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ON THE STRUCTURE OF MULTIPLIER ALGEBRAS

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ABSTRACT. This note gives a characterization of matrix structures for multipliers of a stable C^*-algebra $A \otimes \mathcal{K}$ with any C^*-algebra A. We represent elements in $(A \otimes \mathcal{K})''$, $QM(A \otimes \mathcal{K})$ and $M(A \otimes \mathcal{K})$ as infinite matrices over certain C^*-algebras, respectively. These results generalize the related work of Brown, Lin and Zhang in this area.

1. Introduction and preliminaries. Multiplier algebras play a crucial role in the theory of C^*-algebras and their extensions. In some early work, semicontinuity was used to give characterizations of multipliers, [**1, 2, 3, 4**]. In the early 1980s, Brown [**5**] took another approach to reveal the structures of quasi-multiplier algebra of a stable C^*-algebra. He represented quasi-multipliers of $A\otimes\mathcal{K}$ as certain infinite matrices in the setting of A being a unital C^*-algebra and proved a necessary and sufficient condition for

$$QM(A \otimes \mathcal{K}) = LM(A \otimes \mathcal{K}) + RM(A \otimes \mathcal{K}).$$

Brown's idea and work on this topic were adopted and developed by Lin and Zhang. In [**12**], Zhang gave a representation of multipliers of $A\otimes\mathcal{K}$ in the case where $A \otimes \mathcal{K}$ is stably unital. Lin [**7**] constructed matrix structures of multipliers of A when A is σ-unital, and subsequently, he provided an in-depth series of research on quasi-multipliers and multipliers, see [**8, 9, 10**].

Inspired by the above work, this note is engaged in characterizing the matrix structure of multiplier algebras of stable C^*-algebras. In contrast with previous work we do not require that A be unital or σ-unital. This is an essential difference because $A \otimes \mathcal{K}$ is no longer σ-unital. As a result, we represent elements in $(A \otimes \mathcal{K})''$, $QM(A \otimes \mathcal{K})$

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and $M(A \otimes \mathcal{K})$ as infinite matrices over certain C^*-algebras for any C^*-algebra A, respectively.

Suppose that A is a C^*-algebra and A'' is its enveloping von Neumann algebra. An element x in A'' is called a multiplier of A if $xa, ax \in A$ for any $a \in A$. Similarly, x is a left multiplier if $xa \in A$ for any $a \in A$, x is a right multiplier if $ax \in A$ for any $a \in A$, and x is a quasi-multiplier if $axb \in A$ for all $a, b \in A$. Denote the sets of multipliers, left multipliers, right multipliers and quasi-multipliers by $M(A)$, $LM(A)$, $RM(A)$ and $QM(A)$, respectively.

Recall that $M(A)$ is the completion of A in the strict topology, and $LM(A)$, $RM(A)$ and $QM(A)$ are norm closed subspaces of A''. Moreover,

$$LM(A)^* = RM(A) \quad \text{and} \quad M(A) = LM(A) \cap RM(A).$$

Hence, $M(A)$ is a C^*-algebra.

Let D be a C^*-algebra. Denote the set of infinite matrices over D by

$$M_\infty(D) = \{(x_{ij}) : x_{ij} \in D,\ i, j = 1, 2, \ldots\}.$$

2. Main results. Suppose that H and H_1 are two Hilbert spaces such that H_1 is separable and infinite-dimensional. Let $\{\varepsilon_1,\ \varepsilon_2, \ldots\}$ be an orthonormal basis for H_1 and $\mathcal{K} = \mathcal{K}(H_1)$ the compact operators on H_1. Suppose that $\{e_{ij} : i, j = 1, 2, \ldots\}$ is the standard matrix unit of $\mathcal{K}$ corresponding to $\{\varepsilon_1,\ \varepsilon_2, \ldots\}$. Then, there is an isomorphism

$$\alpha : H \otimes H_1 \longrightarrow \bigoplus_{i=1}^{\infty} H$$

such that $\alpha(x \otimes \varepsilon_i) = (0, \ldots, 0, x, 0, \ldots)$ for any $x \in H$ and $i \in \mathbb{N}$, where x is on the ith entry. Under this isomorphism,

$$B(H \otimes H_1) \cong B\bigg(\bigoplus_{i=1}^{\infty} H\bigg).$$

For every $T \in B(\oplus_{i=1}^{\infty} H)$, there is a unique matrix (T_{ij}) with entries in $B(H)$ such that

$$T\xi = (T_{ij}) \begin{pmatrix} x_1 \\ x_2 \\ \vdots \end{pmatrix} = \Bigg(\sum_{j=1}^{\infty} T_{1j} x_j, \sum_{j=1}^{\infty} T_{2j} x_j, \dots \Bigg),$$

where $\xi = (x_1, x_2, \dots) \in \oplus_{i=1}^{\infty} H$.

In order to differentiate the relation between infinite matrices over $B(H)$ and bounded operators on $\oplus_{i=1}^{\infty} H$, we need the following two propositions. Although they may be known to specialists, we provide them here for the sake of completeness.

Proposition 2.1. *Let $(T_{ij}) \in M_\infty(B(H))$. Then, the following are equivalent*:

(i) *(T_{ij}) represents an element in $B(\oplus_{i=1}^{\infty} H)$*;
(ii) $\sup\{\|(T_{ij})_{1\le i,j\le n}\|_{\oplus_{i=1}^{n} H} : n \in \mathbb{N}\} < +\infty$;
(iii) *$\{\sum_{ij}^{n} T_{ij} \otimes e_{ij}\}_{n=1}^{\infty}$ converges in the* sot *in $B(H \otimes H_1)$ as $n \to \infty$.*

Proof.

(i) $\Leftrightarrow$ (ii). This is from **[6,** 2.6.13].

(iii) $\Rightarrow$ (ii). Since

$$\Bigg\| \sum_{ij}^{n} T_{ij} \otimes e_{ij} \Bigg\| = \Bigg\| \underset{\substack{1\le i\le n \\ 1\le j\le n}}{(T_{ij})} \Bigg\|,$$

by the uniformly bounded theorem,

$$\sup \left\{ \Bigg\| \underset{\substack{1\le i\le n \\ 1\le j\le n}}{(T_{ij})} \Bigg\|_{\oplus_{i=1}^{n} H} : n \in \mathbb{N} \right\} < +\infty.$$

(i) $\Rightarrow$ (iii). Suppose that (T_{ij}) represents an element T in $B(\oplus_{i=1}^{\infty} H)$. Then, for any $\xi = (x_1, x_2, \dots) \in \oplus_{i=1}^{\infty} H$,

$$T\xi = \Bigg(\sum_{j=1}^{\infty} T_{1j} x_j, \sum_{j=1}^{\infty} T_{2j} x_j, \dots \Bigg).$$

For every $x \in H$ and $l \in \mathbb{N}$,

$$\alpha(x \otimes \varepsilon_l) = (0, \ldots, 0, x, 0, \ldots)$$

and, when $n > l$,

$$\left(\sum_{ij}^{n} T_{ij} \otimes e_{ij}\right)(x \otimes \varepsilon_l) = \sum_{i}^{n} T_{il} \otimes \varepsilon_i.$$

It follows that

$$\begin{aligned} T \circ \alpha(x \otimes \varepsilon_l) &= (T_{1l}x,\ T_{2l}x, \ldots) \\ &= \lim_{n\to\infty} \alpha\left(\sum_{i}^{n} T_{il} \otimes \varepsilon_i\right) \\ &= \lim_{n\to\infty} \alpha\left(\sum_{ij}^{n} T_{ij} \otimes e_{ij}\right)(x \otimes \varepsilon_l). \end{aligned}$$

By the above proof, the sequence $\sum_{ij}^{n} T_{ij} \otimes e_{ij}$ is uniformly bounded and

$$\overline{\text{span}}\{x \otimes \varepsilon_i : x \in H,\ i = 1, 2, \ldots\} = H \otimes H_1.$$

Hence,

$$\sum_{ij}^{n} T_{ij} \otimes e_{ij} \xrightarrow{\text{sot}} \alpha^{-1} \circ\ T \circ \alpha, \quad \text{as } n \to \infty. \qquad \square$$

Proposition 2.2. *Suppose that D is a C^*-subalgebra of $B(H)$ and $(T_{ij}) \in M_\infty(B(H))$. Then, (T_{ij}) represents an element T in $D \otimes \mathcal{K}$ if and only if every $T_{ij} \in D$ and*

$$\sum_{ij}^{n} T_{ij} \otimes e_{ij} \xrightarrow{\|\cdot\|} T.$$

Proof. Suppose that (T_{ij}) represents T in $D \otimes \mathcal{K}$. We note that $e_{ij}(\varepsilon_l) = \delta_{jl}\varepsilon_i$, where δ_{jl} is the Kronecker symbol. Then, $(1 \otimes e_{ii})T(1 \otimes e_{jj}) = T_{ij} \otimes e_{ij}$. Since $T \in D \otimes \mathcal{K}$, we have

$$T_{ij} \otimes e_{ij} \in (1 \otimes e_{ii})(D \otimes \mathcal{K})(1 \otimes e_{jj}) = D \otimes e_{ij}.$$

Hence, $T_{ij} \in D$.

Note that

$$\sum_{ij}^{n} T_{ij} \otimes e_{ij} = \sum_{ij}^{n} (1 \otimes e_{ii}) T (1 \otimes e_{jj}) = \Big(\sum_{1}^{n} 1 \otimes e_{ii} \Big) T \Big(\sum_{1}^{n} 1 \otimes e_{jj} \Big),$$

and $\{\sum_1^n 1 \otimes e_{ii}\}_{n=1}^{\infty}$ is an approximate unit of $D \otimes \mathcal{K}$ (which may not be contained in $D \otimes \mathcal{K}$). It follows that $\sum_{ij}^n T_{ij} \otimes e_{ij}$ converges to T in the norm in $D \otimes \mathcal{K}$.

Conversely, since $T_{ij} \in D$ and $\sum_{ij}^n T_{ij} \otimes e_{ij} \to T$ in the norm, then $T \in D \otimes \mathcal{K}$. By Proposition 2.1, (T_{ij}) represents the bounded operator T. $\square$

Let A be a C^*-algebra. Suppose that $\pi : A \to B(H_\pi)$ is the universal representation of A and A'' is the universal enveloping von Neumann algebra of A. Let H be a separable, infinite-dimensional Hilbert space and $\mathcal{K} = \mathcal{K}(H)$ the compact operators on H. Then, we get a representation of $A \otimes \mathcal{K}$,

$$\varphi = \pi \otimes \iota : A \otimes \mathcal{K} \longrightarrow B(H_\pi \otimes H),$$

where ι is the inclusion map from $\mathcal{K}$ into $B(H)$.

Let $A'' \bar{\otimes} B(H)$ be the von Neumann tensor product of A'' and $B(H)$. Then, $(A \otimes \mathcal{K})'' \cong A'' \bar{\otimes} B(H)$ as C^*-algebras. Since π and ι are faithful and non-degenerate, then so is φ. If we identify $A \otimes \mathcal{K}$ with its images under these homomorphisms, then we have the following relation of the above algebras:

$$A \otimes \mathcal{K} \subset (A \otimes \mathcal{K})'' \cong A'' \bar{\otimes} B(H) \subset B(H_\pi \otimes H).$$

Let $M(A)$ be the multiplier algebra of A and $1_{M(A)}$ the unit of $M(A)$. Suppose that $\{\varepsilon_1, \varepsilon_2, \dots\}$ is an orthonormal basis of H and $\{e_{ij} : i, j = 1, 2, \dots\}$ is the standard matrix unit of $\mathcal{K}$ corresponding to $\{\varepsilon_1, \varepsilon_2, \dots\}$. Set $p_n = \sum_1^n e_{ii}$. Then $\{p_n\}$ is an approximate unit of $\mathcal{K}$.

Recall that the strict topology (st) on $B(H_\pi \otimes H)$ is induced by $A \otimes \mathcal{K}$, which is induced by the family of semi-norms of:

$$p_a(x) = \|xa\| + \|x^* a\|, \qquad a \in A \otimes \mathcal{K}, \quad x \in B(H_\pi \otimes H).$$

Hence, $x_\alpha \xrightarrow{\text{st}} x$ in $B(H_\pi \otimes H)$ if and only if, for every a in $A \otimes \mathcal{K}$, $ax_\alpha \xrightarrow{\|\cdot\|} ax$ and $x_\alpha a \xrightarrow{\|\cdot\|} xa$. Set $P_n = 1_{M(A)} \otimes p_n$. Then, $P_n \xrightarrow{\text{st}} 1_{B(H_\pi \otimes H)}$ in $B(H_\pi \otimes H)$.

Next, we try to establish the connection of $B(H_\pi \otimes H)$ and infinite matrices over $B(H_\pi)$ and specialize this connection for several important C^*-subalgebras of $B(H_\pi \otimes H)$.

Theorem 2.3. *Suppose A is a C^*-algebra. Let H_π and $B(H_\pi \otimes H)$ be as above.*

(i) *There is an injection Φ from $B(H_\pi \otimes H)$ into $M_\infty(B(H_\pi))$ with $\Phi(x) = (x_{ij})$, such that $x_{ij} \otimes e_{ij} = (1 \otimes e_{ii})x(1 \otimes e_{jj})$. Now, $\{\sum_{ij}^n x_{ij} \otimes e_{ij}\}$ converges to x in the strong operator topology* (sot).

Conversely, if $(x_{ij}) \in M_\infty(B(H_\pi))$ such that $\{\sum_{ij}^n x_{ij} \otimes e_{ij}\}$ converges to some $x \in B(H_\pi \otimes H)$ in the sot, *then the matrix (x_{ij}) represents x in the above correspondence, i.e., $\Phi(x) = (x_{ij})$.*

(ii) *Let $(x_{ij}) \in M_\infty(B(H_\pi))$. Then, there exists an $x \in (A \otimes \mathcal{K})''$ such that $\Phi(x) = (x_{ij})$ if and only if $x_{ij} \in A''$ and $\sum_{ij}^n x_{ij} \otimes e_{ij} \xrightarrow{\text{sot}} x$.*

Proof.

(i) Let $x \in B(H_\pi \otimes H)$. Since $H = \oplus_{i=1}^\infty \mathbb{C}\varepsilon_i$, we have

$$H_\pi \otimes H = \bigoplus_{i=1}^\infty (H_\pi \otimes \varepsilon_i) \cong \bigoplus_{i=1}^\infty H_\pi.$$

Set $x'_{ij} = (1 \otimes e_{ii})x(1 \otimes e_{jj}) \in B(H_\pi \otimes H)$. Note that $1 \otimes e_{ii}$ and $1 \otimes e_{jj}$ are the projections of $H_\pi \otimes \varepsilon_i$ and $H_\pi \otimes \varepsilon_j$, respectively. Hence, x'_{ij} can be identified with its restriction on $H_\pi \otimes \varepsilon_j$. Then, $x'_{ij} \in B(H_\pi \otimes \varepsilon_j, H_\pi \otimes \varepsilon_i)$. Since $e_{ij}(\varepsilon_j) = \varepsilon_i$, we have

$$B(H_\pi \otimes \varepsilon_j, H_\pi \otimes \varepsilon_i) = B(H_\pi) \otimes e_{ij}.$$

Thus, there is a unique $x_{ij} \in B(H_\pi)$ such that $x'_{ij} = x_{ij} \otimes e_{ij}$ for all $i, j \in \mathbb{N}$.

Define a map Φ from $B(H_\pi \otimes H)$ into $M_\infty(B(H_\pi))$ by $\Phi(x) = (x_{ij})$, where x_{ij} is obtained from the preceding proof.

Note that the sequence $\{P_n\}$ is bounded and the representation

$$\varphi : A \otimes \mathcal{K} \longrightarrow B(H_\pi \otimes H)$$

is non-degenerate. Since $\{P_n\}$ converges to 1 in the strict topology, it converges to 1 in the sot in $B(H_\pi \otimes H)$. Thus, $P_n x P_n \to x$ in the sot as n tends to infinity. Since $P_n x P_n = \sum_{ij}^n x_{ij} \otimes e_{ij}$, then $\sum_{ij}^n x_{ij} \otimes e_{ij} \overset{\text{sot}}{\to} x$.

Let $y \in B(H_\pi \otimes H)$ and $y \neq x$. Suppose (y_{ij}) represents y. Then $(x_{ij}) \neq (y_{ij})$. This is equivalent to saying that there are i, j such that $x_{ij} \neq y_{ij}$. Otherwise, if $x_{ij} = y_{ij}$ for all i, j, then

$$P_n x P_n = \sum_{ij}^n x_{ij} \otimes e_{ij} = \sum_{ij}^n y_{ij} \otimes e_{ij} = P_n y P_n.$$

By the above proof, we have $P_n x P_n \to x$ and $P_n y P_n \to y$, and hence, $x = y$. This is a contradiction. Therefore, the map $x \mapsto (x_{ij})$ is injective.

Conversely, suppose that $(x_{ij}) \in M_\infty(B(H_\pi))$ such that $\sum_{ij}^n x_{ij} \otimes e_{ij}$ converges to some $x \in B(H_\pi \otimes H)$ in the sot. Fix $k, l \in \mathbb{N}$. Then

$$(1 \otimes e_{ll}) \left(\sum_{ij}^n x_{ij} \otimes e_{ij} \right) (1 \otimes e_{kk}) \xrightarrow{\text{sot}} (1 \otimes e_{ll}) x (1 \otimes e_{kk})$$

as $n \to \infty$. When $n \geq \max\{k, l\}$, we have

$$(1 \otimes e_{ll}) \left(\sum_{ij}^n x_{ij} \otimes e_{ij} \right) (1 \otimes e_{kk}) = x_{lk} \otimes e_{lk}.$$

Hence, $(1 \otimes e_{ll}) x (1 \otimes e_{kk}) = x_{lk} \otimes e_{lk}$. Therefore, (x_{ij}) represents x.

(ii) Suppose that x is in $(A \otimes \mathcal{K})''$. By the Kaplansky density theorem, there is a bounded net $\{x_\alpha\} \subset A \otimes \mathcal{K}$ such that $\{x_\alpha\}$ converges to x as α tends to α_0 in the weak operator topology (wot).

Note that $(A \otimes \mathcal{K})'' \cong A'' \bar{\otimes} B(H)$. Since $*$-isomorphisms between von Neumann algebras are continuous with respect to the σ-wot, and it is also known that σ-wot is identified with wot on bounded subsets, we have $x_\alpha \overset{\text{wot}}{\to} x$ in $A'' \bar{\otimes} B(H)$. Then

$$(1 \otimes e_{ii}) x_\alpha (1 \otimes e_{jj}) \xrightarrow{\text{wot}} (1 \otimes e_{ii}) x (1 \otimes e_{jj})$$

in $A'' \bar{\otimes} B(H)$.

By the fact that $x_\alpha \in A \otimes \mathcal{K} \subset A'' \otimes B(H)$, then

$$(1 \otimes e_{ii}) x_\alpha (1 \otimes e_{jj}) \in A \otimes e_{ij} \subset A'' \otimes e_{ij}.$$

Since $A'' \otimes e_{ij}$ is closed in the wot, then $(1 \otimes e_{ii})x(1 \otimes e_{jj}) \in A'' \otimes e_{ij}$. Therefore, there is an $x_{ij} \in A''$ such that $x_{ij} \otimes e_{ij} = (1 \otimes e_{ii})x(1 \otimes e_{jj})$. Since $M_\infty(A'') \subset M_\infty(B(H_\pi))$ and Φ is injective, we have $\Phi(x) = (x_{ij})$ and $\sum_{ij}^n x_{ij} \otimes e_{ij} \overset{\text{sot}}{\to} x$.

On the other hand, since $x_{ij} \in A''$, we have

$$x_{ij} \otimes e_{ij} \in A'' \otimes \mathcal{K} \subset (A \otimes \mathcal{K})''.$$

Since $\sum_{ij}^n x_{ij} \otimes e_{ij} \overset{\text{sot}}{\to} x$, $x_{ij} \in (A \otimes \mathcal{K})''$. By (i), $\Phi(x) = (x_{ij})$. □

Using Theorem 2.3, we can build a C^*-construction on the set of infinite matrices which represent bounded operators such that Φ is a C^*-algebra isomorphism. Obviously, $M_\infty(B(H_\pi))$ can be equipped with an addition, a scalar-multiplication and an involution, as usual, which make it a linear space with an involution. However, the usual multiplication of matrix algebras dose not exist on infinite matrices, in general.

Let $E = \{(x_{ij}) \in M_\infty(B(H_\pi)) : \sup\{\| \sum_{ij}^n x_{ij} \otimes e_{ij}\| : n \in \mathbb{N}\} < +\infty\}$. Then E is a self-adjoint linear subspace of $M_\infty(B(H_\pi))$. We can check that the function

$$\|(x_{ij})\| = \sup\left\{\left\| \sum_{ij}^n x_{ij} \otimes e_{ij}\right\| : n \in \mathbb{N}\right\}$$

transforms E into a linear normed space with $\|(x_{ij})^*\| = \|(x_{ij})\|$.

Next, we define multiplication on E as follows.

For any $(x_{ij}),\ (y_{ij}) \in E$, let

$$(x_{ij})(y_{ij}) = (z_{ij}) \quad \text{where } z_{ij} = (\text{sot}) \sum_{k=1}^{\infty} x_{ik} y_{kj}$$

for $i, j = 1, 2, \ldots$.

Proposition 2.4. *The above map is indeed a multiplication on E, and thus, E constitutes a C^*-algebra which is isomorphic to $B(H_\pi \otimes H)$.*

Proof. Firstly, we need to show that the definition is well defined. For $(x_{ij}),\ (y_{ij}) \in E$, suppose that x and y are the elements in

$B(H_\pi \otimes H)$ which correspond to (x_{ij}) and (y_{ij}), respectively. Then, $\Phi(x) = (x_{ij})$ and $\Phi(y) = (y_{ij})$. Hence,

$$x_{ij} \otimes e_{ij} = (1 \otimes e_{ii})x(1 \otimes e_{jj})$$

and

$$y_{ij} \otimes e_{ij} = (1 \otimes e_{ii})y(1 \otimes e_{jj}).$$

Let $z = xy$, and set $\Phi(z) = (z_{ij})$. Since $1 = (\mathrm{sot}) \sum_{k=1}^{\infty} 1 \otimes e_{kk}$, and the multiplication in $B(H_\pi \otimes H)$ is jointly continuous on bounded subsets in the sot, then

$$\begin{aligned}(\mathrm{sot}) \lim_{n\to\infty} \left((1 \otimes e_{ii})x\left(\sum_{k=1}^{n} 1 \otimes e_{kk}\right)\right)\left(\left(\sum_{k=1}^{n} 1 \otimes e_{kk}\right)x(1 \otimes e_{jj})\right)& \\ = (1 \otimes e_{ii})xy(1 \otimes e_{jj}) = z_{ij} \otimes e_{ij}&.\end{aligned}$$

Note that

$$\begin{aligned}\left(\sum_{k=1}^{n} x_{ik}y_{kj}\right)\otimes e_{ij} &= \left(\sum_{k=1}^{n} x_{ik} \otimes e_{ik}\right)\left(\sum_{k=1}^{n} y_{kj} \otimes e_{kj}\right) \\ &= \left((1 \otimes e_{ii})x\left(\sum_{k=1}^{n} 1 \otimes e_{kk}\right)\right)\left(\left(\sum_{k=1}^{n} 1 \otimes e_{kk}\right)x(1 \otimes e_{jj})\right).\end{aligned}$$

Hence,

$$\left(\sum_{k=1}^{n} x_{ik}y_{kj}\right) \otimes e_{ij} \xrightarrow{\mathrm{sot}} z_{ij} \otimes e_{ij}$$

as $n \to \infty$. Furthermore, $(\mathrm{sot}) \sum_{k=1}^{\infty} x_{ik}y_{kj} = z_{ij}$ for $i, j = 1, 2, \ldots$. Therefore, $\Phi(xy) = \Phi(x)\Phi(y)$.

Secondly, by Proposition 2.1 and Theorem 2.3, Φ is a surjective *-isometry. It follows that E is a C^*-algebra with the operations defined above, and Φ is an isomorphism between $B(H_\pi \otimes H)$ and E. $\square$

Theorem 2.5. *Let (x_{ij}) be in $M_\infty(A'')$. Then:*

(i) *there is an $x \in QM(A\otimes\mathcal{K})$ such that $\Phi(x) = (x_{ij})$ if and only if every $x_{ij} \in QM(A)$ and $\sum_{ij}^n x_{ij}\otimes e_{ij} \overset{\text{sot}}{\to} x$.*

(ii) *Suppose that every $x_{ij} \in LM(A)$,*

$$\sum_{ij}^n x_{ij}\otimes e_{ij} \xrightarrow{\text{sot}} x,$$

and there is an increasing sequence $\{n_k\}$ such that $\{\sum_{(i,j)\in\sigma_n} x_{ij}\otimes e_{ij}\}_{n=1}^\infty$ converges in the norm in $A\otimes\mathcal{K}$. Then $x \in LM(A\otimes\mathcal{K})$, where

$$\sigma_n = \{(i,j) : \textit{there exists } k > l, \textit{ such that } n \ge n_k \ge i > n_{k-1},\\ n \ge n_l \ge j > n_{l-1}\}.$$

(iii) *There is an $x \in A\otimes\mathcal{K}$ such that $\Phi(x) = (x_{ij})$ if and only if every $x_{ij} \in A$ and*

$$\sum_{ij}^n x_{ij}\otimes e_{ij} \xrightarrow{\|\cdot\|} x.$$

Proof.

(i) Let $x \in QM(A\otimes\mathcal{K})$ with $\Phi(x) = (x_{ij})$. By the proof of Theorem 2.3 (ii), we have $x_{ij}\otimes e_{ij} = (1\otimes e_{ii})x(1\otimes e_{jj}) \in A''\otimes e_{ij}$. Suppose that $\{e_\alpha\}$ is an approximate unit of A. Since $e_\alpha \overset{\text{st}}{\to} 1$ in $M(A)$, then

$$e_\alpha\otimes e_{ii} \xrightarrow{\text{st}} 1\otimes e_{ii}$$

in $A''\otimes e_{ii}$, where the strict topology on $A''\otimes e_{ii}$ is inherited from that on $B(H_\pi\otimes H)$. Similarly, $e_\alpha\otimes e_{jj} \overset{\text{st}}{\to} 1\otimes e_{jj}$ in $A''\otimes e_{ii}$. Hence, for any $a, b \in A$,

$$(a\otimes e_{ii})(e_\alpha\otimes e_{ii})x(e_\alpha\otimes e_{jj})(b\otimes e_{jj}) \xrightarrow{\|\cdot\|} (a\otimes e_{ii})x(b\otimes e_{jj}).$$

Since $x \in QM(A\otimes\mathcal{K})$, then $(a\otimes e_{ii})(e_\alpha\otimes e_{ii})x(e_\alpha\otimes e_{jj})(b\otimes e_{jj}) \in A\otimes\mathcal{K}$. Note that

$$\begin{aligned}(ax_{ij}b)\otimes e_{ij} &= (a\otimes e_{ii})(x_{ij}\otimes e_{ij})(b\otimes e_{jj})\\ &= (a\otimes e_{ii})(1\otimes e_{ii})x(1\otimes e_{jj})(b\otimes e_{jj})\\ &= (a\otimes e_{ii})x(b\otimes e_{jj}).\end{aligned}$$

Hence, $(ax_{ij}b) \otimes e_{ij} \in A \otimes \mathcal{K}$. Furthermore,

$$\begin{aligned}(ax_{ij}b) \otimes e_{ij} &= (1 \otimes e_{ii})((ax_{ij}b) \otimes e_{ij})(1 \otimes e_{jj}) \\ &\in (1 \otimes e_{ii})(A \otimes \mathcal{K})(1 \otimes e_{jj}) \\ &= A \otimes e_{ij}.\end{aligned}$$

Therefore, $ax_{ij}b \in A$ and $x_{ij} \in QM(A)$.

By Theorem 2.3 (i), it follows that $\sum_{ij}^n x_{ij} \otimes e_{ij} \overset{\text{sot}}{\to} x$.

Conversely, suppose that (x_{ij}) is in $M_\infty(QM(A))$ such that $\sum_{ij}^n x_{ij} \otimes e_{ij} \overset{\text{sot}}{\to} x$ for some $x \in B(H_\pi \otimes H)$. For any $a, b \in A$ and $l, k, s, t \in \mathbb{N}$, we have

$$(a \otimes e_{lk})\bigg(\sum_{ij}^n x_{ij} \otimes e_{ij}\bigg)(b \otimes e_{st}) \xrightarrow{\text{sot}} (a \otimes e_{lk})x(b \otimes e_{st})$$

as $n \to \infty$. Set $N = \max\{k, s\}$. Then, when $n > N$,

$$(a \otimes e_{lk})\bigg(\sum_{ij}^n x_{ij} \otimes e_{ij}\bigg)(b \otimes e_{st}) = (a \otimes e_{lk})\bigg(\sum_{ij}^N x_{ij} \otimes e_{ij}\bigg)(b \otimes e_{st}).$$

Hence, when $n > N$,

$$(a \otimes e_{lk})x(b \otimes e_{st}) = \sum_{ij}^N ax_{ij}b \otimes e_{lk}e_{ij}e_{st} \in A \otimes \mathcal{K}.$$

Since $\operatorname{span}\{a \otimes e_{lk} : a \in A; l, k \in \mathbb{N}\}$ is dense in the norm in $A \otimes \mathcal{K}$, it follows that $(A \otimes \mathcal{K})x(A \otimes \mathcal{K}) \subset A \otimes \mathcal{K}$. Therefore, $x \in QM(A \otimes \mathcal{K})$. By Theorem 2.3, $\Phi(x) = (x_{ij})$.

(ii) Suppose that $\sum_{(i,j)\in\sigma_n} x_{ij} \otimes e_{ij} \xrightarrow{\|\cdot\|} x_0$ for some x_0 in $A \otimes \mathcal{K}$ as $n \to \infty$. Let $y = x - x_0$. Then $y \in (A \otimes \mathcal{K})''$. Set

$$\lambda_n = \{(i, j) : i, j = 1, 2, \ldots, n\} \setminus \sigma_n, \qquad y_n = \sum_{(i,j)\in\lambda_n} x_{ij} \otimes e_{ij}.$$

Then, $y_n \overset{\text{sot}}{\to} y$.

For any $a \in A$ and $l, k \in \mathbb{N}$, $y_n(a \otimes e_{lk}) \overset{\text{sot}}{\to} y(a \otimes e_{lk})$. Note that y_n is the upper triangular part. Hence, when $n > l$, $y_n(a \otimes e_{lk}) = y_l(a \otimes e_{lk})$.

By the assumption that $x_{ij} \in LM(A)$, it follows that

$$y(a \otimes e_{lk}) = y_l(a \otimes e_{lk}) \in A \otimes \mathcal{K}.$$

Therefore, $y(A\otimes\mathcal{K}) \subset A\otimes\mathcal{K}$ and $y \in LM(A\otimes\mathcal{K})$. Finally, by $y = x - x_0$, we have $x \in LM(A \otimes \mathcal{K})$.

(iii) This follows from Proposition 2.2. □

Theorem 2.6. *Let $x \in (A \otimes \mathcal{K})''$ and $(x_{ij}) \in M_\infty(A'')$ satisfy $\Phi(x) = (x_{ij})$. Consider the statements*:

(i) $x \in M(A \otimes \mathcal{K})$;

(ii) $x_{ij} \in M(A)$ *for any* i, j *such that* $\sum_{ij}^n x_{ij} \otimes e_{ij} \overset{\text{st}}{\to} x$ *in* $M(A\otimes\mathcal{K})$;

(iii) $x_{ij} \in M(A)$ *for any* i, j *such that* $\sum_{ij}^n x_{ij} \otimes e_{ij} \overset{\text{sot}}{\to} x$ *in* $(A \otimes \mathcal{K})''$, *and there are increasing subsequences* $\{n_k\}$ *and* $\{m_l\}$ *such that* $\{\sum_{(i,j)\in\sigma_n} x_{ij} \otimes e_{ij}\}_{n=1}^\infty$ *and* $\{\sum_{(i,j)\in\delta_n} x_{ij} \otimes e_{ij}\}_{n=1}^\infty$ *converge in the norm in* $A \otimes \mathcal{K}$, *where*

$$\sigma_n = \{(i,j) : \textit{there exists } k > l, \textit{ such that } n \geq n_k \geq i > n_{k-1}, \\ n \geq n_l \geq j > n_{l-1}\},$$

$$\delta_n = \{(i,j) : \textit{there exists } k < l, \textit{ such that } n \geq m_k \geq i > m_{k-1}, \\ n \geq m_l \geq j > m_{l-1}\}.$$

Then (i) $\Leftrightarrow$ (ii) *and* (iii) $\Rightarrow$ (i).

Proof.

(i) $\Rightarrow$ (ii). Let $x \in M(A \otimes \mathcal{K})$. For any $r > 0$, the closure of subset $\{a \in A \otimes \mathcal{K} : \|a\| \leq r\}$ in the strict topology is equal to subset $\{y \in M(A\otimes\mathcal{K}) : \|y\| \leq r\}$. Then, there is a bounded net $\{x_\alpha\} \subset A\otimes\mathcal{K}$ such that $x_\alpha \overset{\text{st}}{\to} x$ in $M(A \otimes \mathcal{K})$. Furthermore,

$$(1 \otimes e_{ii})x_\alpha(1 \otimes e_{jj}) \xrightarrow{\text{st}} (1 \otimes e_{ii})x(1 \otimes e_{jj})$$

in $M(A \otimes \mathcal{K})$.

Since $x_\alpha \in A \otimes \mathcal{K}$, then $(1 \otimes e_{ii})x_\alpha(1 \otimes e_{jj}) \in A \otimes e_{ij}$. Hence, there is an $x_\alpha^{ij} \in A$ such that $(1 \otimes e_{ii})x_\alpha(1 \otimes e_{jj}) = x_\alpha^{ij} \otimes e_{ij}$. It follows that

$x_\alpha^{ij}\otimes e_{ij}\overset{\text{st}}{\to}(1\otimes e_{ii})x(1\otimes e_{jj})$, and hence,

$$x_\alpha^{ij}\otimes e_{ij}\xrightarrow{\text{wot}}(1\otimes e_{ii})x(1\otimes e_{jj})$$

since $x_\alpha^{ij}\otimes e_{ij}$ is bounded. Note that the net $\{x_\alpha^{ij}\otimes e_{ij}\}$ is contained in $A''\otimes e_{ij}$, which is a closed subspace in the wot. Thus, $(1\otimes e_{ii})x(1\otimes e_{jj})\in A''\otimes e_{ij}$, and there is an $x_{ij}\in A''$ such that $x_{ij}\otimes e_{ij}=(1\otimes e_{ii})x(1\otimes e_{jj})$ for any $i,j\in\mathbb{N}$.

For every $a\in A$,

$$(x_\alpha^{ij}\otimes e_{ij})(a\otimes e_{jj})\xrightarrow{\|\cdot\|}(x_{ij}\otimes e_{ij})(a\otimes e_{jj}).$$

Then,

$$\|x_\alpha^{ij}a-x_{ij}a\|=\|(x_\alpha^{ij}a-x_{ij}a)\otimes e_{ij}\|\longrightarrow 0.$$

Thus, $x_{ij}a\in A$ for all $a\in A$ and $x_{ij}A\subset A$. Similarly, we have $Ax_{ij}\subset A$. Therefore, $x_{ij}\in M(A)$.

Finally, note that

$$\sum_{ij}^{n}x_{ij}\otimes e_{ij}=\sum_{ij}^{n}(1\otimes e_{ii})x(1\otimes e_{jj})=P_nxP_n,$$

where $P_n=\sum_1^n 1\otimes e_{ii}\in M(A\otimes\mathcal{K})$ and $P_n\overset{\text{st}}{\to}1$ in $M(A\otimes\mathcal{K})$. It follows that $P_nxP_n\overset{\text{st}}{\to}x$. Since the representation $\varphi:A\otimes\mathcal{K}\to B(H_\pi\otimes H)$ is faithful and non-degenerate, the strict topology is stronger than the sot on bounded subsets of $B(H_\pi\otimes H)$. Hence, $x_{ij}\in A''$ and $\sum_{ij}^n x_{ij}\otimes e_{ij}\overset{\text{sot}}{\to}x$. Therefore, by (i), we have $\Phi(x)=(x_{ij})$.

(ii) $\Rightarrow$ (i). Let $x_{ij}\in M(A)$ with $\sum_{ij}^n x_{ij}\otimes e_{ij}\overset{\text{st}}{\to}x$ in $M(A\otimes\mathcal{K})$. Since $\sum_{ij}^n x_{ij}\otimes e_{ij}\in M(A)\otimes\mathcal{K}\subset M(A\otimes\mathcal{K})$ and $M(A\otimes\mathcal{K})$ is complete in the strict topology, then x in $M(A\otimes\mathcal{K})$.

(iii) $\Rightarrow$ (i). Suppose that $x_{ij}\in M(A)$ satisfies the conditions in the assumption. Since $M(A)\subset LM(A)$, by Theorem 2.5 (ii), $x\in LM(A\otimes\mathcal{K})$. Similarly, since $M(A)\subset RM(A)$, by an analogue of Theorem 2.5 (ii), $x\in RM(A\otimes\mathcal{K})$. Therefore,

$$x\in LM(A\otimes\mathcal{K})\cap RM(A\otimes\mathcal{K})=M(A\otimes\mathcal{K}).\qquad\square$$

Remark 2.7. The assumption that $\pi:A\to B(H_\pi)$ is the universal representation of A is not necessary. In fact, since $LM(A\otimes\mathcal{K})$, $RM(A$

$\otimes \mathcal{K})$, $QM(A \otimes \mathcal{K})$ and $M(A \otimes \mathcal{K})$ are isomorphic, respectively, for any faithful and non-degenerate representations. Thus, if we replace the universal representation π of A with any faithful non-degenerate representation ϕ of A and replace the universal enveloping von Neumann algebra A'' with the closure of $\phi(A)$ in the sot, all results given above still hold.

Remark 2.8. In Theorem 2.6, condition (iii) is not necessary for $x \in M(A \otimes \mathcal{K})$.

Let $x_{ij} \in B(H_\pi)$ for $i, j = 1, 2, \ldots$. Set

$$\beta = \begin{pmatrix} x_{11} & 0 & \cdots \\ x_{21} & 0 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} \quad \text{and} \quad \beta_n = \begin{pmatrix} x_{11} & 0 & \cdots & 0 \\ x_{21} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & 0 & \cdots & 0 \end{pmatrix}.$$

Then,

$$\|\beta_n\|^2 = \|\beta_n^* \beta_n\| = \left\| \sum_{i=1}^{n} x_{i1}^* x_{i1} \right\|.$$

Hence, β represents an element in $B(H_\pi \otimes H)$ if and only if

$$\sup_n \left\| \sum_{i=1}^{n} x_{i1}^* x_{i1} \right\| < \infty.$$

Let $A = \mathcal{K}$, and

$$\beta = \begin{pmatrix} e_{11} & 0 & \cdots \\ e_{22} & 0 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}.$$

Let π be the inclusion map from $\mathcal{K}$ into $B(H)$. Then, the representation $\pi \otimes \iota$ is the inclusion map from $A \otimes \mathcal{K}$ into $B(H \otimes H)$. By the above discussion, β represents an element x in $B(H \otimes H) = M(A \otimes \mathcal{K})$, that is, $\sum_{i=1}^{n} e_{ii} \otimes e_{i1} \overset{\text{sot}}{\to} x$.

Note that $\|x - \sum_{i=1}^{n} e_{ii} \otimes e_{i1}\| = 1$ for any $n \in \mathbb{N}$. Hence, for any increasing subsequence $\{n_k\}$, the sequence $\{\sum_{(i,j) \in \sigma_n} e_{ii} \otimes e_{i1}\}_{n=1}^{\infty}$ does not converge in the norm in $A \otimes \mathcal{K}$.

The above example also illustrates that Theorem 2.5 (ii) is not necessary for $x \in LM(A\otimes\mathcal{K})$. However, when A is a unital C^*-algebra, these conditions are sufficient and necessary as Brown, Lin and Zhang have proved.

Corollary 2.9 ([**11,** 5.1.9])**.** *Suppose that A is a unital C^*-algebra and $e_n = \sum_1^n 1 \otimes e_{ii}$ for $n = 1, 2, \ldots$. Then, an infinite matrix (a_{ij}) with $a_{ij} \in A$ represents an element in $M(A \otimes \mathcal{K})$ if and only if*

(i) $\sup\{\| \sum_{ij}^n a_{ij} \otimes e_{ij}\| : n \in \mathbb{N}\} < +\infty$; *and*

(ii) *for any $\varepsilon > 0$ and $l \in \mathbb{N}$, there is an $N > 0$ such that*

$$\|(e_{n+m} - e_n)(a_{ij})e_l\| < \varepsilon \quad \textit{and} \quad \|e_l(a_{ij})(e_{n+m} - e_n)\| < \varepsilon$$

for all $m \in \mathbb{N}$ and all $n > N$.

Proof.

$\Rightarrow$. Suppose that (a_{ij}) represents x in $M(A \otimes \mathcal{K})$. By Theorem 2.5, $\sum_{ij}^n a_{ij} \otimes e_{ij} \overset{\text{sot}}{\to} x$. Therefore, $\sup\{\| \sum_{ij}^n a_{ij} \otimes e_{ij}\| : n \in \mathbb{N}\} < +\infty$.

Since $\{e_n\}$ is an approximate unit of $A \otimes \mathcal{K}$, then $e_n x e_l \to x e_l$ and $e_l x e_n \to e_l x$ for any $l \in \mathbb{N}$ as $n \to \infty$. Hence, (ii) holds.

$\Leftarrow$. By Theorem 2.6, we need to show that $\sum_{ij}^n a_{ij} \otimes e_{ij}$ is a Cauchy sequence in the strict topology in $M(A \otimes \mathcal{K})$. Since $\{\sum_{ij}^n a_{ij} \otimes e_{ij}\}$ is a bounded sequence, it suffices to show that $\{(\sum_{ij}^n a_{ij} \otimes e_{ij})e_l\}$ and $\{e_l(\sum_{ij}^n a_{ij} \otimes e_{ij})\}$ are Cauchy sequences for each e_l in the norm in $A \otimes \mathcal{K}$. This is exactly the statement of condition (ii). $\square$

Corollary 2.10 ([**12,** 1.6.1])**.** *Suppose that A is a unital C^*-algebra and $x \in (A \otimes \mathcal{K})''$ with $\Phi(x) = (x_{ij})$. Let $e_n = \sum_1^n 1 \otimes e_{ii}$ for $n = 1, 2, \ldots$. Then $x \in M(A \otimes \mathcal{K})$ if and only if there are two subsequences $\{e_{n_i}\}$ and $\{e_{m_j}\}$ with $e_{n_0} = e_{m_0} = 0$, such that*

$$\sum_{i=1}^{\infty}(e_{n_i} - e_{n_{i-1}})x(1 - e_{n_{i+1}}),\ \sum_{j=1}^{\infty}(1 - e_{m_{j+1}})x(e_{m_j} - e_{m_{j-1}}) \in A \otimes \mathcal{K}.$$

Proof. The "if" part follows from Theorem 2.6 (iii). The "only if" part follows from the fact that $\{e_n\}$ is contained in $A \otimes \mathcal{K}$ and is an approximate unit for $A \otimes \mathcal{K}$. $\square$

Corollary 2.11 ([**5,** 4.1.9 (ii)]). *Let A be a unital C^*-algebra and $(a_{ij}) \in M_\infty(A'')$. Then, (a_{ij}) represents an element of $LM(A \otimes \mathcal{K})$ if and only if (a_{ij}) is bounded, i.e., represents an element in $(A \otimes \mathcal{K})''$, each $(a_{ij}) \in A$ and there is an increasing subsequence $\{n_k\}$ such that $\{\sum_{(i,j)\in\sigma_n} x_{ij} \otimes e_{ij}\}_{n=1}^{\infty}$ converges in the norm in $A \otimes \mathcal{K}$.*

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