ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 47, Number 3, 2017

CONTINUOUS FIELDS OF POSTLIMINAL C^*-ALGEBRAS

ALDO J. LAZAR

ABSTRACT. We discuss a problem of Dixmier [**6,** Problem 10.10.11] on continuous fields of postliminal C^*-algebras and the greatest liminal ideals of the fibers.

1. Introduction. In [**6,** Problem 10.10.11], Dixmier asked the question: given a continuous field $((A(t)),\Theta)$ of postliminal C^*-algebras over some topological space T, and with $B(t)$ the greatest liminal ideal of $A(t)$, and

$$\Theta' := \{x \in \Theta \mid x(t) \in B(t),\ t \in T\},$$

is $((B(t)),\Theta')$ a continuous field of C^*-algebras? A *tame continuous field* is a continuous field of postliminal C^*-algebras for which the answer to this question is affirmative.

An example of a continuous field that is not tame can be constructed over $T := \mathbb{N} \cup \{\infty\}$. We let $A(n)$, $n \in \mathbb{N}$, be the unitization of $K(H)$, the algebra of all compact operators over an infinite-dimensional Hilbert space H, and $A(\infty) := \mathbb{C}I_H$, I_H the identity operator on H. Let Θ consist of all fields x such that $x(n) = \lambda_n I_H + a_n$ with $\{\lambda_n\}$ a sequence in $\mathbb{C}$ that converges to some $\lambda \in \mathbb{C}$, $\{a_n\}$ a sequence in $K(H)$ that converges to $\{0\}$ and $x(\infty) = \lambda I_H$. Then, $((A(t))_{t\in T},\Theta)$ is a continuous field of postliminal C^*-algebras. Now, the largest liminal ideal of $A(n)$ is $B(n) = K(H)$, and the largest liminal ideal of $A(\infty)$ is $B(\infty) = A(\infty) = \mathbb{C}I_H$. Clearly, $x \in \Theta$ satisfies $x(t) \in B(t)$ for every $t \in T$ if and only if $x(\infty) = 0$ and the continuous field is not tame.

In Theorem 2.3, we show that the continuous fields of postliminal C^*-algebras in a certain class that properly includes the locally trivial continuous fields are always tame. In the last section, we shall exhibit

2010 AMS *Mathematics subject classification.* Primary 46L05.
Keywords and phrases. Continuous fields of C^*-algebras, postliminal C^*-algebras, liminal C^*-algebras.
Received by the editors on December 22, 2014, and in revised form on July 28, 2015.
DOI:10.1216/RMJ-2017-47-3-849

an example of a continuous field of postliminal C^*-algebras such that all of its fibers are mutually isomorphic and its restriction to any open subset of the base space is not tame.

Let A be the C^*-algebra of the continuous field $((A(t)), \Theta)$ of C^*-algebras over the locally compact Hausdorff space T as defined in [**6,** 10.4.1]. By [**8,** Theorem 1.1], for every primitive ideal P of A, there exist a unique $t_P \in T$ and a unique primitive ideal Q_P of $A(t_P)$ such that $P = \{x \in A \mid x(t_P) \in Q_P\}$, and conversely, every pair (t, Q), $t \in T$, $Q \in \operatorname{Prim}(A(t))$ determines a primitive ideal of A in this manner. Obviously, P is minimal in $\operatorname{Prim}(A)$ if and only if Q_P is minimal in $A(t_P)$.

We shall use the terminology and notation for continuous fields as introduced in [**6,** Chapter 10]. The preference to work with continuous fields (rather than the more common Banach bundles) is motivated by the fact that Dixmier's original question was expressed in these terms. The closed unit ball of the Banach space X is denoted X_1. All paracompact spaces considered in the following are implicitly Hausdorff; hence, T_4 by [**7,** Theorem VIII.2.2].

2. Results. The main ingredient in the proof of Proposition 2.1 is Michael's selection theorem [**11,** Theorem 3.2′′]: a multivalued map φ from a paracompact space T to the family of the non-void closed convex subsets of a Banach space X that is lower semicontinuous admits a continuous selection, i.e., there is a continuous function $f : T \to X$ such that $f(t) \in \varphi(t)$ for every $t \in T$. Moreover, if F is a closed subset of T and $g : F \to X$ is a continuous selection for $\varphi|_F$, then one may choose f so that $f|_F = g$ is satisfied. Recall that φ is called *lower semicontinuous* if, for each open subset U of X, the set $\{t \in T \mid \varphi(t) \cap U \neq \emptyset\}$ is open.

Proposition 2.1. *Let T be a paracompact space or a locally compact Hausdorff space and X a Banach space. Denote by $\mathcal{M}$ the space of closed unit balls of all closed subspaces of X endowed with the Hausdorff metric. Suppose that $t \to X(t)_1$, $t \in T$, is a continuous map into $\mathcal{M}$, $X(t)$ a closed subspace of X. With Γ the space of all continuous functions $\varphi : T \to X$ such that $\varphi(t) \in X(t)$, $t \in T$, $((X(t)), \Gamma)$ is a continuous field of Banach spaces.*

Proof. The only evidence we must provide is that, for $t_0 \in T$ and $x_0 \in X(t_0)$, there exists $\varphi \in \Gamma$ such that $\varphi(t_0) = x_0$. Clearly, we may suppose that $x_0 \neq 0$. Set $y_0 := x_0/\|x_0\|$. We claim that $t \to X(t)_1$ is lower semicontinuous as a multivalued map from T to X. In order to see this, let U be an open subset of X,

$$s \in \{t \in T \mid U \cap X(t)_1 \neq \emptyset\} \quad \text{and} \quad z \in U \cap X(s)_1.$$

There is an open ball of X with center z and radius $\varepsilon > 0$ contained in U and a neighborhood V of s in T such that $d(X(s)_1, X(t)_1) < \varepsilon$ for all $t \in V$, d the Hausdorff metric. Thus, for each $t \in V$, there is a $w_t \in X(t)_1$ for which $\|z - w_t\| < \varepsilon$. It follows that

$$V \subset \{t \in T \mid U \cap X(t) \neq \emptyset\};$$

thus, we conclude that $\{t \in T \mid U \cap X(t) \neq \emptyset\}$ is open. We obtain that the map $t \to X(t)_1$ is indeed lower semicontinuous.

Now, suppose that T is paracompact. By Michael's selection theorem, there exists a continuous map $\varphi' : T \to X$ such that $\varphi'(t) \in X(t)_1$ for every $t \in T$ and $\varphi'(t_0) = y_0$. The map φ defined by $\varphi(t) := \|x_0\|$ $\varphi'(t)$ suits the requirements.

Let T be locally compact Hausdorff. Let W be a compact neighborhood of t_0. Again, by Michael's selection theorem, there is a continuous map $\varphi' : W \to X$ such that $\varphi'(t) \in X(t)_1$ for every $t \in W$ and $\varphi'(t_0) = y_0$. Let $f : T \to [0, 1]$ be a continuous function such that $f(t_0) = 1$ and $f(t) = 0$ for $t \notin \operatorname{Int}(W)$. The function $\varphi : T \to X$, defined by

$$\varphi(t) := \begin{cases} \|x_0\| f(t)\varphi'(t) & \text{if } t \in W, \\ 0 & \text{if } t \notin W, \end{cases}$$

is continuous, satisfies $\varphi(t) \in X(t)$ for $t \in T$ and $\varphi(t_0) = x_0$. $\square$

A continuous field of Banach spaces over a paracompact or a locally compact Hausdorff space T isomorphic to a continuous field of Banach spaces as described in Proposition 2.1 will be called *uniform*. Obviously, a trivial continuous field of Banach spaces is uniform. A continuous field of Banach spaces $((X(t)), \Gamma)$ over T is called *locally uniform* if there is a family $\{F_\alpha\}$ of closed subsets of T such that $\{\operatorname{Int}(F_\alpha)\}$ is a cover of T with open non-void sets and the restriction of the field to each F_α is uniform. It is understood that, for $t \in F_\alpha$, the Banach space $X(t)$ is a closed subspace of a Banach space Y_α. Note that, if $t \in F_{\alpha_1} \cap F_{\alpha_2}$

and $x \in X(t)$, then

$$\|x\|_{\alpha_1} = \|x\|_{X(t)} = \|x\|_{\alpha_2}.$$

By [**6,** 10.1.2 (iv)], if x is a vector field that is continuous on each F_α as a function into Y_α, then $x \in \Gamma$. We note that each F_α is paracompact if T is such and locally compact Hausdorff in the second case.

The next proposition gives us a sufficient condition for a field of Banach spaces to be locally uniform.

Proposition 2.2. *Let T be as in Proposition 2.1, $\{U_\alpha\}_{\alpha\in\mathcal{A}}$ an open cover of T and X_α a Banach space, $\alpha \in \mathcal{A}$. Denote by M_α the space of the closed unit balls of all the closed subspaces of X_α endowed with the Hausdorff metric. Suppose that $X(t)$, $t \in T$ is a Banach space that is a closed subspace of X_α whenever $t \in U_\alpha$. Moreover, suppose that the map $t \to X(t)_1$ from U_α into M_α, $\alpha \in \mathcal{A}$, is continuous. Denote by Γ the space of all functions $\varphi : T \to \cup_{\alpha\in\mathcal{A}} X_\alpha$ such that $\varphi(t) \in X(t)$, $t \in T$, and the restriction of φ to U_α is continuous as a map into X_α, $\alpha \in \mathcal{A}$. Then, $((X(t)),\Gamma)$ is a locally uniform continuous field of Banach spaces.*

Proof. For $t \in U_\alpha$, we choose closed sets $E_{t,\alpha}$ and $F_{t,\alpha}$ such that

$$t \in \mathrm{Int}\,(F_{t,\alpha}) \subset F_{t,\alpha} \subset \mathrm{Int}\,(E_{t,\alpha}) \subset E_{t,\alpha} \subset U_\alpha.$$

In the case where T is locally compact, we shall require $E_{t,\alpha}$ to be compact in addition to the above. We shall show that the restriction of $((X(t)),\Gamma)$ to $F_{t,\alpha}$ is a uniform, continuous field of Banach spaces. To this end, let $\varphi : F_{t,\alpha} \to X_\alpha$ be a continuous function such that $\varphi(s) \in X(s)$ for every $s \in F_{t,\alpha}$. We must produce an element φ' of Γ whose restriction to $F_{t,\alpha}$ is φ. Observe that, in both situations for T, $E_{t,\alpha}$ is paracompact, hence, normal. Let $g : E_{t,\alpha} \to (0,\infty)$ be a continuous function such that $g(s) = 1 + \|\varphi(s)\|$ for $s \in F_{t,\alpha}$. The function $\psi : F_{t,\alpha} \to X_\alpha$ given by $\psi(s) := (1 + \|\varphi(s)\|)^{-1}\varphi(s)$, $s \in F_{t,\alpha}$, is a continuous selection for the restriction to $F_{t,\alpha}$ of the lower semicontinuous map $s \to X(s)_1$ from $E_{t,\alpha}$ to X_α. By [**11,** Example 1.3, Theorem 3.2], ψ has a continuous extension $\widetilde{\psi}$ from $E_{t,\alpha}$ to X_α such that $\widetilde{\psi}(s) \in X(s)_1$, $s \in E_{t,\alpha}$. Then, $\widetilde{\varphi} : E_{t,\alpha} \to X_\alpha$ defined by $\widetilde{\varphi}(s) := g(s)\widetilde{\psi}(s)$ is a continuous extension of φ from $E_{t,\alpha}$ to X_α and satisfies $\widetilde{\varphi}(s) \in X(s)$, $s \in E_{t,\alpha}$. In both cases, there is a continuous function $f : T \to [0,1]$ such that $f(s) = 1$ for $s \in F_{t,\alpha}$ and $f(s) = 0$ for $s \notin \operatorname{Int}(E_{t,\alpha})$. The function $\varphi' : T \to \cup_{\beta\in\mathcal{A}} X_\beta$, given by

$$\varphi'(s) := \begin{cases} f(s)\widetilde{\varphi}(s) & \text{if } s \in E_{t,\alpha}, \\ 0 & \text{if } s \notin \operatorname{Int}(E_{t,\alpha}), \end{cases}$$

satisfies $\varphi'(s) \in X(s)$, $s \in T$, and is the required element of Γ. In checking that $\varphi' \mid_{U_\beta}$ as a map into X_β, $\beta \in \mathcal{A}$, is continuous, we rely on the fact that, for $s \in X_\alpha \cap X_\beta$, the norm on $X(s)$ is the same whether $X(s)$ is considered to be a closed subspace of X_α or of X_β. Obviously, on $U_\beta \cap \operatorname{Int}(E_{t,\alpha})$, φ' is continuous. If $s \in U_\beta \setminus \operatorname{Int}(E_{t,\alpha})$ and $\{s_\kappa\}$ is a net in U_β that converges to s, then it is easily seen that

$$\|\varphi'(s_\kappa)\| \longrightarrow 0 = \|\varphi'(s)\|.$$

This establishes the claim for φ'. □

The continuous fields of C^*-algebras are of interest to us, and, for uniform continuous fields of C^*-algebras, we shall require that the Banach space appearing in the definition be a C^*-algebra and the fibers to be C^*-subalgebras of it. It is natural to ask whether a uniform, continuous field of C^*-algebras must be locally trivial. There is some indication [**4,** Theorem 4.3] that this may be the case when the fibers are nuclear and separable. On the other hand, [**3,** Theorem 3.3] provides an example of a uniform, continuous field of (non-separable) nuclear C^*-algebras that is not locally trivial.

Theorem 2.3. *Suppose that* $((A(t)), \Theta)$ *is a locally uniform, continuous field of postliminal* C^**-algebras over a paracompact or locally compact Hausdorff space* T*. Let* $B(t)$ *be the largest liminal ideal of* $A(t)$ *and* $\Theta' := \{x \in \Theta \mid x(t) \in B(t),\ t \in T\}$*. Then,* $((B(t)), \Theta')$ *is a locally uniform, continuous field of* C^**-algebras over* T*.*

Proof. Let $\{F_\alpha\}$ be a family of closed subsets of T as given by the definition of a locally uniform field for $((A(t)), \Theta)$, and set $U_\alpha := \operatorname{Int}(F_\alpha)$. There is no loss of generality if we suppose that over F_α all of the fibers $A(t)$ are C^*-subalgebras of a certain C^*-algebra $\mathfrak{A}_\alpha$, and $t \to A(t)_1$ is continuous for the Hausdorff metric. If t', $t'' \in U_\alpha$ satisfy $d(A(t')_1, A(t'')_1) < s(\leq 1/21)$, then it follows from [**14,** Lemma 1.10] that there is a lattice isomorphism ϕ between the family of closed, two-sided ideals of $A(t')$ and that of closed, two-sided ideals of $A(t'')$ such that $d(\phi(I)_1, I_1) < 7s$ for every ideal I. The restriction of ϕ to $\mathrm{Prim}\,(A(t'))$ is a homeomorphism onto $\mathrm{Prim}\,(A(t''))$. Now, $\mathrm{Prim}\,(B(t'))$ is the largest open subset of $\mathrm{Prim}\,(A(t'))$, that is, T_1 in the relative topology; hence, it is mapped by ϕ onto $\mathrm{Prim}\,(B(t''))$, and we obtain $\phi(B(t')) = B((t''))$. We infer that $d(B(t')_1, B(t'')_1) < 7s$, and we conclude that $t \to B(t)_1$ is continuous on U_α.

Denoting by Θ'', the space of all vector fields x such that $x(t) \in B(t)$, $t \in T$, and for which $x\,|_{U_\alpha}\colon U_\alpha \to \mathfrak{A}_\alpha$ is continuous, by Proposition 2.2, we obtain that $((B(t)), \Theta'')$ is a locally uniform, continuous field.

It remains to show that $\Theta'' = \Theta'$. Clearly, $\Theta' \subset \Theta''$. Let $x \in \Theta''$. We want to show that x as a function from F_α into $\mathfrak{A}_\alpha$ is continuous. Let U be a neighborhood of $t \in F_\alpha$ contained in some U_β. Then x is continuous on U to $\mathfrak{A}_\beta$. The vector field x maps $F_\alpha \cap U$ continuously into $\mathfrak{A}_\alpha \cap \mathfrak{A}_\beta$; thus, we conclude that x is continuous at t as a map from F_α to $\mathfrak{A}_\alpha$. We infer that $x \in \Theta$, and since $x(t) \in B(t)$, $t \in T$, we obtain $x \in \Theta'$. □

We shall now discuss the behavior of locally uniform, continuous fields of postliminal C^*-algebras with respect to two kinds of ideals that give rise to canonical composition series. Recall that one says that a point π in the spectrum $\widehat{A}$ of a C^*-algebra A satisfies the Fell condition if there exist a neighborhood V of π in $\widehat{A}$ and $a \in A^+$ such that $\varrho(a)$ is a projection of rank 1 for every $\varrho \in V$. A *Fell C^*-algebra* is a C^*-algebra for which all points in its spectrum satisfy the Fell condition, see [**1**], [**12,** 6.1], where these algebras were called of Type I_0. Every non trivial postliminal C^*-algebra has a non trivial largest Fell ideal by [**12,** Proposition 6.1.7]. A C^*-algebra A is called *uniformly liminal* if its ideal of all elements $a \in A$ for which the function $\pi \to \pi(a)$ bounded on $\widehat{A}$ is dense in A, see [**2,** page 443] and the references therein. Every non trivial postliminal C^*-algebra has a non trivial largest uniformly liminal ideal by [**2,** Theorems 2.6, 2.8].

Theorem 2.4. *Let $((A(t)), \Theta)$ be a locally uniform, continuous field of postliminal C^*-algebras over a paracompact or locally compact Hausdorff space T. Let $B(t)$ be the largest Fell ideal of $A(t)$ and $\Theta' := \{x \in \Theta \mid x(t) \in B(t),\ t \in T\}$. Then $((B(t)), \Theta')$ is a locally uniform, continuous field of C^*-algebras.*

Proof. Let F_α, U_α, and $\mathfrak{A}_\alpha$ be as in the proof of Theorem 2.3. Now, take t', $t'' \in U_\alpha$ that satisfy $d(A(t')_1, A(t'')_1) < s(< 1/147)$, and let ϕ be the lattice isomorphism between the spaces of ideals of these two C^*-algebras given by [**14,** Lemma 1.10].

The ideal $J := \phi(B(t'))$ of $A(t'')$ satisfies $d(B(t')_1, J_1) < 7s(< 1/21)$. There is a homeomorphism h from $\widehat{A(t')}$ onto $\widehat{A(t'')}$ such that ϕ maps the kernel of $\pi' \in \widehat{A(t')}$ to the kernel of $h(\pi')$, see [**14,** Lemma 1.10]. We have $h(\widehat{B(t')}) = \widehat{J}$. Pick $\pi_0 \in \widehat{B(t')}$, and set $\varrho_0 = h(\pi_0) \in \widehat{J}$. There is an $a \in B(t')_1^+$ and a neighborhood V of π_0 in $\widehat{B(t')}$ such that $\pi(a)$ is a rank 1 projection for $\pi \in V$. We imitate the proof of [**14,** Lemma 2.4] in order to obtain an element $b \in J_1^+$ such that $\varrho(b)$ is a rank 1 projection when $\varrho \in h(V)$. It follows that J is a Fell ideal; therefore, $\phi(B(t')) \subset B(t'')$. There exists a Hermitian $c \in J_1$ such that $\|a - c\| < 7s$. With the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(t) := \begin{cases} 0 & \text{if } t \leq 21s, \\ (t - 21s)/(1 - 42s) & \text{if } 21s \leq t \leq 1 - 21s, \\ 1 & \text{if } t \geq 1 - 21s, \end{cases}$$

we define $b := f(c)$. Let $\pi \in V$, and set $\varrho := h(\pi)$. Then, [**14,** Lemma 2.2] allows us to consider π and ϱ as acting on the same Hilbert space with

$$d(\pi(B(t'))_1, \varrho(J)_1) < 21s(\leq 1/7),$$

and $\|\pi(a) - \varrho(c)\| < 21s$. It is easily seen that the spectrum of $\varrho(b)$ is contained in $\{0, 1\}$; hence, $\varrho(b)$ is a projection. Moreover,

$$\|\pi(a) - \varrho(b)\| \leq \|\pi(a) - \varrho(c)\| + \|\varrho(c) - \varrho(b)\| < 42s;$$

thus, an application of [**14,** Lemma 1.7] allows us to conclude that $\varrho(b)$ is one-dimensional.

Using a similar argument, we obtain $\phi^{-1}(B(t'')) \subset B(t')$; thus, $\phi(B(t')) = B(t'')$ and $d(B(t')_1, B(t'')_1) < 7s$. We find that $t \to B(t)_1$ is continuous on U_α, and subsequently, we proceed as in the last paragraph of the proof of Theorem 2.3. $\square$

In the next result, the hypothesis of separability may be unnecessary; however, we were not able to find a proof that dispenses with it.

Theorem 2.5. *Let $((A(t)), \Theta)$ be a locally uniform, continuous field of separable postliminal C^*-algebras over a paracompact or locally compact Hausdorff space T. Let $B(t)$ be the largest uniformly liminal ideal of $A(t)$ and $\Theta' := \{x \in \Theta \mid x(t) \in B(t),\ t \in T\}$. Then $((B(t)), \Theta')$ is a locally uniform, continuous field of C^*-algebras.*

Proof. Let F be a closed subset of T such that $\mathrm{Int}\,(F) \neq \emptyset$ and the given continuous field of C^*-algebras restricted to F is uniform. We shall also suppose that the family of C^*-algebras

$$\{A(t) \mid t \in F\}$$

is contained in a certain C^*-algebra, and the map $t \to A(t)_1$ is continuous on F for the Hausdorff metric. If $t_1, t_2 \in F$ satisfy $d(A(t_1)_1, A(t_2)_1) < s < 1/420,000$, then $A(t_1)$ and $A(t_2)$ are isomorphic by [**4,** Theorem 4.3]. Of course, every isomorphism between these two C^*-algebras maps $B(t_1)$ onto $B(t_2)$. Hence, the same result of [**4**] tells us that $d(B(t_1)_1, B(t_2)_1) \leq 28s^{1/2}$; thus, the map $t \to B(t)_1$ is continuous on F. Once more, Proposition 2.2 yields the conclusion as in the proof of Theorem 2.3. $\square$

Question 2.6. *Let $((A(t)), \Theta)$ be a uniform, continuous field of postliminal C^*-algebras over T. Can one choose a non-trivial continuous trace ideal $B(t)$, $t \in T$, of $A(t)$ such that, with $\Theta' := \{x \in \Theta \mid x(t) \in B(t), t \in T\}$, $((B(t)),\ \Theta')$ is a continuous field of C^*-algebras?*

Remark 2.7. Given a continuous field $((A(t)), \Theta)$ of postliminal C^*-algebras over a locally compact Hausdorff space T, let A be the C^*-algebra defined by this continuous field; obviously, it is a postliminal C^*-algebra. Let B be its greatest liminal ideal. If the image of B in $A(t)$ by the evaluation map is the greatest liminal ideal of $A(t)$, $t \in T$, then it is easily seen that the given field is tame. Conversely, suppose that $((A(t)), \Theta)$ is tame, and let C be the C^*-algebra defined by the continuous field of the greatest liminal ideals. Then, $C = B$, the greatest liminal ideal of A. Indeed, it is clear that C is a liminal ideal of A, thus $C \subset B$. Now, let $x \in B$. With $t \in T$, $\rho \in \widehat{A(t)}$, we have that $y \to \rho(y(t))$, $y \in A$, is an irreducible representation of A; hence, $\rho(x(t))$ is a compact operator over the space of the representation. We conclude that $x(t) \in C(t)$ by [**6,** 4.2.6]. Thus, $B \subset C$.

The next proposition may already be known; however, in the absence of a reference, we provide its simple proof.

Proposition 2.8. *Let A be a postliminal C^*-algebra, and let $M \subset \mathrm{Prim}\,(A)$ be the set of all minimal primitive ideals. Then $\mathrm{Int}\,(M)$ is the primitive ideal space of the greatest liminal ideal I of A.*

Proof. Dixmier remarked [**5,** page 111, Remark C] that we have $\mathrm{Prim}\,(I) \subset \mathrm{Int}\,(M)$. $\mathrm{Int}\,(M)$ is the primitive ideal space of an ideal, J, say, of A. If $P \in \mathrm{Int}\,(M)$, then the relative closure of $\{P\}$ in $\mathrm{Int}\,(M)$ is merely $\{P\}$ since all ideals in $\mathrm{Int}\,(M)$ are minimal primitive ideals. Thus, $\mathrm{Int}\,(M)$ is T_1 in the relative topology, and we infer that J is a liminal ideal of A. Hence, $J \subset I$, and $\mathrm{Int}\,(M) \subset \mathrm{Prim}\,(I)$. □

The aforementioned set M need not be open; for examples, see [**9**], [**10,** Example 4.3].

Now, let $((A(t)), \Theta)$ be a continuous field of postliminal C^*-algebras over the locally compact Hausdorff space T. Denote by A the C^*-algebra of the field and by M the set of all minimal primitive ideals of A. The set of all minimal primitive ideals of $A(t)$ is $M \cap \mathrm{Prim}\,(A(t))$; thus, in view of Proposition 2.8, we can reformulate Remark 2.7 as: the given field is tame if and only if $\mathrm{Int}\,(M) \cap \mathrm{Prim}\,(A(t))$ is the relative interior in $\mathrm{Prim}\,(A(t))$ of $M \cap \mathrm{Prim}\,(A(t))$ for every $t \in T$.

3. An example. As mentioned in the introduction we shall construct a continuous field of postliminal C^*-algebras over $[0,1]$ whose fibers are mutually isomorphic and which has the additional property that none of its restrictions to the relatively open subsets of $[0,1]$ is tame.

First, we prepare two C^*-algebras that will serve as building blocks of the fibers. Let $\mathbb{N} = \cup_{p=1}^{\infty} S_p$, where the sets $\{S_p\}$ are mutually disjoint and each $S_p = \{n_1^p < n_2^p < \cdots\}$ is infinite. Here, p in n_m^p is a superscript, not an exponent. Let H be a separable Hilbert space with an orthonormal basis $\{\xi_k\}_{k=1}^{\infty}$ and $\mathcal{B}(H)$ the C^*-algebra of all bounded operators on H. Denote by e_{ij}^0 the partial isometry that maps ξ_j to ξ_i and vanishes on each ξ_k with $k \neq j$. The C^*-subalgebra of $\mathcal{B}(H)$ generated by $\{e_{ij}^0 \mid i, j = 1, 2, \ldots\}$ is the ideal of all compact operators, and we shall denote it by A_0. Now, put

$$e^1_{ij} := \sum_{m=1}^{\infty} e^0_{n^i_m n^j_m}, \quad i,j = 1,2,\dots,$$

where the series converges in the strong operator topology. Then, $\{e^1_{ii}\}_{i=1}^{\infty}$ are mutually orthogonal projections,

$$\sum_{i=1}^{\infty} e^1_{ii} = \mathbf{1}_H,$$

e^1_{ij} is a partial isometry from $e^1_{jj}(H)$ onto $e^1_{ii}(H)$, $(e^1_{ij})^* = e^1_{ji}$ and

$$e^1_{ij}e^1_{rs} = \begin{cases} e^1_{is} & j = r, \\ 0 & j \neq r. \end{cases}$$

Hence, the C^*-subalgebra K_1 of $\mathcal{B}(H)$ generated by

$$\{e^1_{ij} \mid i,j = 1,2,\dots\}$$

is isomorphic to A_0 and $A_0 \cap K_1 = \{0\}$. We have

$$e^1_{ij}e^0_{rs} = \begin{cases} e^0_{n^i_m s} & \text{if } r = n^j_m \text{ for some } m, \\ 0 & \text{otherwise} \end{cases} \tag{3.1}$$

and

$$e^0_{rs}e^1_{ij} = \begin{cases} e^0_{r n^j_m} & \text{if } s = n^i_m \text{ for some } m, \\ 0 & \text{otherwise.} \end{cases} \tag{3.2}$$

Since A_0 and $K_1 \sim A_1/A_0$ are postliminal C^*-algebras (actually, liminal C^*-algebras in this case), $A_1 := A_0 + K_1$ is a postliminal C^*-algebra. Each $x \in A_1$ admits a unique decomposition $x = x_0 + x_{K_1}$ with $x_0 \in A_0$ and $x_{K_1} \in K_1$. The map $x \to x_{K_1}$ is a homomorphism; hence, $\|x_{K_1}\| \leq \|x\|$ and $\|x_0\| \leq 2\|x\|$.

From (3.1) and (3.2), it follows that the sequence

$$\left\{ \sum_{i=1}^{m} e^1_{ii} \right\}_{m=1}^{\infty}$$

is an increasing, approximate unit for A_1 consisting of projections.

We now suppose that the C^*-subalgebras $K_1, \dots K_{l-1}$ of $\mathcal{B}(H)$ have been defined such that K_p, $1 \leq p \leq l-1$, is spanned by

$$e^p_{ij} := \sum_{m=1}^{\infty} e^{p-1}_{n^i_m n^j_m}, \quad i, j = 1, 2, \dots .$$

It follows that $\{e^p_{ii}\}_{i=1}^{\infty}$ are mutually orthogonal projections,

$$\sum_{i=1}^{\infty} e^p_{ii} = \mathbf{1}_H,$$

and e^p_{ij} is a partial isometry from $e^p_{jj}(H)$ onto $e^p_{ii}(H)$. With $A_p := A_{p-1} + K_p$, $1 \leq p \leq l-1$, we have $A_{p-1} \cap K_p = \{0\}$, A_p is a postliminal C^*-subalgebra of $\mathcal{B}(H)$, A_{p-1} is an ideal of A_p, and

$$\left\{ \sum_{i=1}^{m} e^p_{ii} \right\}_{m=1}^{\infty}$$

is an increasing approximate unit of A_p consisting of projections. Every element $x \in A_p$ admits a unique decomposition $x = x_{p-1} + x_{K_p}$ where $x_{p-1} \in A_{p-1}$, $x_{K_p} \in K_p$. Moreover, $x \to x_{K_p}$ is a homomorphism; hence, $\|x_{K_p}\| \leq \|x\|$ and $\|x_{p-1}\| \leq 2\|x\|$.

Now, we define

$$e^l_{ij} := \sum_{m=1}^{\infty} e^{l-1}_{n^i_m n^j_m}, \quad i, j = 1, 2, \dots .$$

Then, $\{e^l_{ii}\}_{i=1}^{\infty}$ are mutually orthogonal projections,

$$\sum_{i=1}^{\infty} e^l_{ii} = \mathbf{1}_H,$$

and e^l_{ij} is a partial isometry from $e^l_{jj}(H)$ onto $e^l_{ii}(H)$. We obtain $(e^l_{ij})^* = e^l_{ji}$ and

$$e^l_{ij} e^l_{rs} = \begin{cases} e^l_{is} & \text{if } j = r, \\ 0 & \text{otherwise}; \end{cases} \tag{3.3}$$

hence, the C^*-subalgebra K_l of $\mathcal{B}(H)$ generated by $\{e^l_{ij} \mid i,j=1,2,\ldots\}$ is isomorphic to A_0. We also have

$$e^l_{ij}e^{l-1}_{rs} = \begin{cases} e^{l-1}_{n^i_m s} & \text{if } r = n^j_m \text{ for some } m, \\ 0 & \text{otherwise} \end{cases} \tag{3.4}$$

and

$$e^{l-1}_{rs}e^l_{ij} = (e^l_{ji}e^{l-1}_{sr})^* = \begin{cases} e^{l-1}_{rn^j_m} & \text{if } s = n^i_m \text{ for some } m, \\ 0 & \text{otherwise.} \end{cases} \tag{3.5}$$

Hence, if $x \in K_l$ and $y \in K_{l-1}$, then $xy, yx \in K_{l-1}$. Suppose now that $x \in K_l$, $z \in A_{l-2}$. Then,

$$xz = \lim_{m\to\infty} x\left(\sum_{i=1}^m e^{l-1}_{ii}\right)z = \lim_{m\to\infty}\left(x\sum_{i=1}^m e^{l-1}_{ii}\right)z.$$

We have established that

$$x\sum_{i=1}^m e^{l-1}_{ii} \in K_{l-1},$$

for every m; hence, $xz \in A_{l-2}$. Similarly, $zx \in A_{l-1}$, and we obtain that $A_{l-1} = A_{l-2} + K_{l-1}$, $k \geq 2$, is an ideal in $A_l := A_{l-1} + K_l$ which is a C^*-subalgebra of $\mathcal{B}(H)$ by [**6,** 1.8.4].

We shall now prove that $A_{l-1} \cap K_l = \{0\}$. Denote by B^m_l the finite-dimensional C^*-algebra generated by $\{e^l_{ij} \mid 1 \leq i,j \leq m\}$. Then, $K_l = \overline{\cup_{m=1}^\infty B^m_l}$. From

$$\left\|\sum_{i,j=1}^m \alpha_{ij}e^l_{ij} - \sum_{i,j=1}^m \alpha_{ij}e^l_{ij}\sum_{r=1}^s e^{l-1}_{rr}\right\| = \left\|\sum_{i,j=1}^m \alpha_{ij}e^l_{ij}\right\|,$$

for every s, we obtain $B^m_l \cap A_{l-1} = \{0\}$ for every m. Thus, the quotient map $A_l \to A_l/A_{l-1}$ is isometric on each B^m_l; hence, it is isometric on K_l, and we conclude that $A_{l-1} \cap K_l = \{0\}$. Since A_{l-1} and $A_l/A_{l-1} \sim K_i$ are postliminal C^*-algebras, A_l is a postliminal C^*-algebra.

In this manner, we inductively construct an increasing sequence $\{A_l\}_{l=0}^\infty$ of postliminal C^*-subalgebras of $\mathcal{B}(H)$ such that A_{l-1} is an ideal in A_l. It follows that $A := \overline{\cup_{l=0}^\infty A_l}$ is a postliminal C^*-subalgebra of $\mathcal{B}(H)$ whose greatest liminal ideal is A_0.

Now, set $C_1 := K_1$, $C_2 := C_1 + K_2$. An ideal in $\overline{C_2}$ is C_1; hence, C_2 is closed by [**6,** 1.8.4]. We have $C_1 \cap K_2 = \{0\}$ and $A_0 \cap C_2 = \{0\}$ since $(A_0 + K_1) \cap K_2 = \{0\}$ and $A_0 \cap K_1 = \{0\}$. We inductively define $C_l : C_{l-1} + K_l$. Then C_{l-1} is an ideal of the C^*-algebra C_l, $C_{l-1} \cap K_l = \{0\}$ and $A_0 \cap C_l = \{0\}$. From (3.3), (3.4) and (3.5), we find that $e^l_{ij} \to e^{l-1}_{ij}$, $1 \le l \le p$, $i, j \ge 1$, yields an isomorphism φ_p of C_p onto A_{p-1}. Obviously, φ_{p+1} extends φ_p; hence, we obtain an isomorphism φ from $C := \overline{\cup_{p=1}^{\infty} C_p}$ onto A that extends each φ_p.

Now, let $x \in A$, $x = \lim_{p\to\infty} x_p$ with $x_p \in A_p$, $p \ge 1$. Then $x_p = x^0_p + x^p_p$ where $x^0_p \in A_0$ and $x^p_p \in C_p$. From $\|x^0_p - x^0_q\| \le 2\|x_p - x_q\|$, we conclude that the Cauchy sequence $\{x^0_p\}_{p=1}^{\infty}$ converges to some $x^0 \in A_0$; hence, $\{x^p_p\}_{p=1}^{\infty}$ converges to some $x^C \in C$ that satisfies $x = x^0 + x^C$. Now, $x_p \to x^p_p$ is a homomorphism for each p; therefore, $x \to x^C$ is a homomorphism. We have $\|x^C\| \le \|x\|$ and $\|x^0\| \le 2\|x\|$. The quotient map $A \to A/A_0$ is isometric on each C_p; hence, it is isometric on C. It follows that $A_0 \cap C = \{0\}$, and the decomposition $x = x^0 + x^C$ is unique.

Now, we can begin constructing the continuous field of C^*-algebras which we need. Let $\{r_n\}$ be an enumeration of the set of rational numbers in $[0,1]$. For an irrational number $t \in [0,1]$, we define $A(t) := c_0(A)$, that is, the direct sum of A with itself $\aleph_0$ times. $A(r_n)$ is a C^*-subalgebra of $c_0(A)$ that is also a direct sum of copies of A, except that at the nth spot, we insert C instead of A. Clearly, all fibers are mutually isomorphic postliminal C^*-algebras. The *-algebra Γ of the continuous vector fields consists of all continuous functions $x : [0,1] \to c_0(A)$ such that $x(t) \in A(t)$ for every $t \in [0,1]$.

In order to show that $((A(t))_{t\in[0,1]}, \Gamma)$ so defined is a continuous field, we must check that $\{x(t) \mid x \in \Gamma\} = A(t)$ for $t \in [0,1]$. To this end, let $t_0 \in [0,1]$ and $\{a_n\} \in A(t_0)$. For $n \in \mathbb{N}$, let $f_n : [0,1] \to [0,1]$ be a continuous function such that $f_n(t_0) = 1$ and $f_n(r_n) = 0$ if $r_n \ne t_0$. Define $x(t) := \{f_n(t)a_n\}$, $t \in [0,1]$. Then, x is a continuous function from $[0,1]$ to $c_0(A)$ such that $x(t) \in A(t)$ for $t \in [0,1]$, i.e., $x \in \Gamma$. Moreover, $x(t_0) = \{a_n\}$, and we have proved that a continuous field of C^*-algebras has been constructed.

The greatest liminal ideal $B(t)$ of $A(t)$ is $c_0(A_0)$ when t is irrational. The greatest liminal ideal $B(r_n)$ of $A(r_n)$, $n \in \mathbb{N}$, is again a direct sum whose components are all equal to A_0, except that at the nth place which is equal to K_1. Thus, if $x \in \Gamma$ satisfies $x(t) \in B(t)$ for every $t \in [0,1]$, then the nth component of $x(r_n)$ must vanish since $A_0 \cap C = \{0\}$. It follows that, for the restriction of our continuous field of C^*-algebras to any relatively open subset U of $[0,1]$, the family $(B(t))_{t\in U}$, together with $\{x \in \Gamma \mid x(t) \in B(t), t \in U\}$, does not form a continuous field of C^*-algebras.

Acknowledgments. The author wishes to express his gratitude to Professors E. Christensen and J. Phillips for providing reference [**4**]. The author also wishes to express his gratitude to the referee whose comments helped eliminate some confusion and greatly improved the paper.

REFERENCES

1. R.J. Archbold and D.W.B. Somerset, *Transition probabilities and trace functions for C^*-algebras*, Math. Scand. **73** (1993), 81–111.

2. R.J. Archbold, D.W.B. Somerset and J.S. Spielberg, *Upper multiplicity and bounded trace ideals in C^*-algebras*, J. Funct. Anal. **146** (1997), 430–463.

3. E. Christensen and M.-D. Choi, *Completely order isomorphic and close C^*-algebras need not be *-isomorphic*, Bull. Lond. Math. Soc. **15** (1983), 604–610.

4. E. Christensen, A.M. Sinclair, R.R. Smith, S.A. White and W. Winter, *Perturbations of nuclear C^*-algebras*, Acta Math. **208** (2012), 93–150.

5. J. Dixmier, *Sur les C^*-algèbres*, Bull. Soc. Math. France **88** (1960), 95–112.

6. ———, *C^*-algebras*, North-Holland, Amsterdam, 1977.

7. J. Dugundji, *Topology*, Allyn and Bacon, Boston, 1967.

8. J.M.G. Fell, *The structure of algebras of operator fields*, Acta Math. **106** (1961), 233–280.

9. P. Green, *Minimal primitive ideals of* GCR *C^*-algebras*, Proc. Amer. Math. Soc. **73** (1979), 209–210.

10. A.J. Lazar and D.C. Taylor, *Approximately finite dimensional C^*-algebras and Bratteli diagrams*, Trans. Amer. Math. Soc. **259** (1980), 599–619.

11. E. Michael, *Continuous selections* I, Ann. Math. **63** (1956), 361–382.

12. G.K. Pedersen, *C^*-algebras and their automorphism groups*, Academic Press, London, 1979.

13. J. Phillips, *Perturbations of C^*-algebras*, Indiana Univ. Math. J. **23** (1974), 1167–1176.

14. J. Phillips and I. Raeburn, *Perturbations of C^*-algebras*, II, Proc. Lond. Math. Soc. **43** (1981), 46–72.

Tel Aviv University, School of Mathematical Sciences, Tel Aviv, 69978 Israel
Email address: **aldo@post.tau.ac.il**