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RATIONAL CONVOLUTION ROOTS OF ISOBARIC POLYNOMIALS

HUILAN LI, TRUEMAN MACHENRY AND AURA CONCI

ABSTRACT. In this paper, we exhibit two matrix representations of the rational roots of generalized Fibonacci polynomials (GFPs) under the convolution product, in terms of determinants and permanents, respectively. The underlying root formulas for GFPs and for weighted isobaric polynomials (WIPs), which appeared in an earlier paper by MacHenry and Tudose, make use of two types of operators. These operators are derived from the generating functions for Stirling numbers of the first and second kind. Hence, we call them Stirling operators. In order to construct matrix representations of the roots of GFPs, we use Stirling operators of the first kind. We give explicit examples to show how Stirling operators of the second kind appear in low degree cases for the WIP-roots. As a consequence of the matrix construction we have matrix representations of multiplicative arithmetic functions under the Dirichlet product into its divisible closure.

1. Introduction. In 1975, Carroll and Gioia [**2**] gave a direct construction for an adjoining qth roots, $q \in \mathbb{Q}$, to the group of multiplicative arithmetic functions (MF) under the Dirichlet product. In 2000, MacHenry [**5**] gave a somewhat more general proof of the same result. In 2005, MacHenry and Tudose [**6**] constructed the injective hull of generalized Fibonacci polynomials (GFPs) and extended this construction to the injective hull of the WIP-*module*, that is, the $\mathbb{Z}$-module of all sequences of weighted isobaric polynomials (WIPs) with the convolution product. Isobaric polynomials are the symmetric polynomials over the elementary symmetric polynomial (ESP) basis; the

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isobaric ring is isomorphic to the ring of symmetric polynomials. In 2012, MacHenry and Wong [**9**] showed that GFPs, together with the convolution product, give a faithful representation of the group of MF under the Dirichlet product, which in turn induces the embedding of the MF group into its injective hull, that is, adjoins a qth root to each multiplicative arithmetic function for all non-zero rational numbers q in $\mathbb{Q}$.

In 2013, Li and MacHenry [**4**] gave two matrix representations of WIPs in terms of Hessenberg matrices; they showed that the determinant of one matrix is the permanent of the other, and the determinant and permanent is an element in the WIP-module.

In this paper, we use Hessenberg matrices to give matrix representations of the qth, $q \in \mathbb{Q}$, convolution roots of GFPs, both in terms of determinants and in terms of permanents. The main result of this paper is Theorem 4.1 in Section 4. We introduce $B_j = q(q+1)\cdots(q+j)$ and $B_{-j} = q(q-1)\cdots(q-j)$, the *Stirling operators of first kind and second kind*, respectively. Then we obtain Corollary 4.3:

$F^q_{k,n}$ is the determinant of the following matrix:

$$\begin{pmatrix} B_0t_1 & -1 & 0 & 0 & \cdots & 0 \\ B_0t_2 & \frac{1}{2}\frac{B_1}{B_0}t_1 & -1 & 0 & \cdots & 0 \\ B_0t_3 & \frac{1}{3}(2\frac{B_1}{B_0}-1)t_2 & \frac{1}{3}\frac{B_2}{B_1}t_1 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ s_{n,n} & s_{n,n-1} & s_{n,n-2} & s_{n,n-3} & \cdots & s_{n,1} \end{pmatrix},$$

i.e., the recursion coefficients are

$$s_{n,j} = \frac{1}{n}\left(j\frac{B_{n-j}}{B_{n-j-1}} - (n-j)(j-1)\right)t_j,$$

for $j = 1, \ldots, n-1$, and $s_{n,n} = B_0t_n$. We call these representations *Hessenberg-Stirling representations.*

In order to produce the convolution roots of GFPs and WIPs, we use the Stirling operators of the first and second kind. Currently, we know of no such applications using Stirling generating functions. We would like to point out the unexpected usefulness of Stirling operators. They provide a complete answer to the construction of rational roots of the group of multiplicative arithmetic functions under the Dirichlet product [**2**], which is a concern in arithmetic number theory. We describe how GFPs are used to produce an isomorphism from the group generated by GFPs under the convolution product to the group of multiplicative arithmetic functions under the Dirichlet product [**3**]. The usefulness of Stirling operators in this case suggests that adjoining roots (and powers) to other algebraic structures may also be achieved by using them. Also, Stirling operators may have wider applications, say, to other groups.

This paper is organized as follows. In Section 2, we review basic facts about isobaric polynomials. In Section 3, we review the Hessenberg representations of WIPs. In Section 4, we construct matrix representations of roots of GFPs using Stirling operators. In Section 5, we review the isomorphism from the group generated by GFPs to the group of multiplicative arithmetic functions. We thus have matrix representations of elements in the divisible closure of MF under the Dirichlet product.

2. Isobaric polynomials. An *isobaric polynomial* in k variables $\{t_1, \dots, t_k\}$ of degree n is of the form

$$P_{k,n} = \sum_{\alpha \vdash n} C_\alpha t_1^{\alpha_1} t_2^{\alpha_2} \cdots t_k^{\alpha_k},$$

where $C_\alpha \in \mathbb{Z}$ and $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k) \vdash n$ means that $(1^{\alpha_1}, 2^{\alpha_2}, \dots, k^{\alpha_k})$ is a partition of n with

$$\sum_{j=1}^{k} j\alpha_j = n.$$

An isobaric polynomial may be thought of as a symmetric polynomial written on the elementary symmetric polynomial (ESP) basis.

A sequence of *weighted isobaric polynomials* of weight $\omega = (\omega_1, \omega_2, \dots, \omega_j, \dots)$ with $\omega_j \in \mathbb{Z}$ is defined by

$$P_{\omega,k,n} = \sum_{\alpha \vdash n} \binom{|\alpha|}{\alpha_1, \dots, \alpha_k} \frac{\sum \alpha_i \omega_i}{|\alpha|} t_1^{\alpha_1} t_2^{\alpha_2} \cdots t_k^{\alpha_k},$$

where $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_k$. The union of elements of all sequences of weighted isobaric polynomials is the set of all isobaric polynomials. The index set $\{n\}$ for these polynomials is the set of integers, positive, negative and zero, i.e., $n \in \mathbb{Z}$, in particular, $P_{\omega,k,0} = \omega_k, k \geqslant 1$, and $P_{\omega,k,0} = 1$, $k = 0$ [**3**].

Note that the monomials are indexed by partitions $(1^{\alpha_1}, 2^{\alpha_2}, \ldots, k^{\alpha_k})$ with parts no larger than k. Moreover, the elements in a sequence of weighted isobaric polynomials occur in linear recursions

$$P_{\omega,k,n} = t_1 P_{\omega,k,n-1} + t_2 P_{\omega,k,n-2} + \cdots + t_j P_{\omega,k,n-j} + \cdots + t_k P_{\omega,k,n-k},$$

with respect to the recursion parameters $[t_1, \ldots, t_k]$.

Two important sequences are the generalized Fibonacci polynomials (GFPs)

$$F_{k,n} = \sum_{\alpha \vdash n} \binom{|\alpha|}{\alpha_1, \ldots, \alpha_k} t_1^{\alpha_1} t_2^{\alpha_2} \cdots t_k^{\alpha_k},$$

where the weight vector is $\omega = (1, 1, \ldots, 1 \ldots)$ with $F_{k,0} = 1$ and the generalized Lucas polynomials (GLPs)

$$G_{k,n} = \sum_{\alpha \vdash n} \binom{|\alpha|}{\alpha_1, \ldots, \alpha_k} \frac{n}{|\alpha|} t_1^{\alpha_1} t_2^{\alpha_2} \cdots t_k^{\alpha_k},$$

where the weight vector is $\omega = (1, 2, \ldots, j, \ldots)$ and $G_{k,0} = k$.

Remark 2.1. GFPs are complete symmetric polynomials written on the ESP basis, and GLPs are power sum symmetric polynomials written on the ESP basis; each of these sequences of polynomials is a basis for the ring of symmetric polynomials.

Remark 2.2. WIPs, in general, have special significance in the ring of symmetric polynomials. In order to see how this comes about, it is convenient to consider the notation $[t_1, \ldots, t_k]$ used above to indicate recursion parameters. More generally, we also use $[t_1, \ldots, t_k]$ to indicate the monic polynomial $\mathcal{C}(X) = X^k - t_1 X^{k-1} - \cdots - t_k$, that is,

$$[t_1, \ldots, t_k] = X^k - t_1 X^{k-1} - \cdots - t_k.$$

When we consider t_js as variables, we often refer to $\mathcal{C}(X)$ as the *generic core*, and, when we evaluate t_js over the ring of integers, the term *numerical core* will be used.

Remark 2.3. It is trivial to verify that, when $k = 2$, GFPs are generalizations of the classical "generalized Fibonacci polynomials," and GLPs are generalizations of the classical "generalized Lucas polynomials" [**1**]; when $t_1 = t_2 = 1$, GFPs and GLPs become the classical Fibonacci and Lucas sequences. It is surprising that these older terms persist in the current literature in competition with the true generalizations.

Next, we consider the *companion matrix* of $[t_1, \dots, t_k] = X^k - t_1X^{k-1} - \cdots - t_k$, namely, the $k \times k$-matrix:

$$A_k = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ t_k & t_{k-1} & \cdots & t_2 & t_1 \end{pmatrix}.$$

We use A_k to construct the next infinite matrix by appending the orbit of the row vectors generated by letting A_k act on the right of the last row vector in A_k, and repeating the process on the successive last row vectors. Noting that A_k is non-singular exactly when $t_k \neq 0$ and adding this as an assumption, we can perform the analogous operation on the first row vector of A_k, extending the rows northward, yielding a doubly infinite matrix with k columns. We call this the *infinite companion matrix* and denote it by A_k^∞, or simply as A^∞ when the k is clear. Since it is completely determined by the polynomial $\mathcal{C}(X) = [t_1, \dots, t_k]$, we call $C(X)$ the *core polynomial*.

$$A_k^\infty = \begin{pmatrix} \vdots & \ddots & \vdots & \vdots \\ (-1)^{k-1}S_{(-2,1^{(k-1)})} & \cdots & -S_{(-2,1)} & S_{(-2)} \\ (-1)^{k-1}S_{(-1,1^{(k-1)})} & \cdots & -S_{(-1,1)} & S_{(-1)} \\ (-1)^{k-1}S_{(0,1^{(k-1)}))} & \cdots & -S_{(0,1)} & S_{(0)} \\ (-1)^{k-1}S_{(1,1^{(k-1)})} & \cdots & -S_{(1,1)} & S_{(1)} \\ (-1)^{k-1}S_{(2,1^{(k-1)})} & \cdots & -S_{(2,1)} & S_{(2)} \\ (-1)^{k-1}S_{(3,1^{(k-1)})} & \cdots & -S_{(3,1)} & S_{(3)} \\ (-1)^{k-1}S_{(4,1^{(k-1)})} & \cdots & -S_{(4,1)} & S_{(4)} \\ \vdots & \ddots & \vdots & \vdots \end{pmatrix}$$

$$= ((-1)^{k-j}S_{(n,1^{k-j})}).$$

As pointed out in a number of previous papers, e.g., [**3**], the matrix A_k^∞ has the following remarkable properties:

- the $k \times k$ contiguous blocks of A^∞ are the successive powers in the free abelian group generated by the companion matrix A_k.
- The rows of A^∞ give a vector representation of the successive powers of the zeros of the core polynomial. Essentially, this is a consequence of the Hamilton-Cayley theorem.
- The right hand column of A^∞ is merely the GFPs.
- The traces of the $k \times k$ contiguous blocks give the GLPs in succession.
- The k columns of A^∞ are linearly recursive with respect to the coefficients of the core polynomial as recursion parameters.
- The columns of A^∞ are sequences of weighted isobaric polynomials with weights $\pm(0, \ldots, 0, 1, 1, \ldots, 1, \ldots)$.
- The elements of A^∞ are Schur-hook polynomials $S_{(n,1^r)}$ of arm-length $n-1$ and leg length r, in particular, $F_{k,n} = S_{(n)}$.
- The sequences of WIPs form a free $\mathbb{Z}$-module. The columns of A^∞ form a basis of this module.

Moreover, there is a second matrix that is induced by the core polynomial [**3**]. Consider the derivative of the core polynomial

$$\mathcal{C}'(X) = kX^{k-1} - t_1 X^{k-2} - \cdots - t_{k-1},$$

from which we manufacture the vector $(-t_{k-1}, \ldots, -t_1, k)$. Again, letting the companion matrix A_k act on this vector on the right and appending the resulting orbit as additional row vectors, we get a $k \times k$-matrix, which we call the *different matrix*, denoted by D. From D, we construct an infinite matrix D^∞ as we do for A^∞. We call D^∞ the *infinite different matrix*. It, too, has some useful and remarkable properties [**3**]:

- the determinant $\det D = \Delta$, the discriminant of the core polynomial.
- The right hand column of D^∞ is the sequence GLPs.
- There is a bijection $\mathcal{L}$ from A^∞ to D^∞ which takes the element $a_{i,j}$ in A^∞ to $d_{i,j}$ in D^∞, which has the properties of a logarithm on elements, and which implies that $\mathcal{L}(F_{k,n}) = G_{k,n}$.
- The columns of D^∞ are linear recursions with recursion parameters $\{t_1, \ldots, t_k\}$.

Next, we would like to point out how the sequences discussed here are important. In a series of papers [**3, 4, 6, 7, 8, 9**] it has been shown that subgroups of the ring of arithmetic functions, namely, the Dirichlet group of multiplicative arithmetic functions, and the additive group of additive arithmetical functions have faithful representations using the GFP and the GLP sequences; they also show up in the character theory of symmetric groups and Pólya's theory of counting [**3, 11**]. In the following section, we will recall the matrix representations of the GFP, the GLP and, in general, the WIP sequences [**4**], which give an explicit algorithm for computing these sequences and are useful for calculation.

First, however, it is convenient to introduce the convolution product of weighted isobaric polynomials.

Definition 2.4 ([**3**]). Let $P_{\omega,k,n}$ and $P_{\upsilon,k,n}$ be weighted isobaric polynomials of isobaric degree n. Define the *convolution product* of $P_{\omega,k,n}$ and $P_{\upsilon,k,n}$ by

$$P_{\omega,k,n} * P_{\upsilon,k,n} = \sum_{j=0}^{n} P_{\omega,k,j} P_{\upsilon,k,n-j}.$$

Note that the product is also a weighted isobaric polynomial of isobaric degree n. In the case where we have two integer evaluations of $P_{\omega,k,n}$ and $P_{\upsilon,k,n}$, we denote them as, respectively, P' and P'', and their numerical convolution product is

$$P'_{\omega,k,n} * P''_{\upsilon,k,n} = \sum_{j=0}^{n} P'_{\omega,k,j} P''_{\upsilon,k,n-j}.$$

It is with respect to this product and the ordinary addition of polynomials that the *logarithm operator* $\mathcal{L}$ is defined, see [**3**].

3. Permanent and determinant representations. A formula was given for elements of the divisible closure of the WIP-module [**6**], i.e., each element in WIP-module was given a qth root for all $q \in \mathbb{Q}$, where these roots are unique up to sign. Two interesting representations of the elements of WIP-module were given in terms of determinants and permanents of the following Hessenberg matrices [**4**]:

$$H_{+(\omega,k,n)} = \begin{pmatrix} t_1 & 1 & 0 & \cdots & 0 \\ t_2 & t_1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ t_{n-1} & t_{n-2} & t_{n-3} & \cdots & 1 \\ \omega_n t_n & \omega_{n-1} t_{n-1} & \omega_{n-2} t_{n-2} & \cdots & \omega_1 t_1 \end{pmatrix},$$

and

$$H_{-(\omega,k,n)} = \begin{pmatrix} t_1 & -1 & 0 & \cdots & 0 \\ t_2 & t_1 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ t_{n-1} & t_{n-2} & t_{n-3} & \cdots & -1 \\ \omega_n t_n & \omega_{n-1} t_{n-1} & \omega_{n-2} t_{n-2} & \cdots & \omega_1 t_1 \end{pmatrix}.$$

The principal results are:

$$\operatorname{perm} H_{+(\omega,k,n)} = P_{\omega,k,n} = \det H_{-(\omega,k,n)}.$$

For example, we look at the following matrix when $n = 4$,

$$\begin{pmatrix} t_1 & 1 & 0 & 0 \\ t_2 & t_1 & 1 & 0 \\ t_3 & t_2 & t_1 & 1 \\ \omega_4 t_4 & \omega_3 t_3 & \omega_2 t_2 & \omega_1 t_1 \end{pmatrix},$$

whose permanent is easily seen to be

$$\omega_1 t_1^4 + (2\omega_1 + \omega_2) t_1^2 t_2 + \omega_2 t_2^2 + (\omega_1 + \omega_3) t_1 t_3 + \omega_4 t_4 = P_{\omega,4,4}.$$

Moreover, it is easy to see that there is a nesting of the Hessenberg matrices from the lower right hand corner to the upper left. We call these representations *Hessenberg representations*. It turns out that we can use these to go further and give two useful representations of the qth convolution roots of generalized Fibonacci polynomials in terms of Hessenberg matrices.

4. Convolution roots. MacHenry and Tudose [**6,** Theorems 5.1, 5.7] gave a general expression for the qth, $q \in \mathbb{Q}$, convolution roots of GFPs and a more general expression for the qth convolution roots of WIPs.

The formula for qth roots of polynomials in the GFP is given by

$$F_{k,n}^{q}=\sum_{\alpha\vdash n}\frac{1}{|\alpha|!}B_{|\alpha|-1}\binom{|\alpha|}{\alpha_1,\ldots,\alpha_k}t_1^{\alpha_1}t_2^{\alpha_2}\cdots t_k^{\alpha_k}.$$

For $n=3$ and $n=4$, we have the following determinantal representations:

$$F_{k,3}^{q}=\det\begin{pmatrix} qt_1 & -1 & 0\\ qt_2 & \frac{1}{2}(q+1)t_1 & -1\\ qt_3 & \frac{1}{3}(2q+1)t_2 & \frac{1}{3}(q+2)t_1\end{pmatrix}$$

and

$$F_{k,4}^{q}=\det\begin{pmatrix} qt_1 & -1 & 0 & 0\\ qt_2 & \frac{1}{2}(q+1)t_1 & -1 & 0\\ qt_3 & \frac{1}{3}(2q+1)t_2 & \frac{1}{3}(q+2)t_1 & -1\\ qt_4 & \frac{1}{4}(3q+1)t_3 & \frac{1}{4}(2q+2)t_2 & \frac{1}{4}(q+3)t_1\end{pmatrix},$$

where B_j is the polynomial generating function for Stirling numbers of the first kind evaluated at q. (B_{-j}, the analogue, is determined by the polynomial generating function for Stirling numbers of the second kind), namely,

$$B_j=q(q+1)\cdots(q+j)\quad\text{and}\quad B_{-j}=q(q-1)\cdots(q-j).$$

We call B_j and B_{-j} *Stirling operators of the first kind and second kind*, respectively.

The main theorem of this paper is a generalization to arbitrary n of the two matrices which appear above. The first five such roots, starting with $F_{k,0}^{q}$ for an arbitrary q, are as follows:

(4.1)

$$\begin{aligned}
F_{k,0}^{q}&=1,\\
F_{k,1}^{q}&=qt_1,\\
F_{k,2}^{q}&=\frac{1}{2}q(q+1)t_1^2+qt_2,\\
F_{k,3}^{q}&=\frac{1}{3!}q(q+1)(q+2)t_1^3+q(q+1)t_1t_2+qt_3,\\
F_{k,4}^{q}&=\frac{1}{4!}q(q+1)(q+2)(q+3)t_1^4+\frac{1}{2}q(q+1)(q+2)t_1^2t_2
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2}q(q+1)t_2^2+q(q+1)t_1t_3+qt_4,\\
F_{k,5}^q &= \frac{1}{5!}q(q+1)(q+2)(q+3)(q+4)t_1^5\\
&+\frac{1}{6}q(q+1)(q+2)(q+3)t_1^3t_2+\frac{1}{2}q(q+1)(q+2)t_1t_2^2\\
&+\frac{1}{2}q(q+1)(q+2)t_1^2t_3+q(q+1)t_2t_3+q(q+1)t_1t_4+qt_5
\end{aligned}
$$

and, in the Stirling operator notation, these translate into:

$$
\begin{aligned}
F_{k,0}^q &= 1,\\
F_{k,1}^q &= B_0t_1,\\
F_{k,2}^q &= \frac{1}{2!}B_1t_1^2+B_0t_2,\\
F_{k,3}^q &= \frac{1}{3!}B_2t_1^3+\frac{1}{2!}2B_1t_1t_2+B_0t_3,\\
F_{k,4}^q &= \frac{1}{4!}B_3t_1^4+\frac{1}{3!}3B_2t_1^2t_2+\frac{1}{2!}B_1t_2^2+\frac{1}{2!}2B_1t_1t_3+B_0t_4,\\
F_{k,5}^q &= \frac{1}{5!}B_4t_1^5+\frac{1}{4!}4B_3t_1^3t_2+\frac{1}{3!}3B_2t_1t_2^2+\frac{1}{3!}3B_2t_1^2t_3\\
&\quad+\frac{1}{2!}2B_1t_2t_3+\frac{1}{2!}2B_1t_1t_4+B_0t_5.
\end{aligned}
\tag{4.2}
$$

A rule of thumb for writing the qth convolution roots is as follows: first, write the polynomial F_n as a function of t_j, $j=1,\ldots,k$. Then, observing the exponent sum $|\alpha|$, monomial-by-monomial, enter the fraction $1/|\alpha|!$ and the Stirling operators $B_{|\alpha|-1}$. There will usually be some cancellations among the fractions for the most economical expression.

For example,

$$F_{k,3}=t_1^3+2t_1t_2+t_3,$$

and

$$F_{k,3}^q=\frac{1}{3!}B_2t_1^3+\frac{1}{2!}2B_1t_1t_2+B_0t_3.$$

Theorem 4.1. *$F^q_{k,n}$ is the determinant of*

$$\begin{pmatrix} qt_1 & -1 & 0 & \cdots & 0 \\ qt_2 & \frac{1}{2}(q+1)t_1 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ qt_{n-1} & \frac{1}{n-1}((n-2)q+1)t_{n-2} & \frac{1}{n-1}((n-3)q+2)t_{n-3} & \cdots & -1 \\ qt_n & \frac{1}{n}((n-1)q+1)t_{n-1} & \frac{1}{n}((n-2)q+2)t_{n-2} & \cdots & \frac{1}{n}(q+(n-1))t_1 \end{pmatrix}$$

and the permanent of

$$\begin{pmatrix} qt_1 & -1 & 0 & \cdots & 0 \\ qt_2 & \frac{1}{2}(q+1)t_1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ qt_{n-1} & \frac{1}{n-1}((n-2)q+1)t_{n-2} & \frac{1}{n-1}((n-3)q+2)t_{n-3} & \cdots & 1 \\ qt_n & \frac{1}{n}((n-1)q+1)t_{n-1} & \frac{1}{n}((n-2)q+2)t_{n-2} & \cdots & \frac{1}{n}(q+(n-1))t_1 \end{pmatrix}$$

Proof. Note that the determinants and permanents are nested, that is, $F^q_{k,j}$ is the $j \times j$ principal minor in the upper left hand corner of the matrices. This allows us to use induction in the proof. We shall carry out computations for the determinant case. The proof for the permanent case is similar. $\square$

Lemma 4.2. *The $F^q_{k,n}$ satisfies the recursive formula*:

$$F^q_{k,n} = s_{n,1}F^q_{k,n-1} + s_{n,2}F^q_{n-2} + s_{n,3}F^q_{k,n-3} + \cdots + s_{n,n-1}F^q_{k,1} + s_{n,n}F^q_{k,0},$$

where the recursion parameters are $s_{n,j} = (1/n)(jq + n - j)t_j$.

Proof. The nesting of the matrices, and hence of the determinants and permanents, implies the recursion. Let M_n be the determinant of

$$\begin{pmatrix} qt_1 & -1 & 0 & \cdots & 0 \\ qt_2 & \frac{1}{2}(q+1)t_1 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ qt_{n-1} & \frac{1}{n-1}((n-2)q+1)t_{n-2} & \frac{1}{n-1}((n-3)q+2)t_{n-3} & \cdots & -1 \\ qt_n & \frac{1}{n}((n-1)q+1)t_{n-1} & \frac{1}{n}((n-2)q+2)t_{n-2} & \cdots & \frac{1}{n}(q+(n-1))t_1 \end{pmatrix}$$

$M_0 = 1$ and $m_{i,j}$ the (i,j)th entry in the matrix. In order to prove the recursive formula is equivalent to proving

$$M_n = m_{n,n}M_{n-1} + m_{n,n-1}M_{n-2} + \cdots + m_{n,2}M_1 + m_{n,1}M_0.$$

We compute the cofactor expansion along the last column from bottom to top, and we obtain:

$$\begin{aligned}
M_n &= m_{n,n}M_{n-1} + \det\begin{pmatrix} m_{1,1} & -1 & 0 & \cdots & 0 \\ m_{2,1} & m_{2,2} & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ m_{n-2,1} & m_{n-2,2} & m_{n-2,3} & \cdots & -1 \\ m_{n,1} & m_{n,2} & m_{n,3} & \cdots & m_{n,n-1} \end{pmatrix} \\
&= m_{n,n}M_{n-1} + m_{n,n-1}M_{n-2} \\
&\quad + \det\begin{pmatrix} m_{1,1} & -1 & 0 & \cdots & 0 \\ m_{2,1} & m_{2,2} & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ m_{n-3,1} & m_{n-3,2} & m_{n-3,3} & \cdots & -1 \\ m_{n,1} & m_{n,2} & m_{n,3} & \cdots & m_{n,n-2} \end{pmatrix} \\
&\quad\vdots \\
&= m_{n,n}M_{n-1} + m_{n,n-1}M_{n-2} + \cdots + m_{n,2}M_1 + m_{n,1}M_0.
\end{aligned}$$

Let $s_{n,j} = m_{n,n-j+1}$. We then have

$$M_n = s_{n,1}M_{n-1} + s_{n,2}M_{n-2} + \cdots + s_{n,n-1}M_1 + s_{n,n}M_0.$$

Putting $F^q_{k,n-j} = M_{n-j}$, we assume inductively that $M_{n-j} = F^q_{k,n-j}$, $j = 0, 1, \ldots, n-1$. We have

$$M_n = s_{n,1}F^q_{k,n-1} + s_{n,2}F^q_{k,n-2} + \cdots + s_{n,n-1}F^q_{k,1} + s_{n,n}F^q_{k,0}.$$

Now, we only need to show that $M_n = F^q_{k,n}$.

Recall that $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k) \vdash n$ means $\alpha_1 + 2\alpha_2 + \cdots + k\alpha_k = n$ and $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_k$.

In order to prove

$$\begin{aligned}
F^q_{k,n} &= \sum_{\alpha\vdash n} \frac{1}{|\alpha|!} B_{|\alpha|-1} \binom{|\alpha|}{\alpha_1, \ldots, \alpha_k} t_1^{\alpha_1} t_2^{\alpha_2} \cdots t_k^{\alpha_k} \\
&= \sum_{\alpha\vdash n} \frac{B_{|\alpha|-1}}{\alpha_1!\alpha_2!\cdots\alpha_k!} t_1^{\alpha_1} t_2^{\alpha_2} \cdots t_k^{\alpha_k}
\end{aligned}$$

is the determinant or permanent of the matrices in Theorem 4.1, we only need to show that $F^q_{k,n}$ satisfies the recursive formula in

Lemma 4.2, which is equivalent to showing that

$$\frac{B_{|\alpha|-1}}{\alpha_1!\alpha_2!\cdots\alpha_k!} = \sum_{i=1}^{k} \frac{1}{n}(iq+n-i)\frac{B_{|\alpha|-2}}{\alpha_1!\alpha_2!\cdots(\alpha_i-1)!\cdots\alpha_k!}.$$

$$\begin{aligned}
&\sum_{i=1}^{k} \frac{1}{n}(iq+n-i)\frac{B_{|\alpha|-2}}{\alpha_1!\alpha_2!\cdots(\alpha_i-1)!\cdots\alpha_k!} \\
&\quad = \frac{(q+n-1)B_{|\alpha|-2}}{n(\alpha_1-1)!\alpha_2!\cdots\alpha_k!} + \frac{(2q+n-2)B_{|\alpha|-2}}{n\alpha_1!(\alpha_2-1)!\cdots\alpha_k!} + \cdots \\
&\qquad + \frac{(kq+n-k)B_{|\alpha|-2}}{n\alpha_1!\alpha_2!\cdots(\alpha_k-1)!} \\
&\quad = \frac{B_{|\alpha|-2}}{n\alpha_1!\alpha_2!\cdots\alpha_k!}[\alpha_1(q+n-1)+\alpha_2(2q+n-2)+\cdots \\
&\qquad\qquad + \alpha_k(kq+n-k)] \\
&\quad = \frac{B_{|\alpha|-2}}{n\alpha_1!\alpha_2!\cdots\alpha_k!}[\alpha_1 q+n\alpha_1-\alpha_1+2\alpha_2 q+n\alpha_2-2\alpha_2+\cdots \\
&\qquad\qquad + k\alpha_k q+n\alpha_k-k\alpha_k] \\
&\quad = \frac{B_{|\alpha|-2}}{n\alpha_1!\alpha_2!\cdots\alpha_k!}[(\alpha_1+2\alpha_2+\cdots+k\alpha_k)q+n(\alpha_1+\alpha_2+\cdots+\alpha_k) \\
&\qquad\qquad - (\alpha_1+2\alpha_2+\cdots+k\alpha_k)] \\
&\quad = \frac{B_{|\alpha|-2}}{n\alpha_1!\alpha_2!\cdots\alpha_k!}(nq+n|\alpha|-n) = \frac{B_{|\alpha|-2}(q+|\alpha|-1)}{\alpha_1!\alpha_2!\cdots\alpha_k!} \\
&\quad = \frac{B_{|\alpha|-1}}{\alpha_1!\alpha_2!\cdots\alpha_k!} \qquad \square
\end{aligned}$$

It is of interest to see the matrix which represents the convolution roots in a form which explicitly displays the Stirling operators B_j, which we now do in:

Corollary 4.3. *$F^q_{k,n}$ is the determinant of the following matrix:*

$$\begin{pmatrix} B_0t_1 & -1 & 0 & 0 & \cdots & 0 \\ B_0t_2 & \frac{1}{2}\frac{B_1}{B_0}t_1 & -1 & 0 & \cdots & 0 \\ B_0t_3 & \frac{1}{3}(2\frac{B_1}{B_0}-1)t_2 & \frac{1}{3}\frac{B_2}{B_1}t_1 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ s_{n,n} & s_{n,n-1} & s_{n,n-2} & s_{n,n-3} & \cdots & s_{n,1} \end{pmatrix}$$

where the recursion coefficients are

$$s_{n,j} = \frac{1}{n}\left(j\frac{B_{n-j}}{B_{n-j-1}} - (n-j)(j-1)\right)t_j$$

for $j = 1, \ldots, n-1$, and $s_{n,n} = B_0t_n$. □

We call these the *Hessenberg-Stirling representations.* The Stirling part is due to the role that the Stirling operators play in the construction of the roots of the GFPs.

The root formula for the WIPs is a generalization of the root formula for the GFP and is a bit more complicated.

Theorem 4.4 ([**6**])**.**

$$P^q_{\omega,k,n} = \sum_{\alpha\vdash n} L_{k,n,\omega}(\alpha)t_1^{\alpha_1}\cdots t_k^{\alpha_k},$$

where

$$L_{\omega,k,n}(\alpha) = \sum_{j=0}^{|\alpha|-1}\frac{1}{(\Pi_{i=1}^k\alpha_i)!}\binom{|\alpha|-1}{j}B_{-j}D_{|\alpha|-j-1}(\omega_1^{\alpha_1}\cdots\omega_k^{\alpha_k})$$

and $D_j(\omega^\alpha) = D_j(\omega_1^{\alpha_1}\cdots\omega_k^{\alpha_k})$ is the total derivative of the expression j times.

The *total differential operator* D_j is inductively defined by $D_j = D_1(D_{j-1})$ with

$$D_1(\omega_1^{\alpha_1}\cdots\omega_k^{\alpha_k}) = \sum_{i=1}^k \partial_i(\omega_1^{\alpha_1}\cdots\omega_k^{\alpha_k}) = \sum_{i=1}^k \alpha_i(\omega_1^{\alpha_1}\cdots\omega_i^{\alpha_i-1}\cdots\omega_k^{\alpha_k}).$$

For example, $D_2(\omega_1^3\omega_2^2) = 6\omega_1\omega_2^2 + 12\omega_1^2\omega_2 + 2\omega_1^3$.

Here, we give some low-dimensional examples for the qth roots of weighted isobaric polynomials:

$$\begin{aligned}
P^q_{\omega,k,0} &= 1,\\
P^q_{\omega,k,1} &= q\omega_1 t_1,\\
P^q_{\omega,k,2} &= [q\omega_1 + \frac{1}{2}q(q-1)\omega_1^2]t_1^2 + q\omega_2 t_2,\\
P^q_{\omega,k,3} &= [q\omega_1 + q(q-1)\omega_1^2 + \frac{1}{3!}q(q-1)(q-2)\omega_1^3]t_1^3\\
&\quad + [q(\omega_1+\omega_2) + q(q-1)\omega_1\omega_2]t_1t_2 + q\omega_3 t_3,
\end{aligned}$$

and, in the Stirling operator notation, these translate into:

$$\begin{aligned}
P^q_{\omega,k,0} &= 1,\\
P^q_{\omega,k,1} &= B_0\omega_1 t_1,\\
P^q_{\omega,k,2} &= [B_0\omega_1 + \frac{1}{2}B_{-1}\omega_1^2]t_1^2 + B_0\omega_2 t_2,\\
P^q_{\omega,k,3} &= [B_0\omega_1 + B_{-1}\omega_1^2 + \frac{1}{3!}B_{-2}\omega_1^3]t_1^3\\
&\quad + [B_0(\omega_1+\omega_2) + B_{-1}\omega_1\omega_2]t_1t_2 + B_0\omega_3 t_3.
\end{aligned}$$

Remark 4.5. A more precise notation for the roots is P^{*q}, with the emphasis that this root is to be taken with respect to the convolution product, that is, to retrieve the original function after having taken the qth root, one must take the convolution product $1/q$ times. We shall use the shorter form P^q with the meaning $P^q = P^{*q}$.

In the next section, we describe how GFPs are used to produce an isomorphism from the WIP-module for the group of multiplicative arithmetic functions **[3]**.

5. Multiplicative arithmetic functions. The ring (UFD) of arithmetic functions consists of the functions $\alpha : \mathbb{Z} \to \mathbb{Q}$. The *Dirichlet product* of two arithmetic functions α and β is given by

$$\alpha * \beta(n) = \sum_d \alpha(d)\beta\left(\frac{n}{d}\right),$$

where $d \mid n$ **[10]**.

The *multiplicative arithmetic functions* (MF) are those functions α such that

$$\alpha(mn) = \alpha(m)\alpha(n),$$

whenever $(m, n) = 1$. This is equivalent to saying that a multiplicative function is completely determined by its values at the primes. We shall say that such functions are determined locally, so that we are interested in the products

$$\alpha * \beta(p^n) = \sum_{i=0}^{n} \alpha(p^i)\beta(p^{n-i}).$$

If we consider the group generated by GFPs under the convolution product as multiplication, then we also obtain an abelian group. And, if we consider all of the evaluations of the variables t_j over the integers, we produce a group that is locally isomorphic to the MF group[1] [**3, 8**]. It was shown that this induces a mapping from the divisible closure of the group generated by GFPs to the divisible closure of MF, and this mapping is a local isomorphism [**6**].

Thus, the matrix representations of $F^q_{k,n}$ carry over to matrix representations of the divisible closure of MF.

ENDNOTES

1. There are analogous results for the additive group GLPs and the group of additive arithmetic functions.

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Shandong Normal University, School of Mathematics and Statistics, Ji'nan, Shandong, 250358 P.R. China
Email address: **huilan.li@gmail.com**

York University, Department of Mathematics and Statistics, 4700 Keele Street, Toronto ON, M3J 1P3 Canada
Email address: **machenry@mathstat.yorku.ca**

Universidade Federal Fluminense, Department of Computer Science, Niteroi, Rio de Janeiro, 24210-346 Brasil
Email address: **aconci@ic.uff.br**