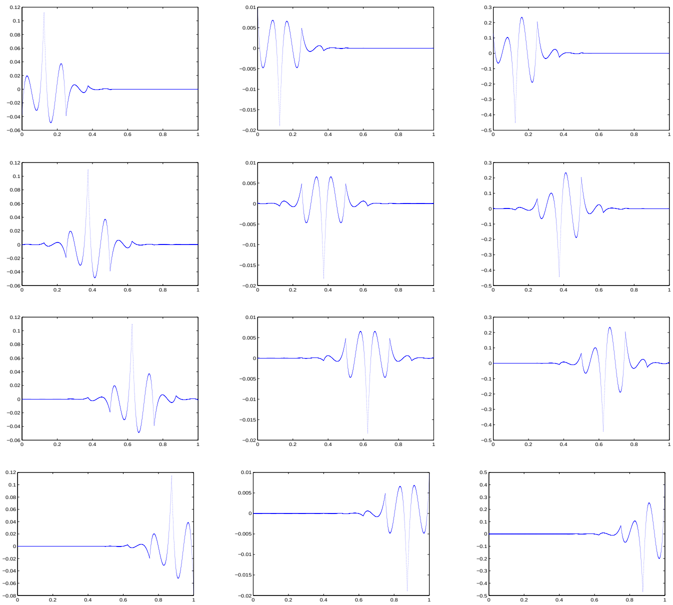

FIGURE 7. $\psi_{(k,i),j}$, $0 \leq k \leq 3$, $0 \leq i \leq 2$ in the space $V^3_{4,2}$.

that, by Lemma 3.3 (ii), function $f$ belongs to the space $V_{8,\mathbf{t}^9_8}$ in which $\mathbf{t}^9_8 = \{k2^{-9}\}_{k=0}^{2^9}$ with $n^j_0 = n^j_{2^9} = 9, n^j_{2^8} = 7$, and $n^j_k = 1$ for the other nodal points. Hence, a good approximation solution is expected even in the presence of thresholding. Furthermore, if there are more singularity points in Figure 9, we may expect more difference in errors between MKBSWs and the other wavelets.

**Example 2.** Consider the function $f : [0,1] \to \mathbb{R}$, defined by

$$f(x) = \begin{cases} 3x^2 - 1/2x + 5 & x \in [0, 1/2), \\ -3x^2 + 4x + 17/4 & x \in [1/2, 1]. \end{cases} \tag{6.2}$$

Figure 10 shows the function $f$ and its approximated function by MKBSW. It contains a singularity at $x = 1/2$. Note that the function $f$ lies in the space $V_{3,\mathbf{t}^9_3}$ by Lemma 3.3 (ii) where $V_{3,\mathbf{t}^9_3}$ in which $\mathbf{t}^9_3 = \{k2^{-9}\}_{k=0}^{2^9}$ with $n^j_0 = n^j_{2^9} = 9$, $n^j_{2^8} = 7$ and $n^j_k = 1$ for the other

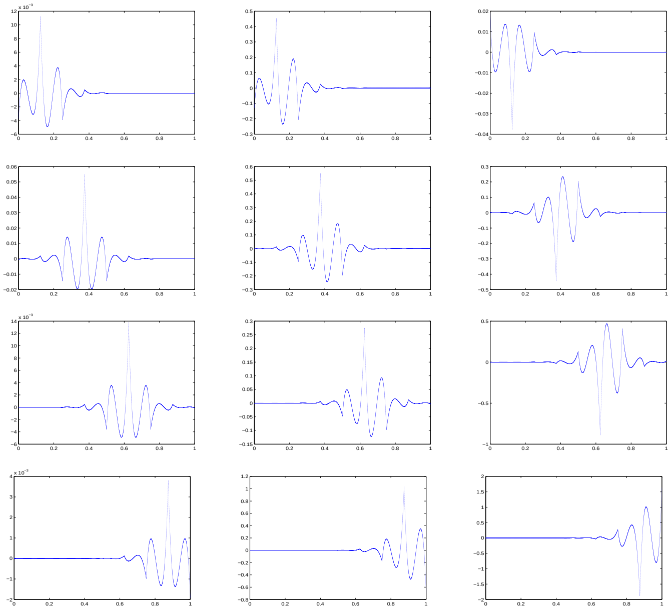

FIGURE 8. The basis functions of $\psi_{(k,i),j}$, $0 \leq k \leq 3$, $0 \leq i \leq 2$, with $\Lambda_{k^*,i^*} := \alpha_{k^*,i^*}^{\Lambda_{k^*,i^*}} = \{-1, 0.1, 2, 1, 5, -3, 2, 2.5, -0.75, 4, 9, -0.2\}$ in the space $V_{4,2}^3$.

TABLE 1. A comparison of errors and number of zeros in percent.

| Method | $\epsilon$ | Number of zeros |
|---|---|---|
| MKBSW | 0.00004 | 91.01 |
| Haar | 0.0013 | 3.26 |
| db2 | 0.0050 | 0.00 |
| db3 | 0.0023 | 84.21 |
| db4 | 0.0016 | 83.83 |
| db5 | 0.0018 | 81.80 |
| coif2 | 0.0020 | 79.17 |
| bior2.2 | 0.0032 | 72.41 |

nodal points. Function $f$ is only continuous at this point, and there is no higher degree of continuity. Hence, the multiple knot function with multiplicity $n = 2$ is considered. Moreover, we use hard global

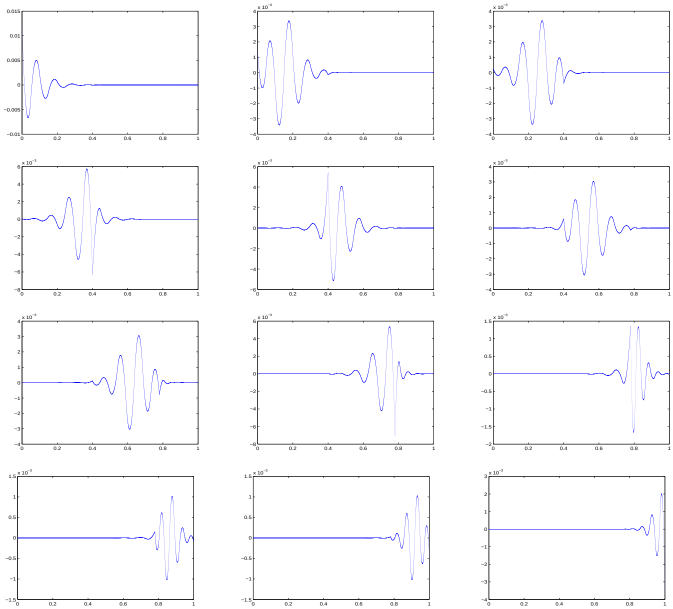

FIGURE 9. $\psi_{(0,0)}, \psi_{(1,0)}, \ldots, \psi_{(12,0)}$.

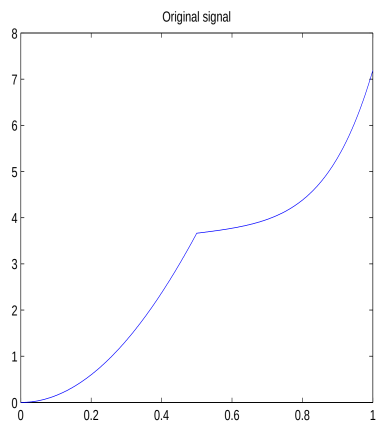


FIGURE 10. The function $f$ defined by (6.1).

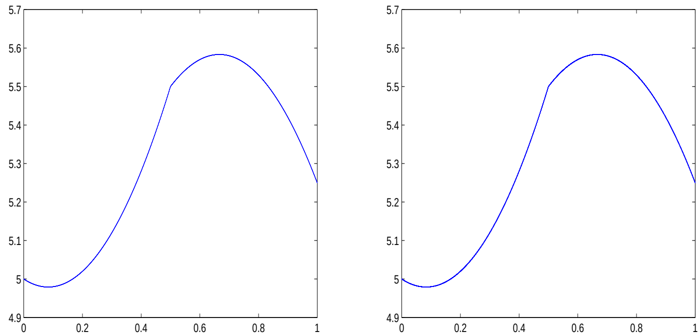


FIGURE 11. The original function $f$ (left) defined by (6.2) and an approximation of it (right).

TABLE 2. A comparison of errors and number of zeros in percent.

| Method | $\epsilon$ | Number of zeros |
|---|---|---|
| MKBSW | $1.0012e-015$ | 99.61 |
| Haar | $1.4465e-013$ | 1.72 |
| db2 | $7.7619e-012$ | 0.00 |
| db3 | $5.2660e-010$ | 83.85 |
| db4 | $2.3856e-011$ | 78.03 |
| db5 | $4.9877e-008$ | 73.47 |
| coif2 | $3.45382e-009$ | 70.58 |
| bior2.2 | $2.0332e-013$ | 0.00 |

thresholding thr $= 10^{-7}$. A comparison of the errors and number of zeros is given in Table 2. In Table 2, we observe that MKBSW has less error and more zeros over the other wavelets.

**7. Conclusions and future work.** In this paper, a new multiple knot B-spline wavelet was introduced. In fact, this paper generalizes the construction of multiple knot B-spline wavelet that was introduced by Chui and Quak [**2**]. The MKBSW of Chui and Quak has been constructed based on the knot vectors that are multiple only on the endpoints, while the MKBSWs introduced in the present paper are constructed based on the knot vectors that are multiple not only in the endpoints but also in the middle knots. This work fulfills the construction of multiple knot B-spline wavelets on the interval. Multiple knot B-spline wavelets are only semi-orthogonal wavelets and thus, for future work, the bi-orthogonal wavelets $\{\widetilde{\psi}_{(k,i),j}\}_{(k,i)\in\mathcal{J}_j^*}$ may be built such that

$$(\widetilde{\psi}_{(k,i),j}, \psi_{(k',i'),j'}) = \delta_{k,k'}\delta_{i,i'}\delta_{j,j'}.$$

This is the subject of our upcoming work [**14**].

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