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A CHARACTERIZATION OF NON-NOETHERIAN BFDS AND FFDS

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ABSTRACT. Characterizations of bounded and finite factorization domains are given using topological notions. Using our characterizations, the almost Dedekind domain and Prüfer domain constructed by Grams **[3]** are shown to be a BFD and an FFD, respectively. For a class of almost Dedekind (not Dedekind) domains it is shown that satisfying the ascending chain condition for principal ideals implies BFD.

1. Motivation. The study of factorization in integral domains has a rich history. In particular, the literature on factorization in Dedekind domains is quite extensive. This paper is motivated in two regards. First, while the literature contains numerous examples of non-Noetherian domains satisfying or failing to satisfy various factorization properties, there has been no attempt to classify the subset of non-Noetherian domains that possesses finite factorization, bounded factorization and the ascending chain condition for principal ideals (ACCP). This paper presents characterization of both finite factorization domains (FFDs) and bounded factorization domains (BFDs).

In an integral domain, it is known that

$$\text{FFD} \implies \text{BFD} \implies \text{ACCP} \implies \text{atomic},$$

and none of these arrows may be reversed, see **[1]**. All Dedekind domains are FFDs; hence, all of the arrows may be reversed in the class of Dedekind domains. A question of interest is when some (or possibly all) of these arrows can be reversed. Using our characterizations, we show that there exists a class of almost Dedekind (not Dedekind) domains in which ACCP implies BFD.

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The study of almost Dedekind and Prüfer domains has been the topic of much research. A small sample of such research includes [**5, 6, 7**]. From a factorization point of view, the constructions in [**3**] are of great interest. Grams' construction was the first non-Noetherian example of a domain satisfying ACCP. This construction is almost Dedekind. Moreover, she constructed a non-Noetherian atomic Prüfer domain. We show that Grams' almost Dedekind domain is a BFD, and her Prüfer domain is an FFD.

2. ACCP, BFD and FFD in integral domains. Let D be an integral domain, and let $U(D)$ be the set of units of D. Let D^* denote $D \setminus \{0\}$. We say that an integral domain D is an FFD if, for all $b \in D^* \setminus U(D)$, the set $Z(b) = \{d \in D \setminus U(D) : d \mid b\}$ is finite, that is, in an FFD, every nonzero element has finitely many divisors up-to-associates. We establish that D is a BFD if D is atomic and if, for all $b \in D^* \setminus U(D)$, there exists a $\pi(b) \in \mathbb{N}$, such that, whenever $b = a_1 a_2 \cdots a_k$ is a factorization of b into a product of irreducibles (atoms), then $k \leq \pi(b)$. Also, D is said to satisfy the ascending chain condition on principal ideals (ACCP), if every chain of strictly increasing principal ideals terminates.

Let D be a domain, and let $\operatorname{Max}(D)$ denote the set of maximal ideals of D. We say that D is almost Dedekind if, for all $M \in \operatorname{Max}(D)$, the localization D_M is a Noetherian valuation domain. A domain is said to be Prüfer if D_M is a valuation domain for all $M \in \operatorname{Max}(D)$. For $b \in D$, we will denote the set of maximal ideals that contain b by $\max(b)$.

Definition 2.1. Let D be an integral domain, and let $b \in D^*$. We say $Z(b)$ is disconnected if there exists an $\{a_i\}_{i=1}^{\infty} \subseteq Z(b)$ such that $\max(a_i) \cap \max(a_j) = \emptyset$ whenever $i \neq j$. We say that $Z(b)$ is connected if it is not disconnected.

We extend this definition to a domain, then present a lemma that will show the usefulness of connectedness.

Definition 2.2. An integral domain D is connected if, for all $b \in D$, $Z(b)$ is connected. We will say that D is disconnected if there exists a $b \in D$ such that $Z(b)$ is disconnected.

Lemma 2.3. *Let D be an integral domain, and let $d \in D^*$ with $a, b \in Z(d)$. If* $\max(a) \cap \max(b) = \emptyset$*, then $ab \in Z(d)$.*

Proof. We will use the fact that $D = \cap_{M \in \mathrm{Max}(D)} D_M$. We first observe that both d/a, $d/b \in D_M$ for all $M \in \mathrm{Max}(D)$. Now, since $b \notin M$ for all $M \notin \max(b)$, it is the case that $d/ab \in D_M$ for all $M \notin \max(b)$. Then, since $d/b \in D_M$ for all M and $a \notin M \in \max(b)$, we have that $d/ab \in D_M$ for all $M \in \max(b)$. Thus, $d/ab \in D_M$ for all M. We conclude that $ab \in Z(d)$. $\square$

We now show that connectedness is a necessary condition for a domain to be ACCP.

Theorem 2.4. *If D satisfies* ACCP*, then D is connected.*

Proof. Suppose that D is disconnected. Then, there exists a $d \in D$ such that $Z(d)$ is disconnected. We find $\{a_i\}_{i=1}^{\infty} \subset Z(d)$ such that $\max(a_i) \cap \max(a_j) = \emptyset$ for all $i \neq j$. Now, using Lemma 2.3, we see that

$$(d) \subsetneq \left(\frac{d}{a_1}\right) \subsetneq \left(\frac{d}{a_1 a_2}\right) \subsetneq \left(\frac{d}{a_1 a_2 a_3}\right) \cdots$$

is an infinite strictly increasing chain of principal ideals. Hence, D does not satisfy ACCP. $\square$

Further, since ACCP is a consequence of FFD and BFD, we see that FFDs and BFDs need to be connected. One might ask whether connectedness is sufficient for any of these conditions. The answer is no; in fact, a domain can be connected and not even be atomic.

Example 2.5. The domain $D = \mathbb{Z}_{(2)} + x\mathbb{Q}[[x]]$ is connected but not atomic. In order to see this, observe that D is quasi-local, and x can never be factored as a finite product of atoms.

We define another useful topological notion regarding $Z(b)$.

Definition 2.6. Let D be an integral domain, and let $b \in D^*$. We say that $S = \{M_1, M_2, \ldots, M_k\} \subset \max(b)$ is a finite covering of $Z(b)$ if, for all $d \in Z(b)$, there exists an $i \in \{1, \ldots, k\}$ such that $d \in M_i$. We further state that D is finitely coverable if, for all $b \in D^*$, $Z(b)$ has a finite covering.

Example 2.5 shows that an integral domain can be finitely coverable and yet fail to be atomic. However, if D is almost Dedekind and finitely coverable, then D is a BFD. Let D be almost Dedekind, and denote the local valuation map from D_M into $\mathbb{N}_0$ by ν_M. Recall that, if $b \in M$, then $\nu_M(b) > 0$ and

$$\nu_M\left(\frac{b}{d}\right) = \nu_M(b) - \nu_M(d).$$

For more on factoring in almost Dedekind domains, see [**4**].

Theorem 2.7. *Let D be an almost Dedekind domain. If D is finitely coverable, then D is a* BFD.

Proof. Let $b \in D^*$. Now, find $S = \{M_1, M_2, \ldots, M_k\}$ that covers $Z(b)$. Next, since every divisor d of b is contained in some M_i, the value of b/d is decreased by at least one in M_i. Thus,

$$\pi(b) = \sum_{i=1}^{k} \nu_{M_i}(b)$$

is a bound on the length of factorizations of b. $\square$

In [**6**], Lucas and Loper introduced the notion of dull and sharp maximal ideals. For a one-dimensional Prüfer domain a maximal ideal is said to be *sharp* if it is a radical of a finitely generated ideal. Maximal ideals that are not sharp are said to be *dull*. We wish to extend these notions to a general integral domain.

Let D be an integral domain, and let $\mathcal{F} = \{b \in D : |\max(b)| < \infty\}$. Now, clearly, if two elements b and c are in only finitely many maximal ideals, their product bc is in only finitely many maximal ideals. Further, if $b \in \mathcal{F}$ and c divides b, we must have $b = cl$ for some l. It is clear from the equation that c can only be in finitely many maximal ideals. Thus, $\mathcal{F}$ is a multiplicatively closed saturated set. Therefore, in a one-dimensional integral domain, $\mathcal{F}$ must be the set complement of the union of maximal ideals.

Thus,

$$\mathcal{F} = \left(\bigcup_{M \in M^\infty} M \right)^c,$$

for some $M^\infty \subset \mathrm{Max}(D)$. Therefore, we see that

$$\mathcal{F}^c = \bigcup_{M \in M^\infty} M,$$

that is, if $b \in M$ for some $M \in M^\infty$, then $|\max(b)| = \infty$.

We partition the divisors of $b \in D$ along the same lines. More precisely, let

$$Z^\infty(b) = \{d \in Z(b) : |\max(d)| = \infty\}$$

and

$$Z^F(b) = \{d \in Z(b) : |\max(d)| < \infty\}.$$

Theorem 2.8. *Let D be a connected domain. Then, $Z^F(b)$ is finitely covered for all $b \in D^*$.*

Proof. Let $H = \{a_1, \dots a_l\} \subset Z^F(b)$ be a set that is maximal with respect to $\max(a_1), \max(a_2), \dots, \max(a_l)$ being mutually disjoint. We know that this set must be finite, else D would be disconnected. Now, set $S = \cup_{i=1}^l \max(a_i)$, and note that S is finite since each of the $\max(a_i)$ are finite. Further, if $d \mid b$, we must have that $d \in M$ for some $M \in S$, else H would not be maximal with respect to the $\max(a_i)$s being mutually disjoint. $\square$

An almost Dedekind domain is one-dimensional, giving us the next theorem.

Theorem 2.9. *Let D be an almost Dedekind domain with $M^\infty = \{M_1, M_2, \dots, M_l\}$. The following are equivalent*:

(i) *D is connected*;
(ii) *D satisfies* ACCP;
(iii) *D is a* BFD.

Proof. Suppose that D is connected. Then, for all $b \in D$, $Z^F(b)$ can be finitely covered by some set S. Now, $S \cup M^\infty$ is a finite covering of $Z(b)$. Thus, D is a BFD. It is well known that BFD implies ACCP in any integral domain. We have already established that ACCP implies connected. $\square$

The almost Dedekind domain constructed in [**3**] satisfies ACCP and has only one maximal ideal in M^∞. Thus, we obtain the following corollary.

Corollary 2.10. *The almost Dedekind domain constructed in* [**3**] *is a* BFD.

Now, in order to achieve a characterization of BFDs and FFDs we need to introduce more definitions. We let $Z_M(b) = \{d \in M : d \mid b\}$. Clearly, for a finite factorization domain, the cardinality of this set needs to be finite.

Definition 2.11. Let D be an integral domain and $b \in D^*$. We say that $Z(b)$ behaves finitely if $|Z_M(b)| < \infty$ for all $M \in \max(b)$. We also say that an integral domain is finitely behaved if, for all $b \in D^*$, $Z(b)$ behaves finitely.

Definition 2.12. Let D be an integral domain, and let $b \in D^*$. We say that $Z(b)$ is l-bounded at $M \in \max(b)$ if there exists an $l_M \in \mathbb{N}$ such that, given any $d_1, d_2, \ldots, d_{M_l} \in Z_M(b)$, the product $d_1 d_2 \cdots d_{l_M}$ does not divide b. Moreover, we say that $Z(b)$ is l_∞-bounded if there exists an $l_\infty \in \mathbb{N}_0$ such that, given any $d_1, d_2, \ldots, d_{l_\infty} \in Z^\infty(b)$, the product $d_1 d_2 \cdots d_{l_\infty}$ does not divide b.

Definition 2.13. An integral domain D is l-bounded, if, for all $b \in D^*$, $Z(b)$ is both l- and l_∞-bounded.

We now present characterizations for both BFDs and FFDs. Following the theorems, we give an example to demonstrate why the conditions cannot be relaxed.

Theorem 2.14. *Let D be an integral domain. D is an* FFD *if and only if D is finitely coverable and finitely behaved.*

Proof. Suppose that D is an FFD. It should be clear that D is finitely behaved. Let $b \in D^*$. Now, b has only finitely many divisors, say, $d_1, d_2, \dots, d_k$. Choosing $M_1 \in \max(d_1), M_2 \in \max(d_2), \dots, M_k \in \max(d_k)$, we see that $S = \{M_1, M_2, \dots, M_k\}$ is a finite cover of $Z(b)$

Now, suppose that $Z(b)$ has a finite cover and is finitely behaved. Let $S = \{M_1, M_2, \dots, M_k\}$ be a finite cover of $Z(b)$. Further,

$$|Z(b)| \le \sum_{i=1}^{k} |Z_{M_i}(b)|,$$

showing that $Z(b)$ is finite. We conclude that D is an FFD. $\square$

Theorem 2.15. *Let D be an integral domain. D is a* BFD *if and only if D is connected and l-bounded.*

Proof. It should be clear that, if D is neither connected nor l-bounded, then D is not a BFD. Suppose D is connected and l-bounded, and let $b \in D^*$. Since D is connected, we have from Theorem 2.8 that $Z^F(b)$ is finitely covered, say, by $\{M_1, M_2, \dots, M_k\}$. Now, the length of the factorization of $b \le \pi(b) = l_\infty + \sum_{i=1}^{k} l_{M_i}$. Thus, D is a BFD. $\square$

An almost Dedekind domain D is said to be a sequence domain if $(D) = \{M_1, M_2, \dots\} \cup M^*$ such that each M_i is principal and M^* is a dull maximal ideal. Moreover, D has a nonzero Jacobson radical $\mathcal{J}$. Now, given $b \in \mathcal{J}$, $\nu_{M_i}(b)$ and $\nu_{M^*}(b)$ are bounds showing that $Z(b)$ is l-connected. However, D fails to be finitely coverable, finitely behaved or connected. All sequence domains fail to be atomic, see [**4**]. This shows that l-bounded is not enough to force bounded factorization. A brief discussion on sequence domains may be found in [**5**].

In [**3**], a Prüfer domain was constructed by taking a union of Dedekind domains. The constructed domain is of finite character, that is, a domain such that $\max(b)$ is finite for all nonzero $b \in D$. Furthermore, this one-dimensional domain contains one idempotent maximal ideal M^* such that

$$M^* \subseteq \bigcup_{M \in \mathrm{Max}(D) \setminus M^*} M.$$

We now show that this domain is an FFD.

Corollary 2.16. *The Prüfer domain constructed in* [**3**] *is an* FFD.

Proof. Since D is of finite character, D is finitely coverable for all $b \in D^*$. Now, the value group of D_M is $\mathbb{Z}$ for all $M \neq M^* \in \mathrm{Max}(D)$. Further, $b \in D$ if and only if $\nu_M(b) \geq 0$ for all $M \neq M^* \in \mathrm{Max}(D)$. Suppose that $b \notin M^*$. Denote $\max(b) = \{M_1, M_2, \dots M_k\}$.

Now, the number of divisors of b is bounded by

$$\prod_{i=1}^{k} (\nu_{M_i}(b) + 1),$$

which is finite.

Now, suppose that $b \in M^*$. Let $\max(b) = \{M_1, M_2, \dots, M_k, M^*\}$. Suppose that D is not an FFD. Since

$$M^* \subseteq \bigcup_{M \in \mathrm{Max}(D) \setminus M^*} M,$$

it must be the case that there exist $d_1, d_2 \in Z(b)$ such that $\nu_{M_i}(d_1) = \nu_{M_i}(d_2)$ for all $i = 1, 2, \dots, k$. (There are only $\prod_{i=1}^{k} (\nu_{M_i}(b) + 1) < \infty$ choices for the values of any divisor on the set $\{M_1, M_2, \dots, M_k\}$.) Now, without loss of generality, assume that $\nu_{M^*}(d_1) > \nu_{M^*}(d_2)$. However, then $\nu_M(d_1/d_2) = 0$ for all $M \neq M^* \in \mathrm{Max}(D)$ and $\nu_{M^*}(d_1/d_2) > 0$, which is a contradiction since $\max(d_1/d_2) = \{M^*\}$. We conclude that D is an FFD. $\square$

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