ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 47, Number 3, 2017

IF B AND $f(B)$ ARE BROWNIAN MOTIONS, THEN f IS AFFINE

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ABSTRACT. It is shown that, if the processes B and $f(B)$ are both Brownian motions (without a random time change), then f must be an affine function. As a by-product of the proof it is shown that the only functions which are solutions to both the Laplace equation and the eikonal equation are affine.

1. Statement of results. Suppose that the process B is a Brownian motion and that the function f is affine. Then the process $f(B)$ is again a Brownian motion. This short note proves the converse: if both B and $f(B)$ are Brownian motions, then f must be affine.

To be precise, we will use the following definition of Brownian motion.

Definition 1.1. The continuous process $B = (B_t)_{t \geq 0}$ is called an n-dimensional Brownian motion in a filtration $\mathcal{F} = (\mathcal{F}_t)_{t \geq 0}$ if and only if there exist an n-dimensional vector b and an $n \times n$ non-negative definite matrix A such that, for all $0 \leq s \leq t$, the conditional distribution of the increment $B_t - B_s$ given $\mathcal{F}_s$ is normal with mean $(t-s)b$ and covariance matrix $(t-s)A$.

A Brownian motion is standard if and only if $B_0 = 0$, $b = 0$ and A is the $n \times n$ identity matrix.

The main result of this note is the following theorem.

Theorem 1.2. *Suppose that B is an n-dimensional Brownian motion in the filtration $\mathcal{F}$ with non-singular diffusion matrix A. Suppose that*

2010 AMS *Mathematics subject classification.* Primary 31B05, 35Q60, 60J65.
Keywords and phrases. Brownian motion, harmonic function, Laplace equation, eikonal equation.
Received by the editors on on July 1, 2015, and in revised form on August 14, 2015.
DOI:10.1216/RMJ-2017-47-3-947

the process $f(B) = (f(B_t))_{t\geq 0}$ *is an* m*-dimensional Brownian motion in the same filtration* $\mathcal{F}$ *for a measurable function* $f : \mathbb{R}^n \to \mathbb{R}^m$*. Then there exist an* $m \times n$ *matrix* P *and vector* $q \in \mathbb{R}^m$ *such that*

$$f(x) = Px + q$$

for almost every x*.*

There are already a number of similar results in the literature. For instance, Dudley [**2**] showed that, if B is a one-dimensional standard Brownian and $f : \mathbb{R} \to \mathbb{R}$ is a continuous function such that the law of the process $f(B)$ is absolutely continuous with respect to the law of B, then, necessarily, $f(x) = x$ or $f(x) = -x$. This implies Theorem 1.2 in the case $n = 1$.

When B is an n-dimensional standard Brownian motion, Bernard, Campbell and Davie [**1**] studied functions $f : \mathbb{R}^n \to \mathbb{R}^m$ such that $f(B)$ is a standard Brownian motion up to a random time change. For instance, it is easy to see by the Dambis, Dubins and Schwarz theorem, for instance, [**4,** subsection 3.4.B] that, in the case $m = 1$, it is sufficient that f is harmonic with $f(0) = 0$. In particular, we do not allow for time change in Theorem 1.2, and hence, more structure is imposed on the function f.

Letac and Pradines [**5**] proved that, if $f : \mathbb{R}^n \to \mathbb{R}^m$ is such that $f(x + \sqrt{t}Z)$ has the normal distribution for all $x \in \mathbb{R}^n$ and $t \geq 0$, where Z is an n-dimensional standard normal random vector, then f is necessarily equal to an affine function almost everywhere. At first look, it would seem that Letac and Pradines' result would imply Theorem 1.2 since, if $f(B)$ is a Brownian motion, then $f(B_t)$ is normally distributed for all $t \geq 0$. However, the implication is not entirely obvious, due to the following, perhaps surprising, result.

Theorem 1.3. *Let* B *be an* n*-dimensional standard Brownian motion with* $n \geq 2$*. There exists a continuous non-linear function* $g : \mathbb{R}^n \to \mathbb{R}^n$ *such that the random vectors* $g(B_t)$ *and* B_t *have the same law for each* $t \geq 0$*.*

Indeed, the reason that Letac and Pradines' result does not contradict Theorem 1.3 is that they impose normality for all $x \in \mathbb{R}^n$, whereas the mean is fixed at $x = B_0 = 0$ in Theorem 1.3.

The idea of the proof of Theorem 1.2 is simply an application of the following form of Jensen's inequality: if G is strictly convex and $\int G(x)\, d\mu = G\left(\int x\, d\mu\right)$ for a probability measure μ, then μ is a point mass. A similar argument yields a related theorem. We will use the notation $\|\cdot\|$ for the Euclidean norm and $\langle\cdot,\cdot\rangle$ for the Euclidean inner product on $\mathbb{R}^n$.

Theorem 1.4. *Let $D \subseteq \mathbb{R}^n$ be an open, connected set, and suppose that $u : D \to \mathbb{R}$ is a classical solution to both the Laplace equation*

$$\Delta u = 0$$

and the eikonal equation

$$\|\nabla u\| = 1.$$

Then, $u(x) = \langle p, x\rangle + q$ for some constants $p \in \mathbb{R}^n$ and $q \in \mathbb{R}$, where $\|p\| = 1$.

Theorem 1.4 is contained in the recent paper of Garnica, Palmas and Ruiz-Hernandez [**3,** Lemma 4.1]. Their proof appeals to methods of differential geometry, while the proof given below only uses Jensen's inequality.

Remark 1.5. There is little loss in assuming that u is a classical solution to the Laplace equation. Indeed, if u is only assumed to be locally integrable and a solution to the Laplace equation in the sense of distributions, then u automatically has an infinitely differentiable version which is, in particular, a classical solution to the Laplace equation. See [**6,** subsection 9.3].

2. Proofs.

In this section, we prove the results presented above.

Proof of Theorem 1.2. Since every component of a vector-valued Brownian motion is a scalar Brownian motion, it is sufficient to consider the case $m = 1$.

First, we show that f is smooth. Now, since the conditional distribution of $f(B_t)$ given $\mathcal{F}_0$ is normal, we can conclude that

$$\mathbb{E}\left[|f(B_t)| \mid \mathcal{F}_0\right] < \infty$$

almost surely for all $t \geq 0$. In particular, we have the growth bound

$$(*) \qquad x \longmapsto f(x)e^{-\epsilon\|x\|^2} \text{ is Lebesgue integrable on } \mathbb{R}^n$$

for all $\epsilon > 0$. Now, since $f(B)$ is a Brownian motion, there is a constant $\mu \in \mathbb{R}$ such that

$$\mathbb{E}[f(B_t) \mid \mathcal{F}_s] = (t-s)\mu + f(B_s),$$

and hence, for almost every x, we have the representation

$$f(x) = -\tau\mu + \int f(y)\phi(\tau, x, y)\, dy,$$

where

$$\phi(\tau, x, y) = (2\pi\tau)^{-n/2} \det(A)^{-1/2} \exp\left(-\frac{1}{2\tau}\langle y - b\tau - x, A^{-1}(y - b\tau - x)\rangle\right)$$

is the Brownian transition density. However, by the boundedness property $(*)$ and the smoothness of $x \mapsto \phi(t, x, y)$ combined with the dominated convergence theorem, the function f has a differentiable version. Furthermore, its gradient ∇f has the representation

$$\nabla f(x) = \int \nabla f(y)\phi(\tau, x, y)\, dy$$

and also satisfies the boundedness property $(*)$. By iterating this argument, we see that f is infinitely differentiable.

Now, we show that f must satisfy an eikonal equation. Note that Itô's formula states that

$$df(B_t) = \langle \nabla f(B_t), dB_t\rangle + \frac{1}{2}\Delta f(B_t)\, dt.$$

Since $f(B)$ is a Brownian motion, the quadratic variation is

$$[f(B)]_t = \int_0^t \|\nabla f(B_s)\|^2 ds = \sigma^2 t$$

for some constant $\sigma \geq 0$. Hence, ∇f is a solution of the eikonal equation

$$\|\nabla f\| = \sigma$$

almost everywhere. However, since f is smooth, it solves the eikonal equation everywhere.

Now, note that

$$\sigma^2 = \|\nabla f(x)\|^2 = \int \|\nabla f(y)\|^2 \phi(\tau, x, y)\, dy.$$

Since the squared Euclidean norm is strictly convex, Jensen's inequality states that, for every x, there exists a vector $p_x \in \mathbb{R}^n$, possibly dependent upon x, such that $\nabla f(y) = p_x$ almost everywhere $y \in \mathbb{R}^n$. Since ∇f is continuous, we must have $\nabla f(y) = p$ for all y and for some constant vector p. Hence, $f(y) = \langle p, y \rangle + q$, as claimed. $\square$

We now proceed to the proof of Theorem 1.4. It follows the same pattern but differs in a few details which we spell out for completeness.

Proof of Theorem 1.4. We will show that there is a unit vector p such that $\nabla u(x) = p$ everywhere in D. Below, we will use the notation $B = \{x \in \mathbb{R}^n : \|x\| < 1\}$ to denote the open unit ball in $\mathbb{R}^n$, and hence, $x + rB$ denotes the ball of radius $r \geq 0$ centered at the point $x \in \mathbb{R}^n$.

Since u is harmonic, it is well known again, see [**6,** subsection 9.3] that u has the mean-value property: for every constant $r > 0$ such that $x + rB \subseteq D$, we have

$$u(x) = \frac{1}{r^n V} \int_{rB} u(x+y)\, dy,$$

where

$$V = \frac{\pi^{n/2}}{\Gamma(n/2)}$$

denotes the Lebesgue measure of the unit ball B. Since u is continuously differentiable in D, the gradient ∇u is bounded on compact sets; thus, the dominated convergence theorem allows us to differentiate both sides of the above equation, yielding

$$\nabla u(x) = \frac{1}{r^n V} \int_{rB} \nabla u(x+y)\, dy.$$

Now, for each $x \in \mathbb{R}^n$, note that

$$1 = \|\nabla u(x)\|^2 = \frac{1}{r^n V} \int_{rB} \|\nabla u(x+y)\|^2 dy.$$

Again, since the squared Euclidean norm is strictly convex, Jensen's inequality states that there is a vector p_x, possibly dependent upon x, such that $\nabla u(z) = p_x$ almost everywhere $z \in x + rB$ and $\|p_x\| = 1$. Since ∇u is continuous, we must have $\nabla u(z) = p_x$ for all z such that $\|x - z\| \leq r$.

Furthermore, fix two points x and x' in D. Since D is open and connected, there exists a path $C \subseteq D$ connecting them. Hence, there exist a finite number of points $x = x_1, \dots, x_N = x' \in D$ and radii $r_1, \dots, r_N > 0$ such that $\{x_i + r_i B\}_{i=1}^N$ is a cover of the compact set $C \subseteq D$. In particular, $p_x = p_{x'}$, and hence, ∇u is constant on D, as claimed. $\square$

Lastly, we construct an example of the function g claimed to exist in Theorem 1.3.

Proof of Theorem 1.3. Let $S^{n-1} = \{x \in \mathbb{R}^n : \|x\| = 1\}$ be the unit $(n-1)$-dimensional sphere, and let λ be the uniform probability measure on S^{n-1}. Let $h : S^{n-1} \to S^{n-1}$ be a continuous λ-preserving transformation. Finally, let $g(0) = 0$ and

$$g(x) = \|x\| \; h\left(\frac{x}{\|x\|}\right),$$

when $x \neq 0$. Fix a bounded and measurable function $\varphi : \mathbb{R}^n \to \mathbb{R}$ and $t \geq 0$. Using the assumption that the transformation h preserves the measure λ, we obtain

$$\begin{aligned}
\mathbb{E}[\varphi \circ g(B_t)] &= \int_{\mathbb{R}^n} \varphi\left[\sqrt{t}\|x\| h\left(\frac{x}{\|x\|}\right)\right] \frac{e^{-\|x\|^2/2}}{(2\pi)^{n/2}}\, dx \\
&= \int_0^\infty \int_{S^{n-1}} \varphi[\sqrt{t} r h(u)] \frac{r^{n-1} e^{-r^2/2}}{2^{n/2-1}\Gamma(n/2)}\, \lambda(du)\, dr \\
&= \int_0^\infty \int_{S^{n-1}} \varphi(\sqrt{t} r u) \frac{r^{n-1} e^{-r^2/2}}{2^{n/2-1}\Gamma(n/2)}\, \lambda(du)\, dr \\
&= \int_{\mathbb{R}^n} \varphi(\sqrt{t} x) \frac{e^{-\|x\|^2/2}}{(2\pi)^{n/2}}\, dx \\
&= \mathbb{E}[\varphi(B_t)],
\end{aligned}$$

where we have used the polar coordinates $x = ru$ with $r \geq 0$ and $u \in S^{n-1}$. Hence, $g(B_t)$ and B_t have the same law for each $t \geq 0$.

In order to show that there exists at least one function h which is non-linear, it is sufficient to consider the case $n = 2$ since we may restrict attention to the first two coordinates of B. Now, let $h : S^1 \to S^1$ be defined by $h(\cos(\theta), \sin(\theta)) = (\cos(2\theta), \sin(2\theta))$. It is well known that this transformation h is measure preserving. Explicitly, the function g in this case is:

$$g(x_1, x_2) = \left(\frac{x_1^2 - x_2^2}{\sqrt{x_1^2 + x_2^2}}, \frac{2x_1 x_2}{\sqrt{x_1^2 + x_2^2}} \right),$$

when $x \neq 0$. $\square$

Acknowledgments. The author would like to thank Pat Fitzsimmons for feedback on an early draft of this note, and, in particular, for alerting me to several papers in the literature. The author would also like to thank Chris Rogers and Arun Thillaisundaram for interesting discussions of this problem, and the members of the Laboratoire de Probabilités et Modèles Aléatoires at Université Paris 6 for their hospitality during much of the writing of this note. Finally, I would like to thank an anonymous referee for helpful comments.

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