ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 47, Number 3, 2017

DISCRETE CONDUCHÉ FIBRATIONS AND C^*-ALGEBRAS

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ABSTRACT. The k-graphs in the sense of Kumjian and Pask [**7**] are discrete Conduché fibrations over the monoid $\mathbb{N}^k$, satisfying a finiteness condition. We examine the generalization of this construction to discrete Conduché fibrations with the same finiteness condition and a lifting property for completions of cospans to commutative squares, over any category satisfying a strong version of the right Ore condition, including all categories with pullbacks and right Ore categories in which all morphisms are monic.

1. Introduction. In 2000, Kumjian and Pask introduced k-graphs [**7**] in order to generalize the higher-rank Cuntz-Krieger algebras of Robertson and Steger [**14**], and the graph algebras of Kumjian, et al., [**9**]. Their construction of C^*-algebras from k-graphs yields algebras which are uncommonly tractable and include a wide variety of examples, see, for example, [**11**]. This combination has led to significant interest in k-graph C^*-algebras.

A k-graph consists of a category Λ that is fibred over $\mathbb{N}^k$ by a degree functor $d : \Lambda \to \mathbb{N}^k$, satisfying the unique factorization condition: if $d(\lambda) = m + n$, then there exist unique μ, ν with $d(\mu) = m$, $d(\nu) = n$ and $\lambda = \mu\nu$ [**7**]. It is suggested [**7**] that it might be interesting to study categories Λ that are fibred over a cancellative abelian monoid. Recently, [**3, 16**] used such a construction as a technical tool for studying the primitive ideals and abelian subalgebras of k-graphs, respectively. A category fibred over a cancellative abelian monoid (in particular, a k-graph) is an example of a previously studied notion in category theory called a *discrete Conduché fibration*, see Definition 2.1. In this paper, our goal is to develop a theory of C^*-algebras associated with discrete Conduché fibrations, and thus, generalize k-graph C^*-

2010 AMS *Mathematics subject classification.* Primary 18D30, 22A22, 46L05.
Keywords and phrases. Conduché fibrations, Cuntz-Krieger algebra, groupoids.
Received by the editors on October 24, 2014, and in revised form on July 13, 2015.
DOI:10.1216/RMJ-2017-47-3-711

algebras. While extensions of the theory of [**7**] to categories fibred over a cancellative abelian monoid are relatively straightforward, substantial difficulties arise when the base of the fibration is a more general category.

More recently, Spielberg [**15**] proposed a different construction of C^*-algebras from what he calls *categories of paths*. These are special categories which, among other things, contain no inverses to non-identity morphisms and are both left and right cancellative. He also does not require any fibration of the category corresponding to the degree functor, and so his construction has a different flavor from [**7**]. We allow our categories to have inverses. Thus, a discrete group H equipped with the identity functor $\mathrm{Id}_H : H \to H$ is an admissible fibration for our theory, but neither category is a category of paths in the sense of Spielberg. It turns out that the C^*-algebra associated by our construction to $\mathrm{Id}_H : H \to H$ is the full group C^*-algebra of H. We also consider categories that are not necessarily cancellative.

In Section 2, we introduce a discrete Conduché fibration $F : \mathcal{E} \to \mathcal{B}$ and show that there is a universal C^*-algebra, $C^*(F)$, for the analogous Cuntz-Kreiger relations.

In Section 3, we restrict our attention to fibrations over strong right Ore categories, see Definition 3.1. This restriction allows us to define an infinite path as a section of the functor induced by F on a slice category. Our infinite paths behave analogously to those of [**7**], see Proposition 3.17. Then, in Proposition 3.19, we represent $C^*(F)$ on the set of infinite paths and use this to show that $C^*(F) \neq \{0\}$.

In Section 4, we place a topology on the set of infinite paths and then use a special class of local homomorphisms to construct a locally compact Hausdorff étale groupoid G_F.

In Section 5, we show that, if morphisms in the category are monic, then $C^*(G_F) \cong C^*(F)$. We use this to show that, if morphisms are also epi, then the category $\mathcal{E}$ can be faithfully represented as a subcategory of operators on a Hilbert space.

Throughout our discussion, all categories are small, and composition in categories is written in the customary anti-diagrammatic order as multiplication; thus, the composition of the morphisms in $C \xrightarrow{\beta} B \xrightarrow{\alpha} A$ is written $\alpha\beta$. We denote the target or codomain of a morphism α

by $r(\alpha)$ and its source or domain by $s(\alpha)$. Given an object X in a category $\mathcal{C}$, we denote the set of morphisms in $\mathcal{C}$ with range X by $X\mathcal{C}$. We denote the set of objects in $\mathcal{C}$ by $\mathrm{Obj}(\mathcal{C})$, and, for a morphism γ in $\mathcal{C}$, we let $\gamma\mathcal{C} = \{\gamma\mu : s(\gamma) = r(\mu), \mu \in \mathcal{C}\}$. We will often identify categories with their set of morphisms, so that $\gamma \in \mathcal{C}$ will mean that γ is a morphism of $\mathcal{C}$.

2. Discrete Conduché fibrations.

Definition 2.1. A *discrete Conduché fibration* (dCf) is a functor $F : \mathcal{E} \to \mathcal{B}$ with the unique factorization lifting property or Conduché condition: for every morphism $\phi : Y \to X$ in $\mathcal{E}$, every factorization of $F(\phi)$ in $\mathcal{B}$

$$F(Y) \xrightarrow{\lambda} B \xrightarrow{\rho} F(X)$$

lifts uniquely to a factorization of ϕ:

$$Y \xrightarrow{\widetilde{\lambda}} Z \xrightarrow{\widetilde{\rho}} X,$$

with $F(\widetilde{\lambda}) = \lambda$, $F(\widetilde{\rho}) = \rho$ and $F(Z) = B$.

As an aside, we note that the more categorically natural notion of a Conduché fibration, in which the factorization is unique only up to isomorphism, appears to be inadequate for our purposes, although in most of our examples, there are no non-identity isomorphisms in $\mathcal{E}$, and in such cases, the two notions coincide.

A simple argument shows that unique factorization extends to finitely many factors.

Lemma 2.2. *If $F : \mathcal{E} \to \mathcal{B}$ is a discrete Conduché fibration, $\mu : Y \to X$ is any morphism of $\mathcal{E}$ and $F(\mu) = cba$, that is,*

$$F(Y) \xrightarrow{a} B_1 \xrightarrow{b} B_2 \xrightarrow{c} F(X)$$

is a factorization of $F(\mu) : F(Y) \to F(X)$ in $\mathcal{B}$, then there exists a unique factorization

$$Y \xrightarrow{\alpha} Z_1 \xrightarrow{\beta} Z_2 \xrightarrow{\gamma} X$$

of μ in $\mathcal{E}$ such that $F(\alpha) = a$, $F(\beta) = b$, $F(\gamma) = c$, $F(Z_1) = B_1$ and $F(Z_2) = B_2$.

Moreover, in general, any factorization of a morphism in $\mathcal{B}$ into a finite number of composands uniquely lifts to a factorization of any morphism in its preimage in $\mathcal{E}$ into the same number of composands.

Lemma 2.3. *If $F : \mathcal{E} \to \mathcal{B}$ is a discrete Conduché fibration and $F(\phi : X \to Y) = \mathrm{Id}_B$ for some object B in $\mathcal{B}$, then ϕ is an identity morphism in $\mathcal{E}$.*

Proof. By functoriality of F, we have $F(X) = F(Y) = B$. Now, Id_B factors as $\mathrm{Id}_B(\mathrm{Id}_B)$, and both $\mathrm{Id}_Y(\phi)$ and $\phi(\mathrm{Id}_X)$ are lifts of this factorization. Therefore, by the uniqueness condition of dCFs, $\mathrm{Id}_Y = \phi = \mathrm{Id}_X$, and we see that ϕ is an identity morphism. □

Following Kumjian, Pask and Raeburn [**8**] we define:

Definition 2.4. A functor $F : \mathcal{E} \to \mathcal{B}$ is *row-finite* if, for every object X in $\mathcal{E}$, and every morphism $\beta : B \to F(X)$ in $\mathcal{B}$, the class of morphisms with target X whose image under F is β is a finite set.

The next surjectivity condition takes the place of the no sources condition in the study of graphs and k-graphs.

Definition 2.5. A functor $F : \mathcal{E} \to \mathcal{B}$ between small categories is *strong surjective* if it is surjective on objects, and, given any object X in $\mathcal{E}$, the map induced from the set of morphisms targeted at X to the set of morphisms targeted at $F(X)$ in $\mathcal{B}$ is surjective.

Now, we may rephrase the definition presented in [**7**].

Definition 2.6. A row-finite *k-graph* with no sources is a countable category $\mathcal{E}$ equipped with a strong surjective, row-finite, dCf to the additive monoid $\mathbb{N}^k$, regarded as a category with one object.

Now, it is easy enough to describe generators and relations after the manner of [**4, 7, 8**] using an arbitrary strong surjective, row-finite functor between small categories as data.

Definition 2.7. Given a strong surjective, row-finite discrete Conducé fibration, $F : \mathcal{E} \to \mathcal{B}$, a *Cuntz-Krieger system associated to* F in a C^*-algebra D is:

- a projection P_X for each object X of $\mathcal{E}$;
- a partial isometry S_α for each morphism $\alpha : Y \to X$ of $\mathcal{E}$;

satisfying the relations:

(i) for $X \neq Y$, $P_X \perp P_Y$;
(ii) if α and β are composable, then $S_{\alpha\beta} = S_\alpha S_\beta$;
(iii) for all x, $P_x = S_{\mathrm{Id}_x} = S^*_{\mathrm{Id}_x}$;
(iv) for all $\alpha : y \to x$, $S^*_\alpha(S_\alpha) = P_y$;
(v) if $f(\alpha) = f(\beta)$ and $\alpha \neq \beta$, then $S^*_\beta(S_\alpha) = 0$;
(vi) for all X, and for all morphisms $b : B \to F(X)$ in $\mathcal{B}$

$$\sum_{\{\alpha \in X\mathcal{E} : F(\alpha) = b\}} S_\alpha S^*_\alpha = P_X.$$

Remark 2.8. While item (v) follows from item (vi), we will be using item (v) often enough that we include it in the axioms for emphasis.

We denote the smallest C^*-subalgebra of D containing P, S by $C^*(P, S)$.

In the next proposition, we show that there exists a universal C^*-algebra for these relations. The proof of this proposition follows [**5,** Theorem 2.1].

Proposition 2.9. *Let $F : \mathcal{E} \to \mathcal{B}$ be a strong surjective, row-finite discrete Conduché fibration. Then there exists a C^*-algebra $C^*(F)$ generated by a Cuntz-Krieger system $\{p_X, s_\alpha\}$ such that, for any Cuntz-Krieger system $\{Q_X, T_\alpha\}$ associated to F in a C^*-algebra B, there is a unique $*$-homomorphism from $C^*(F)$ to B extending the map $s_\alpha \mapsto T_\alpha$.*

Proof. Let K_F be the free complex $*$-algebra generated by the morphisms of $\mathcal{E}$, that is, K_F is a vector space over $\mathbb{C}$ with basis given by the set of words $\mathcal{W}$ in symbols

$$\{\alpha : \alpha \in \mathcal{E}\} \cup \{\beta^* : \beta \in \mathcal{E}\} \cup \{X : X \in \mathrm{Obj}(\mathcal{E})\},$$

multiplication induced by concatenation of words and involution induced by the maps $\alpha \in \mathcal{E} \mapsto \alpha^*$ and $c \in \mathbb{C} \mapsto \overline{c}$. Define a seminorm $\|\cdot\|_I$ on K_F by

$$\Big\| \sum_{w \in N} c_w w \Big\|_I = \sum_{w \in N} |c_w|,$$

where N is a finite subset of $\mathcal{W}$ and c_w are complex numbers.

Let J be the ideal in K_F generated by the Cuntz-Krieger relations. Then, $\Gamma(F) = K_F/J$ and $\|\cdot\|_I$ induces a seminorm $\|\cdot\|_\Gamma$ on $\Gamma(F)$ by

$$\|a + J\|_\Gamma = \inf_{b+J=a+J} \|b + J\|_I.$$

Let

$$\begin{aligned} \|a\|_0 = \sup\{\|\rho(a)\| : \rho \text{ is a } \Gamma - \text{seminorm decreasing} \\ * -\text{representation of } \Gamma(F) \text{ on a Hilbert space}\}, \end{aligned}$$

for $a \in \Gamma(F)$. Since $\|\cdot\|_0$ is bounded by $\|\cdot\|_\Gamma$, this gives a C^*-seminorm on $\Gamma(F)$.

Now, define $C^*(F)_0 := \Gamma(F)/\ker(\|\cdot\|_0)$. Then, $\|\cdot\|_0$ induces a C^*-norm $\|\cdot\|$ on $C^*(F)_0$. Complete $C^*(F)_0$ in this norm to obtain a C^*-algebra $C^*(F)$. Denote the images of objects X and morphisms μ of $\mathcal{E}$ in $C^*(F)$ by p_X and s_μ, respectively.

It remains to show that this C^*-algebra satisfies the required universal property. Suppose that $\{Q_X\}_{X \in \mathtt{Obj}(\mathcal{E})}$ and $\{T_\mu\}_{\mu \in \mathcal{E}}$ is a Cuntz-Krieger family in a C^*-algebra B. By the Gelfand-Nainmark theorem, we can assume that B is a subalgebra of the bounded operators on a Hilbert space. By the universal property of the free algebra K_F, there exists a $*$-homomorphism $\widetilde{\rho}$ from $\Gamma(F)$ to B. By the definition of $\|\cdot\|_0$, this representation is norm decreasing on $C^*(F)_0$, and thus, extends to $*$-homomorphism ρ from $C^*(F)$ to B, as desired. By construction, ρ must send p_X to Q_X and s_μ to T_μ, and thus, is determined on the dense subalgebra $C^*(F)_0$. Thus, ρ is unique. $\square$

A question immediately arises: under what circumstances is the map $s_{(\cdot)} : \mathcal{E} \to C^*(F)$ injective?

The upshot of results in **[7]** is that, when F is a row-finite, strong surjective dCF over $\mathbb{N}^k$, i.e., a k-graph, the map $s_{(\cdot)} : \mathcal{E} \to C^*(F)$ is injective. This result uses an infinite path construction. We will use a similar construction to prove Proposition 3.19, which gives some mild conditions that ensure that $C^*(F) \neq \{0\}$, and Proposition 5.4, which gives conditions that guarantee that the map $\mathcal{E} \to C^*(F)$ given by $\alpha \mapsto s_\alpha$ is injective.

3. Kumjian-Pask fibrations and infinite paths. Recall that a morphism α is *epi* if $\beta\alpha = \gamma\alpha$ implies $\beta = \gamma$; α is *monic* if $\alpha\beta = \alpha\gamma$ implies $\beta = \gamma$. A category $\mathcal{B}$ is *right* (*respectively, left*) *cancellative* if every morphism is epi (respectively, monic).

One initially might think that cancellation conditions after the manner of **[15]** are necessary. However, as we shall see, weaker conditions suffice for most of the construction, although for some results, cancellation conditions on the base category are required.

Following Johnstone **[6]**, define:

Definition 3.1. A category *satisfies the right Ore condition* or, for brevity, is *a right Ore category*, if every cospan $A \overset{m}{\to} B \overset{n}{\leftarrow} C$ can be completed to a commutative square.

We extend this notion to:

Definition 3.2. A category is *strong right Ore* if it is right Ore and, moreover, for every cospan $m, n \in \mathcal{B}$ with $p_1, p_2, q_1, q_2 \in \mathcal{B}$ with $mp_i = nq_i$ for $i = 1, 2$, there exist $a, b \in \mathcal{B}$ with $p_1 a = p_2 b$ and $q_1 a = q_2 b$.

Such categories are plentiful:

Proposition 3.3.

(i) *Any category with pullbacks is strong right Ore*;
(ii) *any left cancellative right Ore category is strong right Ore.*

Proof. For item (i), the existence of pullbacks implies that the category is right Ore. In order to show that such a category is strong right Ore, suppose that (P, t, w) is the pullback of the cospan (m, n) so that $mt = nw$ and (Z_1, p_1, q_1) and (Z_2, p_2, q_2) are two completions of (m, n) so that $mp_i = nq_i$. By the universal property of pullbacks, $u_i : Z_i \to P$ exists such that $tu_i = p_i$ and $wu_i = q_i$. Now, (u_1, u_2) is a cospan, so let (Q, a, b) be the pullback of this cospan. By definition, we have $p_1a = p_2b$ and $q_1a = q_2b$.

For item (ii), given a cospan m, n, let $mp_i = nq_i$ for $i = 1, 2$ be two completions to commutative squares, and let a and b complete the cospan mp_1, mp_2 to a commutative square $mp_1a = mp_2b$. Now, since m is monic, it follows that $p_1a = p_2b$. Since the completed cospan could also be considered to be nq_1, nq_2, and n is monic, it follows that $q_1a = q_2b$. $\square$

Since all lattices in the order-theoretic sense (including the lattices of open sets in topological spaces) have pull-backs, they are all strong right Ore. Similarly, since all groups, groupoids and the positive cones of lattice-ordered groups are left cancellative, they are strong right Ore as well.

Proposition 3.4. *If $\mathcal{C}_i$ for $i \in \mathcal{I}$ is a set-indexed family of categories, each of which is strong right Ore, then the product category $\prod_{i\in\mathcal{I}} \mathcal{C}_i$ is strong right Ore.*

Proof. The required completions of diagrams exist in the product because their components exist in the factors. $\square$

Definition 3.5. Given a category $\mathcal{C}$ and an object X in $\mathcal{C}$, the *slice category* $\mathcal{C}/X$ is the category whose objects are morphisms of $\mathcal{C}$ with target X, and morphisms given by commutative triangles, that is, for $\alpha, \beta \in X\mathcal{C} = \mathrm{Obj}(\mathcal{C}/X)$, a morphism from β to α is given by a $\gamma \in \mathcal{C}$ with $\alpha\gamma = \beta$. We can thus view morphisms in $\mathcal{C}/X$ as ordered pairs $(\alpha, \gamma) \in X\mathcal{C} \times \mathcal{C}$ with $r(\gamma) = s(\alpha)$. The range of (α, γ) is α, and its source is $\alpha\gamma$. Composition $(\alpha, \gamma)(\beta, \delta)$ can only occur if $\beta = \alpha\gamma$, and, in this case, we have $(\alpha, \gamma)(\alpha\gamma, \delta) = (\alpha, \gamma\delta)$. Let π_i be the projection onto the ith factor of $\mathcal{C}/X$. Note that $\pi_1(\alpha, \gamma)\pi_2(\alpha, \gamma) = \alpha\gamma$ and that $\pi_2((\alpha, \gamma)(\alpha\gamma, \delta)) = \pi_2(\alpha, \gamma)\pi_2(\alpha\gamma, \delta)$. (Slice categories are a special instance of a more general construction, known as a *comma category*, see **[10]** for more details.)

For any object X in $\mathcal{C}$, a functor $F : \mathcal{C} \to \mathcal{D}$ will induce a natural functor from $\mathcal{C}/X$ to $\mathcal{D}/F(X)$, which, by abuse of notation, we also denote by F.

Definition 3.6. A functor $F : \mathcal{E} \to \mathcal{B}$ is *locally split* if, for every object $X \in \mathcal{E}$, the induced functor (also denoted F, by abuse of notation) $F : \mathcal{E}/X \to \mathcal{B}/F(X)$ admits a splitting (or section) $x : \mathcal{B}/F(X) \to \mathcal{E}/X$, that is, a functor such that $F \circ x = \mathrm{Id}_{\mathcal{B}/F(X)}$. (Contrary to the usual practice in category theory of denoting functors by capital letters, we will use lower case Latin letters near the end of the alphabet to denote local splittings of the fibration, as these will play the role of infinite paths, denoted by such letters in **[7]**.)

Note that a locally split functor which is surjective on objects is strong surjective.

Let $x : \mathcal{B}/F(X) \to \mathcal{E}/X$ be a splitting of F as above and $(a, b) \in \mathcal{B}/F(X)$. Then, a is an object in $\mathcal{B}/F(X)$. By the functoriality of x and the unique factorization lifting property of F, for $x(a, b) = (x_1(a, b), x_2(a, b))$, we have

$$x(a) = x_1(a, b) = x_2(\mathrm{Id}_{F(X)}, a).$$

Definition 3.7. A functor $F : \mathcal{E} \to \mathcal{B}$ targeted at a strong right Ore category is a *Kumjian-Pask fibration* (KPf) if it is a locally split dCf.

Note that we do not require strong surjectivity; however, we have:

Proposition 3.8. *If* $F : \mathcal{E} \to \mathcal{B}$ *is a* KPf*, then so is* $\widehat{F} : \mathcal{E} \to F(\mathcal{E})$*. Moreover,* $\widehat{F}$ *is strong surjective.*

Proof. Since F is a locally split dCF, $\widehat{F}$ is as well; $\widehat{F}$ is surjective on objects, and therefore, strong surjective. In order to show that $\widehat{F}$ is a KPf, it suffices to show that $F(\mathcal{E})$ is strong right Ore. We begin by showing that $F(\mathcal{E})$ is right Ore. Let (m, n) be a cospan in $F(\mathcal{E})$ and $A = r(m)$. Since $m \in F(\mathcal{E})$, we can choose $\mu \in \mathcal{E}$ with $F(\mu) = m$. Thus, $F(r(\mu)) = r(m) = A$. Let $X = r(\mu)$. Since F is locally split, choose a section $x : \mathcal{B}/A \to \mathcal{E}/X$. Since $m, n \in \mathcal{B}$ and $\mathcal{B}$ is right Ore, there exists $p, q \in \mathcal{B}$ such that $mp = nq$. Now, $F(x_2(m, p)) = p$ and $F(x_2(n, q)) = q$. Thus, $p, q \in F(\mathcal{E})$, that is, $F(\mathcal{E})$ is right Ore. Similarly, if $mp_i = nq_i$, then, since $\mathcal{B}$ is strong right Ore, there exist $a, b \in \mathcal{B}$ with $p_1 a = p_2 b$ and $q_1 a = q_2 b$. Therefore, $F(x_2(mp_1, a)) = a$

and $F(x_2(np_2, b)) = b$. Thus, $a, b \in F(\mathcal{E})$, that is, $F(\mathcal{E})$ is strong right Ore as well. $\square$

There are many examples:

Example 3.9. If $d : \Lambda \to \mathbb{N}^k$ is a row-finite k-graph with no sources, then d is a KPf.

Example 3.10. If $\mathcal{B}$ is any strong right Ore category, $\mathrm{Id}_{\mathcal{B}} : \mathcal{B} \to \mathcal{B}$ is a KPf.

Of particular interest is a special case of the last example:

Example 3.11. If H is a discrete group, regarded as a one object category, then the identity group homomorphism Id_H regarded as a functor is a KPf. In this example, the C^*-algebra $C^*(\mathrm{Id}_H)$ is particularly easy to compute. Suppose that (S, P) is a Cuntz-Krieger system for Id_H in a C^*-algebra A. Since H has only one object e, and we have $P_e S_t = S_t P_e$ for all $t \in H$, thus (S, P) is a Cuntz-Krieger system in the unital C^*-algebra PAP with unit P. Because we are using the identity functor, the sum in relation (6) has only one summand and, combining with relation (iv), we have $S_t S_t^* = P = S_t^* S_t$ for all $t \in H$. This gives that S_t is unitary. Now, by relation (2), we get that $t \mapsto S_t$ is a group homomorphism, and thus is a unitary representation of H. Since $C^*(H)$ is universal for unitary representations of H, and $C^*(\mathrm{Id}_H)$ is universal for Cuntz-Krieger systems, we obtain that $C^*(\mathrm{Id}_H) \cong C^*(H)$.

As a preliminary to our next examples, recall that any poset $(P, \leq)$ induces a category, which, by abuse of notation, we also denote by P, whose objects are elements of P and whose morphisms are ordered pairs $(p, q) \in P \times P$ with $p \leq q$, with $r((p, q)) = q$ and $s((p, q)) = p$ and composition defined by $(q, r)(p, q) = (p, r)$, cf., **[10]**. Note that reversing the ordering gives another poset. There is an arbitrary choice in this convention, which was made so that the abstract maps of a poset would correspond in the case of concrete posets of subobjects of a mathematical object to the inclusion maps from one subobject to another which contains it. (The convention, unfortunately, appears to be at odds with that adopted in **[7]** for their category Ω_k, in which the morphisms have the greater k-tuple as source and the lesser as target. In fact, Ω_k is the opposite category of $(\mathbb{N}^k, \leq)$, regarded as a category in the standard way, a fact either intentionally or fortuitously emphasized by the fact that Kumjian and Pask [**7**] denote the unique morphism from p' to p when $p' \geq p$ by (p, p'), the same notation by which the standard convention for a poset would denote the unique morphism in the opposite direction. As we will see in Example 3.15, Ω_k is better thought of as arising not from the poset of k-tuples of natural numbers but as the slice category of the monoid $(\mathbb{N}^k, +, 0)$, as a one-object category, over its unique object, which naturally gives the morphisms in the direction from greater to lesser, although it will also suggest the use of a different notation for them.)

Example 3.12. Let X be a topological space and $\mathfrak{S}$ a presheaf of sets on X, that is, a contravariant functor from the lattice of open sets $\mathcal{T}_X$ ordered by inclusion to **Sets**, the category of sets and set-functions. As usual, we regard the images of elements of $\mathfrak{S}(U)$ under maps $\mathfrak{S}(V, U)$ as restrictions, denoting $\mathfrak{S}(V, U)(\sigma)$ by $\sigma|_V$ for $\sigma \in \mathfrak{S}(U)$.

Let $\mathcal{S}$ be the set of local sections, that is, pairs (U, σ) where U is an open set and $\sigma \in \mathfrak{S}(U)$. Then, $\mathcal{S}$ is partially ordered by restriction, with $(V, \tau) \leq (U, \sigma)$ exactly when $V \subseteq U$ and $\tau = \sigma|_V$. Now,

$$((V, \tau), (U, \sigma)) \longmapsto (V, U)$$

defines a functor $F : \mathcal{S} \to \mathcal{T}_X$. If $((V, \tau), (U, \sigma)) \in \mathcal{S}$ and $V \subseteq W \subseteq U$ so that $(V, W)(W, U)$ is a factorization of $F((V, \tau), (U, \sigma)) = (V, U)$, then $((V, \tau), (W, \sigma|_W))((W, \sigma|_W), (U, \sigma))$ is the unique factorization of $((V, \tau), (U, \sigma))$ lifting the factorization of (V, U) as $(V, W)(W, U)$.

Since $\mathcal{T}_X$ has pullbacks, it is strong right Ore. Finally, for any object (U, σ) in $\mathcal{S}$, the induced functor $F : \mathcal{S}/(U, \sigma) \to \mathcal{T}_X/U$ admits a splitting given by

$$(V, U) \longmapsto ((V, \sigma|_V), (U, \sigma)),$$

and thus, $F : \mathcal{S} \to \mathcal{T}_X$ is a KPf.

In fact, this last example generalizes:

Example 3.13. If $\mathcal{B}$ is a right Ore poset (i.e., a poset in which every pair of elements which admits an upper bound also admits a lower bound, necessarily strong right Ore since all morphisms are monic) and $S : \mathcal{B}^{\mathrm{op}} \to \mathbf{Sets}$ is a presheaf of sets on $\mathcal{B}$, then the poset of sections $\mathcal{E}_S$ with elements all pairs (U, a) with $U \in \mathrm{Ob}\,(\mathcal{B})$ and $a \in S(U)$, and $(V, b) \leq (U, a)$ exactly when $V \leq U$ and $b = a|_V$ (using the usual notation in the context of presheaves on a space) admits an obvious monotone map $(U, a) \mapsto U$ to $\mathcal{B}$ which, regarded as a functor, is a KPf.

Among the last two classes of examples, those arising from presheaves of finite sets will be row-finite.

Definition 3.14. Let $F : \mathcal{E} \to \mathcal{B}$ be a row-finite, Kumjian-Pask fibration.

(i) For an object X of $\mathcal{E}$, an *infinite path to* X is a section $x : \mathcal{B}/F(X) \to \mathcal{E}/X$ of the functor $F : \mathcal{E}/X \to \mathcal{B}/F(X)$.
(ii) We denote the set of all infinite paths to X by $Z(X)$ and the set of all infinite paths, that is, $\bigcup_{X \in \mathrm{Ob}(\mathcal{E})} Z(X)$, by F^∞.
(iii) For a morphism $\mu : Y \to X$ in $\mathcal{E}$, an *infinite path ending in* μ is an infinite path x to X for which $x(F(\mu)) = \mu$. We denote the set of all infinite paths ending in μ by $Z(\mu)$.
(iv) When an infinite path x is introduced without explicitly stating that it is a functor from $\mathcal{B}/F(X)$ to $\mathcal{E}/X$ for a specified object X, we will denote the object over which its target category is a slice category by $r(x)$.

These notions are consistent with the usual notion of infinite paths for row-finite source free k-graphs. Indeed, the next example is the motivation for regarding splittings of the induced functors on slice categories as infinite paths.

Example 3.15. Let $d : \Lambda \to \mathbb{N}^k$ be a row-finite k-graph, for some $k > 0$, with no sources. The base category is the additive monoid $\mathbb{N}^k$, regarded as a category with a single object $*$ in the usual way.

Using the conventions of Definition 3.5, the objects of $\mathbb{N}^k/*$ are elements of $\mathbb{N}^k$, while morphisms are ordered pairs $(p, q) \in \mathbb{N}^k \times \mathbb{N}^k$, with range p and source $p+q$. Note that, if p and p' are two objects of $\mathbb{N}^k/*$, there is a unique morphism from p' to p exactly when $p' \geq p$ in the product partial order on $\mathbb{N}^k$, and thus, the slice category is isomorphic to the category denoted by Ω_k [**7**]. The unique map from p' to p, when one exists (denoted by (p, p') in [**7**]) is denoted by $(p, p' - p)$ in our convention for slice categories. The functor by which Kumjian and Pask [**7**] endow Ω_k with a k-graph structure maps all objects to $*$, and the morphism which we denote by $(p, p' - p)$ and [**7**] denote (p, p') to $p' - p$. This is precisely the natural forgetful functor from the slice category $\mathbb{N}^k/*$ to $\mathbb{N}^k$.

It remains to show that an infinite path in the sense of [**7**] that is a degree preserving functor from Ω_k to Λ, or equivalently from $\mathbb{N}/*$ to Λ, is equivalent to a splitting of the induced functor $d : \Lambda/v \to \mathbb{N}/*$. Given any functor $F : \mathcal{C} \to \mathcal{D}$, and an object X of $\mathcal{C}$, the square formed by the given F, the induced $F : \mathcal{C}/X \to \mathcal{D}/F(X)$ and the forgetful functors from the slice categories to the $\mathcal{C}$ and $\mathcal{D}$ commute.

Applying this observation to the degree functor of Λ, we see that, if $\widetilde{x}$ is a splitting of $d : \Lambda/v \to \mathbb{N}/*$, and $U : \Lambda/v \to \Lambda$ is the forgetful functor, then $x := U\widetilde{x}$ is a degree-preserving functor from $\mathbb{N}/*$ to Λ since $dU\widetilde{x} = \Upsilon d\widetilde{x} = \Upsilon$, where $\Upsilon : \mathbb{N}/* \to \mathbb{N}$ is the forgetful functor which coincides with Kumjian and Pask's degree functor on Ω_k by the preceding discussion, the first equation holding by the commutativity of the square of functors just observed and the second by the fact that $\widetilde{x}$ splits the induced d. In particular, x will be an infinite path in the sense of [**7**] to the object v of Λ.

Conversely, if $x : \mathbb{N}^k \to \Lambda$ is a degree preserving functor, we construct a factorization of x of the form $U\widetilde{x}$ for $\widetilde{x} : \mathbb{N}^k/* \to \Lambda/x(0)$, a splitting of the induced $d : \Lambda/x(0) \to \mathbb{N}^k/*$ as follows:

For an object q of $\mathbb{N}^k/*$, that is, a morphism of $\mathbb{N}^k$, let $\widetilde{x}(q) = x(q, 0)$. For a morphism (p, q) of $\mathbb{N}^k/*$ from $p + q$ to q, let $\widetilde{x}(p, q) = (x(p, q), x(q, 0))$. The functoriality of $\widetilde{x}$ follows easily from that of x. It follows from the preservation of degree by x that $d\widetilde{x}(q) = q$ and $d\widetilde{x}(p, q) = d(x(p, q), x(q, 0)) = (p, q)$, so that $\widetilde{x}$ is a splitting of the induced functor d on slice categories. Finally, the forgetful functor from a slice category to the underlying category is given in our notation by the second projection $U\widetilde{x}(q) = Ux(q, 0) = x(q)$ and $U\widetilde{x}(p, q) = U(x(p, q), x(q, 0)) = x(p, q)$.

It is easy to see that the two constructions are inverse to each other.

The next lemma is easy to verify.

Lemma 3.16. *Let $F : \mathcal{E} \to \mathcal{B}$ be a Kumjian-Pask fibration. If $b : B \to F(X)$ is a morphism in $\mathcal{B}$, then*

$$Z(X) = \bigcup_{\{\beta \in X\mathcal{E}:\ and\ F(\beta)=b\}} Z(\beta),$$

and the union is disjoint.

The hypothesis that F is locally split does not seem to guarantee any infinite paths ending in a particular morphism. However, the next proposition, in particular, implies that, for a Kumjian-Pask fibration $F : \mathcal{E} \to \mathcal{B}$ and $\alpha \in \mathcal{E}$, there exists an infinite path $x \in Z(\alpha)$. Proposition 3.17 defines maps which are crucial to the rest of our study, so we enumerate their properties here.

Proposition 3.17. *If $F : \mathcal{E} \to \mathcal{B}$ is a row-finite Kumjian-Pask fibration, for any morphism $\mu : Y \to X \in \mathcal{E}$, there exist bijections*

$$\mathrm{ind}_\mu : Z(Y) \longrightarrow Z(\mu) \quad and \quad \mathrm{res}_\mu : Z(\mu) \longrightarrow Z(Y),$$

satisfying the properties:

(i) $\mathrm{ind}_\mu(x)(F(\mu)a) = \mu x(a)$ *for any* $a \in F(s(\mu))\mathcal{B}$;
(ii) $\mathrm{ind}_\mu \circ \mathrm{res}_\mu = \mathrm{Id}_{Z(\mu)}$ *and* $\mathrm{res}_\mu \circ \mathrm{ind}_\mu = \mathrm{Id}_{Z(r(\mu))}$;
(iii) *for X an object in $\mathcal{E}$,* $\mathrm{ind}_{\mathrm{Id}_X} = \mathrm{res}_{\mathrm{Id}_X} = \mathrm{Id}_{Z(X)}$;
(iv) *for $\mu, \nu \in \mathcal{E}$ with $s(\mu) = r(\nu)$,* $\mathrm{res}_\mu(Z(\mu\nu)) = Z(\nu)$ *and* $\mathrm{ind}_\mu(Z(\nu)) = Z(\mu\nu)$;
(v) *for $\mu, \nu \in \mathcal{E}$ with $s(\mu) = r(\nu)$, then the domain of* $\mathrm{res}_\nu \circ \mathrm{res}_\mu$ *is* $Z(\mu\nu)$ *and* $\mathrm{res}_\nu \circ \mathrm{res}_\mu = \mathrm{res}_{\mu\nu}$;
(vi) *for $\mu, \nu \in \mathcal{E}$ with $s(\mu) = r(\nu)$, then the domain of* $\mathrm{ind}_\mu \circ \mathrm{ind}_\nu$ *is* $Z(s(\nu))$ *and* $\mathrm{ind}_\mu \circ \mathrm{ind}_\nu = \mathrm{ind}_{\mu\nu}$.

Proof. We begin by defining the maps ind_μ and res_μ. Since res_μ is easier to describe, we define it first.

Let x be an infinite path ending in μ, $x : \mathcal{B}/F(X) \to \mathcal{E}/X$. Recall that, for $(c, d) \in \mathcal{B}/F(X)$, we denote $x(c, d) = (x_1(c, d), x_2(c, d))$, and then $x(cd)$ is a morphism in $\mathcal{E}$ whose factorization with respect to cd is $x_1(c, d)x_2(c, d)$. The idea of $\mathrm{res}_\mu(x)$ is that it removes the first μ part of the morphism $x(F(\mu)ab)$.

As objects can be identified with identity morphisms, it is enough to describe $\mathrm{res}_\mu(x)$ on pairs of the form (a,b) with $r(a) = F(s(\mu))$ and $r(b) = s(a)$. Now, $(F(\mu)a, b) \in \mathcal{B}/F(X)$, and

$$\begin{aligned} x(F(\mu)a,b) &= (x_1(F(\mu)a,b), x_2(F(\mu)a,b)) \\ &= (x(F(\mu)a), x_2(F(\mu)a,b)) \\ &= (\mu x_2(F(\mu),a), x_2(F(\mu)a,b)) \in \mathcal{E}/X \subset \mathcal{E} \times \mathcal{E}. \end{aligned}$$

We then define

$$\mathrm{res}_\mu(x)(a,b) := (x_2(F(\mu),a), x_2(F(\mu)a,b)).$$

Since

$$r(x_2(F(\mu),a)) = s(x(F(\mu))) = s(\mu)$$

and

$$r(x_2(F(\mu)a,b)) = s(x(F(\mu)a)) = s(x_2(F(\mu),a)),$$

we obtain that $\mathrm{res}_\mu(x)$ maps into $\mathcal{E}/s(\mu)$. As x is a section,

$$F(\mathrm{res}_\mu(x)(a,b)) = (F(x_2(F(\mu),a)), F(x_2(F(\mu)a,b))) = (a,b);$$

thus, $\mathrm{res}_\mu(x)$ is a section of $F : \mathcal{E}/s(\mu) \to \mathcal{B}/F(s(\mu))$. Lastly, in order to see that x is a functor, if $(a,b),(ab,c) \in \mathcal{B}/F(s(\mu))$, then

$$\begin{aligned} \mathrm{res}_\mu(x)(a,b)\,\mathrm{res}_\mu(ab,c) &= (x_2(F(\mu),a), x_2(F(\mu)a,b)) \\ &\quad \cdot (x_2(F(\mu),ab), x_2(F(\mu)ab,c)) \\ &= (x_2(F(\mu),a), x_2(F(\mu)a,b)x_2(F(\mu)ab,c)) \\ &= (x_2(F(\mu),a), x_2(F(\mu)a,bc)) \\ &= \mathrm{res}_\mu(x)(a,bc). \end{aligned}$$

Thus, $\mathrm{res}_\mu(x) : \mathcal{B}/F(s(\mu)) \to \mathcal{E}/s(\mu)$, as desired.

We turn our attention to defining ind_μ. The idea of ind_μ is to add μ onto the beginning of infinite paths. Suppose that $x \in Z(s(\mu))$. Consider $(a,b) \in \mathcal{B}/F(X)$. Now, $F(\mu), ab$ is a cospan in $\mathcal{B}$. Since $\mathcal{B}$ is right Ore, $(c,d) \in \mathcal{B}$ exists such that $F(\mu)c = abd$. By unique factorization, unique morphisms $\alpha_{a,b}^{c,d}$, $\beta_{a,b}^{c,d}$, $\delta_{a,b}^{c,d} \in \mathcal{E}$ exist such that

$$\mu x(c) = \alpha_{a,b}^{c,d}\,\beta_{a,b}^{c,d}\,\delta_{a,b}^{c,d},$$

and $F(\alpha^{c,d}_{a,b}) = a$, $F(\beta^{c,d}_{a,b}(b)) = b$, and $F(\delta^{c,d}_{a,b}(d)) = d$. Take

$$\mathrm{ind}_\mu(x)(a,b) = (\alpha^{c,d}_{a,b}, \beta^{c,d}_{a,b}).$$

We need to show that this does not depend on the choice of the span c, d. For this, we use the strong right Ore condition. Suppose that there exist c_i and d_i that complete the cospan $F(\mu), ab$ into a commuting square. By strong right Ore, $t, u \in \mathcal{B}$ exist such that $c_1 t = c_2 u$ and $d_1 t = d_2 u$. Thus, it suffices to show that $(\alpha^{c_1,d_1}_{a,b}, \beta^{c_1,d_1}_{a,b}) = (\alpha^{c_1t,d_1t}_{a,b}, \beta^{c_1t,d_1t}_{a,b})$. Consider $\mu x(c_1 t) = \mu x(c_1) x_2(c_1, t)$. By definition,

$$\mu x(c_1) x_2(c_1,t) = \mu x(c_1 t) = \alpha^{c_1t,d_1t}_{a,b} \beta^{c_1t,d_1t}_{a,b} \delta^{c_1t,d_1t}_{a,b},$$

with $F(\alpha^{c_1t,d_1t}_{a,b}) = a$, $F(\beta^{c_1t,d_1t}_{a,b}) = b$ and $F(\delta^{c_1t,d_1t}_{a,b}) = d_1 t$. By unique factorization, γ, τ exist with $\gamma\tau = \delta^{c_1t,d_1t}_{a,b}$ and $F(\gamma) = d_1$, and $F(\tau) = t$. Since $F(\alpha^{c_1t,d_1t}_{a,b}\beta^{c_1t,d_1t}_{a,b}\gamma) = abd_1 = F(\mu x(c_1))$, by unique factorization, we have

$$\alpha^{c_1,d_1}_{a,b} \beta^{c_1,d_1}_{a,b} \delta^{c_1,d_1}_{a,b} = \mu x(c_1) = \alpha^{c_1t,d_1t}_{a,b} \beta^{c_1t,d_1t}_{a,b} \gamma.$$

Thus, by unique factorization, again we have $\alpha^{c_1,d_1}_{a,b} = \alpha^{c_1t,d_1t}_{a,b}$ and $\beta^{c_1,d_1}_{a,b} = \beta^{c_1t,d_1t}_{a,b}$, as desired. Therefore, $\mathrm{ind}_\alpha(x)$ is well defined.

We must check that $\mathrm{ind}_\mu(x)$ is a section of $F : \mathcal{E}/Y \to \mathcal{B}/F(Y)$. Note that $r(\alpha^{c,d}_{a,b}) = r(\mu)$ and $r(\beta^{c,d}_{a,b}) = s(\alpha^{c,d}_{a,b})$ by definition. It follows that $\mathrm{ind}_\mu(x)(a,b) \in \mathcal{E}/Y$. Also, by definition, $F(\mathrm{ind}_\mu(x)(a,b)) = (a,b)$. It remains to show that $\mathrm{ind}_\mu(x)$ is a functor. Suppose that $(a,b)(ab,t) \in \mathcal{B}/F(Y)$. Consider the cospan $F(\mu)$ and abt. By right Ore, $c, d \in \mathcal{B}$ exist with $F(\mu)c = abtd$. Thus, by unique factorization, unique $\alpha, \beta, \tau, \delta$ exist such that $F(\alpha) = a$, $F(\beta) = b$, $F(\tau) = t$ and $F(\delta) = d$, and $\mu x(c) = \alpha\beta\tau\delta$. By definition,

$$\begin{aligned}
\alpha^{c,td}_{a,b} &= \alpha = \alpha^{c,d}_{a,bt},\\
\beta^{c,td}_{a,b} &= \beta,\\
\alpha^{c,d}_{ab,t} &= \alpha\beta,\\
\beta^{c,d}_{ab,t} &= \tau,\\
\beta^{c,d}_{a,bt} &= \beta\tau.
\end{aligned}$$

Thus,

$$\operatorname{ind}_\mu(x)(a,b)\operatorname{ind}_\mu(x)(ab,t) = (\alpha,\beta)(\alpha\beta,\tau) = (\alpha,\beta\tau) = \operatorname{ind}_\mu(a,bt),$$

so that ind_μ is indeed a functor.

In order to see that $\operatorname{ind}_\mu(x) \in Z(\mu)$ and res_μ is the inverse for ind_μ we need the next claim:

$$(\operatorname{ind}_\mu(x))_1(F(\mu)a,b) = \mu x(a) \tag{3.1}$$

and

$$(\operatorname{ind}_\mu(x))_2(F(\mu)a,b) = x_2(a,b)$$

for all $(a,b) \in \mathcal{B}/F(Y)$. Next note that $F(\mu), F(\mu)ab$ is a cospan, and the span $ab, s(b)$ completes it to a commuting square. Thus, we have, by the definition of $\operatorname{ind}_\mu(x)$, that

$$\mu x(a)x_2(a,b) = \mu x(ab) = (\operatorname{ind}_\mu(x))_1(F(\mu)a,b)(\operatorname{ind}_\mu(x))_2(F(\mu)a,b).$$

Unique factorization now gives equation (3.1).

Since $\operatorname{ind}_\mu(x)(F(\mu)a) = \operatorname{ind}_\mu(x)_1(F(\mu)a, F(s(\mu))) = \mu x(a)$, item (i) follows immediately from equation (3.1); in particular, taking $a = s(\alpha)$ gives $\operatorname{ind}_\mu(x) \in Z(\mu)$.

We now show item (ii), that is, ind_μ is the inverse of res_μ. Let $x \in Z(\mu)$ and $(a,b) \in \mathcal{B}/F(r(\mu))$; we want to show $\operatorname{ind}_\mu(\operatorname{res}_\mu(x))(a,b) = x(a,b)$. We compute $\operatorname{ind}_\mu(\operatorname{res}_\mu(x))(a,b)$. Now, $F(\mu), ab$ is a cospan in $\mathcal{B}$, so there exist $c, d \in \mathcal{B}$ with $F(\mu)c = abd$. Thus,

$$\begin{aligned} x_1(a,b)x_2(a,b)x_2(ab,d) &= x_1(ab,d)x_2(ab,d) \\ &= x(abd) \\ &= x(F(\mu)c) \\ &= \mu x_2(F(\mu),c) \quad \text{since } x \in Z(\mu) \\ &= \mu \operatorname{res}_\mu(x)(c) \quad \text{by definition of } \operatorname{res}_\mu \\ &= (\operatorname{ind}_\mu(\operatorname{res}_\mu(x)))_1(a,b)(\operatorname{ind}_\mu(\operatorname{res}_\mu(x)))_2(a,b)\delta^{c,d}_{a,b} \end{aligned}$$

by definition of ind_μ. Therefore, $\operatorname{ind}_\mu(\operatorname{res}_\mu(x))(a,b) = x(a,b)$ by unique factorization.

Now suppose that $x \in Z(s(\mu))$ and $(a,b) \in \mathcal{B}/F(s(\mu))$. We want to show that $\operatorname{res}_\alpha(\operatorname{ind}_\mu(x))(a,b) = x(a,b)$. Using equation (3.1), we compute:

$$\begin{aligned}\operatorname{res}_\mu(\operatorname{ind}_\mu(x)(a,b)) &= (\operatorname{ind}_\mu(x)_2(F(\mu),a), \operatorname{ind}_\mu(x)_2(F(\mu)a,b)))\\ &= (x_2(s(\mu),a), x_2(a,b))\\ &= (x_1(a,b), x_2(a,b)) = x(a,b),\end{aligned}$$

as desired.

The other properties now follow fairly quickly.

For item (iii), note that

$$\begin{aligned}\operatorname{res}_{\mathrm{Id}_X}(x)(a,b) &= (x_2(\mathrm{Id}_X, a), x_2(\mathrm{Id}_X\, a,b))\\ &= (x_1(a,b), x_2(a,b)) = x(a,b),\end{aligned}$$

so that $\operatorname{res}_{\mathrm{Id}_X} = \mathrm{Id}_{Z(X)}$. Now,

$$\operatorname{ind}_{\mathrm{Id}_X} = (\operatorname{res}_{\mathrm{Id}_X})^{-1} = \mathrm{Id}_{Z(X)}^{-1} = \mathrm{Id}_{Z(X)}\,.$$

For item (iv), we first show that

$$\operatorname{ind}_\mu(Z(\nu)) \subset Z(\mu\nu) \quad \text{and} \quad \operatorname{res}_\mu(Z(\mu\nu)) \subset Z(\nu). \tag{3.2}$$

For the former, note that, if $x \in Z(\nu)$, $\operatorname{ind}_\mu(x)(F(\mu)F(\nu)) = \mu x(F(\nu)) = \mu\nu$ by item (i). For the latter, if $x \in Z(\mu\nu)$, then

$$\mu\nu = x(F(\mu)F(\nu)) = x_1(F(\mu), F(\nu))x_2(F(\mu), F(\nu)).$$

Since $F(x_2(F(\mu), F(\nu))) = F(\nu)$, we have $x_2(F(\mu), F(\nu)) = \nu$ by unique factorization. Therefore,

$$\operatorname{res}_\mu(x)(F(\nu)) = \operatorname{res}_\mu(x)_1(F(\nu), s(\nu))) = x_2(F(\mu), F(\nu)) = \nu,$$

that is, $\operatorname{res}_\mu(x) \in Z(\nu)$.

It now suffices to show the reverse inclusions. If $x \in Z(\nu)$, then $\operatorname{ind}_\mu(x) \in Z(\mu\nu)$. Thus, $x = \operatorname{res}_\mu(\operatorname{ind}_\mu(x)) \in \operatorname{res}_\mu(Z(\mu\nu))$, that is, $Z(\nu) \subset \operatorname{res}_\mu(Z(\mu\nu))$, and thus, $\operatorname{res}_\mu(Z(\mu\nu)) = Z(\nu)$. Similarly, if $x \in Z(\mu\nu)$, then $\operatorname{res}_\mu(x) \in Z(\nu)$, so that $x = \operatorname{ind}_\mu(\operatorname{res}_\mu(x)) \in \operatorname{ind}_\mu(Z(\nu))$, that is, $Z(\mu\nu) \subset \operatorname{ind}_\mu(Z(\nu))$, and thus, $\operatorname{ind}_\mu(Z(\nu)) = Z(\mu\nu)$.

For item (v), we first show that the domain of $\operatorname{res}_\nu \circ \operatorname{res}_\mu = Z(\mu\nu)$. We want to show that $x \in Z(\mu\nu)$ by letting x be in the domain of

$\mathrm{res}_\nu \circ \mathrm{res}_\mu$. Now, $x = \mathrm{ind}_\mu \circ \mathrm{ind}_\nu(\mathrm{res}_\nu \circ \mathrm{res}_\mu(x))$ so that

$$\begin{aligned} x(F(\mu)F(\nu)) &= \mathrm{ind}_\mu \circ \mathrm{ind}_\nu(\mathrm{res}_\nu \circ \mathrm{res}_\mu(x))(F(\mu)F(\nu)) \\ &= \mu\, \mathrm{ind}_\nu(\mathrm{res}_\nu \circ \mathrm{res}_\mu(x))(F(\nu)) = \mu\nu \end{aligned}$$

where we used item (i) twice. Thus, $x \in Z(\mu\nu)$. Now, if $x \in Z(\mu\nu)$, then by item (iv), $\mathrm{res}_\mu(x) \in Z(\nu)$, and thus, x is in the domain of $\mathrm{res}_\nu \circ \mathrm{res}_\mu$, as desired.

Next, we show that $\mathrm{res}_\nu \circ \mathrm{res}_\mu = \mathrm{res}_{\mu\nu}$ is equal. For this, we compute

$$\begin{aligned} \mathrm{res}_\nu \circ \mathrm{res}_\mu(x)(a,b) &= ((\mathrm{res}_\mu(x))_2(F(\nu),a), (\mathrm{res}_\mu(x))_2(F(\nu)a,b)) \\ &= (x_2(F(\mu\nu),a), x_2(F(\mu\nu)a,b)) \\ &= \mathrm{res}_{\mu\nu}(x)(a,b), \end{aligned}$$

as desired.

For item (vi), notice that by item (v), we have $\mathrm{res}_\nu \circ \mathrm{res}_\mu = \mathrm{res}_{\mu\nu}$. Thus,

$$\mathrm{ind}_\mu \circ \mathrm{ind}_\nu = (\mathrm{res}_\nu \circ \mathrm{res}_\mu)^{-1} = (\mathrm{res}_{\mu\nu})^{-1} = \mathrm{ind}_{\mu\nu},$$

which completes the proof. □

Remark 3.18. Note that, defining res_β merely requires that $F : \mathcal{E} \to \mathcal{B}$ be a dCF. We only use that F is a KPf to define ind_β.

We can now show that $C^*(F) \neq 0$ for row finite Kumjian-Pask fibrations.

Proposition 3.19. *Let $F : \mathcal{E} \to \mathcal{B}$ be a row finite Kumjian-Pask fibration. Then, $C^*(F) \neq 0$.*

Proof. Consider $\ell^2(F^\infty)$. For $\mu \in \mathcal{E}$, define

$$T_\mu x = \begin{cases} \mathrm{ind}_\mu x & \text{if } x \in Z(s(\mu)), \\ 0 & \text{otherwise,} \end{cases}$$

and let Q_X be the projection onto the subspace spanned by $Z(X)$. A quick computation shows that

$$T_\mu^* x = \begin{cases} \mathrm{res}_\mu x & \text{if } x \in Z(\mu), \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 3.17 shows that Q and T satisfy all of the conditions of a Cuntz-Krieger F-family except (6). But, (6) follows immediately from Lemma 3.16. Thus, by the universal property, $C^*(Q,T)$ is a quotient of $C^*(F)$. Since $C^*(Q,T)$ is nonzero, we have that $C^*(F) \neq 0$. □

4. The groupoid of a Kumjian-Pask fibration. In this section, we use the infinite path space of a Kumjian-Pask fibration to construct a groupoid.

A *groupoid* G is a small category in which every morphism is invertible. We identify the objects in G with the identity morphisms and denote both by $G^{(0)}$. As G is a category, we can send any morphism γ to its range and source and denote these maps by r and s, respectively. A *topological groupoid* is a groupoid with a topology in which composition is continuous and inversion is a homeomorphism. It follows that r and s are also continuous. An open subset B of a topological groupoid G is called a *bisection* if $r|_B$ and $s|_B$ are injective and open. We say that a topological groupoid is *étale* if it has a basis of bisections.

We are interested in locally compact Hausdorff étale groupoids because there is a well-developed theory of the C^*-algebras constructed from them. We will show that Kumjian-Pask fibrations give rise to étale groupoids. First, however, we sketch the construction of locally compact Hausdorff étale groupoid C^*-algebras for the convenience of the reader; for details, see **[12]**.

Let G be a locally compact Hausdorff étale groupoid. Define a convolution algebra structure on the continuous compactly supported functions $C_c(G)$ on G by

$$f * g(\gamma) = \sum_{\eta : r(\eta) = r(\gamma)} f(\eta) g(\eta^{-1}\gamma), \quad f^*(\gamma) = \overline{f(\gamma^{-1})}.$$

The sum is finite because f is compactly supported, and the inverse image of a point under an étale map (here, $r^{-1}(\gamma)$) is discrete. We say that a sequence of functions $\{f_i\}$ in $C_c(G)$ converges to f in the inductive limit topology if

$$\|f_i - f\|_\infty = \sup_{\gamma \in G}\{|f_i(\gamma) - f(\gamma)|\} \longrightarrow 0,$$

and there exists a compact set $K \subset G$ such that $\operatorname{supp}(f_i) \subset K$

eventually. Define $\operatorname{Rep}(C_c(G))$ to be the set of $*$-homomorphisms from $C_c(G)$ to $B(\mathcal{H})$ such that the image of inductive limit convergent sequences is weak-$*$ convergent. Define a norm on $C_c(G)$ by

$$\|f\| = \sup\{\|\pi(f)\| : \pi \in \operatorname{Rep}(C_c(G))\}.$$

We then define $C^*(G)$ to be the completion of $C_c(G)$ in $\|\cdot\|$. (It is non trivial that the supremum defining the norm exists, see [**12**].) Given a unit $u \in G^{(0)}$, there is a representation $L^u : C^*(G) \to B(\ell^2(Gu))$ given by

$$L^u(f)\,\delta_\xi = \sum_{s(\eta)=r(\xi)} f(\eta)\,\delta_{\eta\xi}.$$

A quick computation shows that $L^u \in \operatorname{Rep}(C_c(G))$. We define $I_\lambda = \bigcap_{u\in G^{(0)}} \ker(L^u)$ and $C^*_r(G) = C^*(G)/I_\lambda$.

In order to describe the groupoid constructed from a row finite Kumjian-Pask fibration, we first need to topologize the infinite path space. In Definition 3.14, we described for each $\alpha \in \mathcal{E}$ a set of infinite paths $Z(\alpha) = \{x \in F^\infty : x(F(\alpha)) = \alpha\}$. In this section, we show that, under a mild countability hypothesis, the collection of these sets forms a basis of compact open sets for a locally compact Hausdorff topology on F^∞.

Lemma 4.1. *Let $F : \mathcal{E} \to \mathcal{B}$ be a row finite Kumjian-Pask fibration.*

(i) *If $\alpha, \delta \in \mathcal{E}$ with $r(\delta) = s(\alpha)$, then $Z(\alpha\delta) \subset Z(\alpha)$.*
(ii) *If $\alpha, \beta \in \mathcal{E}$ with $F(\alpha) = F(\beta)$, then $Z(\alpha) \cap Z(\beta) \neq \emptyset$ if and only if $\alpha = \beta$.*

Proof. For item (i), observe that, if $x \in Z(\alpha\delta)$, then $x(F(\alpha)F(\delta)) = \alpha\delta$; thus, by unique factorization $x(F(\alpha)) = \alpha$.

For item (ii), if $x \in Z(\alpha) \cap Z(\beta)$, then $\alpha = x(F(\alpha)) = x(F(\beta)) = \beta$. $\square$

Proposition 4.2. *Let $F : \mathcal{E} \to \mathcal{B}$ be a row finite Kumjian-Pask fibration and $\alpha, \beta \in \mathcal{E}$ with $Z(\alpha) \cap Z(\beta) \neq \emptyset$. Then, there exist $c, d \in \mathcal{B}$ such that*

$$F(\alpha)c = F(\beta)d,$$
$$I = \{\alpha\gamma : s(\alpha) = r(\gamma), F(\gamma) = c\} \cap \{\beta\delta : s(\beta) = r(\delta), F(\delta) = d\} \neq \emptyset$$

and

$$Z(\alpha) \cap Z(\beta) = \bigcup_{\mu \in I} Z(\mu).$$

Further, the sets $Z(\mu)$ for $\mu \in I$ are mutually disjoint.

Proof. If $x \in Z(\alpha) \cap Z(\beta)$, then $r(\alpha) = r(x) = r(\beta)$. Thus, $F(\alpha), F(\beta)$ is a cospan in $\mathcal{B}$ and, since $\mathcal{B}$ is right Ore, there exist $c, d \in \mathcal{B}$ with $F(\alpha)c = F(\beta)d$. For $x \in Z(\alpha) \cap Z(\beta)$, $\alpha x_2(F(\alpha), c) = x(F(\alpha)c) = x(F(\beta)d) = \beta x_2(F(\beta), d)$, we then have $\alpha x_2(F(\alpha), c) \in I$ and $x \in Z(\alpha x_2(F(\alpha), c))$. Thus, $Z(\alpha) \cap Z(\beta) \subset \bigcup_{\mu \in I} Z(\mu)$. Now, the reverse inclusion follows from Lemma 4.1. Finally, Lemma 4.1 also gives that $Z(\mu)$ is mutually disjoint. □

Corollary 4.3. *Let $F : \mathcal{E} \to \mathcal{B}$ be a Kumjian-Pask fibration. Then the set $\{Z(\alpha) : \alpha \in \mathcal{E}\}$ is a basis for a topology on F^∞. Furthermore, if $\mathcal{B}$ is countable and F is row-finite, then F^∞ is second countable. If each slice category $\mathcal{B}/B$ is countable and F is row-finite, then each $Z(\beta)$ and each $Z(X)$ is second countable.*

Proof. Since $F^\infty = \bigcup_{X \in \mathrm{Obj}(\mathcal{E})} Z(X)$, Proposition 4.2 shows that $\{Z(\alpha) : \alpha \in \mathcal{E}\}$ is a basis. For the second statement, if $\mathcal{B}$ is countable and $F : \mathcal{E} \to \mathcal{B}$ is row-finite, then $\mathcal{E}$ and $\{Z(\alpha) : \alpha \in \mathcal{E}\}$ are countable. □

Remark 4.4. $\mathcal{B}$ countable implies that $\mathcal{B}/B$ is countable for all objects B in $\mathcal{B}$. The latter can happen without $\mathcal{B}$ being countable, for instance, if $\mathcal{B}$ is the uncountable disjoint union of countable categories.

Henceforth, we regard F^∞ as a topological space with topology induced by the basis $\{Z(\beta) : \beta \in \mathcal{E}\}$, and the subsets $Z(\beta)$ and $Z(X)$ for morphisms β and object X of $\mathcal{E}$ as spaces in the subspace topology.

The next lemma will be quite useful in what follows.

Lemma 4.5. *Let $F : \mathcal{E} \to \mathcal{B}$ be a row-finite* KPf*, X an object in $\mathcal{E}$, $x, y \in Z(X)$ and $(a, b) \in \mathcal{B}/F(X)$. The following are equivalent*:

(i) *$x(a, b) = y(a, b)$ as morphisms in $\mathcal{E}/X$;*

(ii) *$x(a) = y(a)$ and $x(ab) = y(ab)$ as objects in $\mathcal{E}/X$;*
(iii) *$x(ab) = y(ab)$ as objects in $\mathcal{E}/X$.*

Consequently, if x and y agree on all objects of $\mathcal{B}/F(X)$, then $x = y$.

Proof. For the first statement, note that (i) implies (ii), since equality of morphisms implies equality of their sources and targets, and (ii) implies (iii), trivially. To see that (iii) implies (i), recall that $x_1(a,b)x_2(a,b) = x(ab) = y(ab) = y_1(a,b)y_2(a,b)$. Thus, by unique factorization, $x_i(a,b) = y_i(a,b)$ for $i = 1, 2$, and therefore, $x(a,b) = y(a,b)$.

The second statement follows from the first, since agreement on the source of a morphism implies agreement on the morphism. □

Proposition 4.6. *Let $F : \mathcal{E} \to \mathcal{B}$ be a row-finite KPf. Then F^∞ is Hausdorff.*

Proof. Suppose that $x \in Z(X)$ and $y \in Z(Y)$ with $x \neq y$. If $X \neq Y$, then $Z(X) \cap Z(Y) = \emptyset$, and we are done. If $X = Y$, then by Lemma 4.5, there is a morphism (object in the slice category) $b : B \to F(X)$ such that $x(b) \neq y(b)$. But, then $x \in Z(x(b))$ and $y \in Z(y(b))$ give open neighborhoods that are disjoint. □

Theorem 4.7. *If $F : \mathcal{E} \to \mathcal{B}$ is a row-finite, strong surjective* KPf *and, moreover, every slice category $\mathcal{B}/B$ is countable, then for each $\beta \in \mathcal{E}$,*

(i) *$Z(\beta)$ is compact; and*
(ii) *F^∞ is totally disconnected.*

Proof.

(i) implies (ii), because F^∞ is Hausdorff, and thus, $Z(\beta)$ compact implies that it is closed. Since the $\mathcal{B}/B$ are each countable, each $Z(\beta)$ is second countable; thus, it suffices to show that each $Z(\beta)$ is sequentially compact.

Fix a morphism $\beta : Y \to X$ in $\mathcal{E}$, and order the objects of $\mathcal{B}/F(X)$ so as to give a sequence $\{b_i : B_i \to F(X)\}_{i=0}^\infty$ with $b_0 = F(\beta)$.

Given a sequence $\{x_n\}_{n=0}^\infty$ in $Z(\beta)$, we will construct a convergent subsequence by a diagonalization argument. Let $\{x_{n,0}\}_{n=0}^\infty :=$

$\{x_n\}_{n=0}^{\infty}$, and observe that, for all n,

$$x_{n,0}(b_0) = \beta,$$

that is, $x_{n,0}(b_0)$ is constant, and thus, a fortiori, eventually constant.

Now, suppose that we have constructed sequences $\{x_{n,i}\}_{n=0}^{\infty}$ for $i = 0, \dots, k$, such that

(i) $\{x_{n,j+1}\}_{n=0}^{\infty}$ is a subsequence of $\{x_{n,j}\}_{n=0}^{\infty}$; and
(ii) $x_{n,i}(b_j)$ is constant for all $j \leq i$.

Note that the nesting of subsequences ensures that $j \leq i < h \leq k$ implies $x_{n,i}(b_j) = x_{n,h}(b_j)$.

We construct a subsequence $x_{n,k+1}$ of $x_{n,k}$ satisfying item (ii). Consider the list of objects in $\mathcal{E}/X$ given by $\{x_{n,k}(b_{k+1})\}_{n=0}^{\infty}$. Each element of this list is a preimage of b_{k+1}, but, by row-finiteness, there are only finitely many distinct preimages. Thus, by the pigeonhole principle, $x_{N,k}(b_{k+1})$ occurs infinitely often. Let $\{x_{n,k+1}\}_{n=0}^{\infty}$ be the subsequence of all $x_{n,k}$s for which $x_{n,k}(b_{k+1}) = x_{N,k}(b_{k+1})$. Now, since it is a subsequence of $\{x_{n,k}\}_{n=0}^{\infty}$, our new sequence is constant on the b_j for $j \leq k$, and, by construction, it is constant on b_{k+1}. Thus, $x_{n,k+1}$ satisfies item (ii), as desired.

We claim the diagonal sequence $\{x_{n,n}\}$, necessarily a subsequence of our original sequence $\{x_n\}_{n=0}^{\infty}$, is convergent. By construction, for each $b : B \to F(X)$, $x_{n,n}(b)$ is eventually constant, and, by Lemma 4.5, for any morphism (c, d) in $\mathcal{B}/F(X)$, $x_{n,n}(c, d)$ is eventually constant.

We define the limiting infinite path x by $x(\omega) = \lim_{n\to\infty} x_{n,n}(\omega)$, whether ω is an object or morphism of $\mathcal{B}/F(X)$, and the limit exists by eventual constancy. In order to see that x is a functor, let $(a, b), (ab, c) \in \mathcal{B}/F(X)$. Then n exists sufficiently large so that

$$x(a, b)x(ab, c) = x_n(a, b)x_n(ab, c) = x_n(a, bc) = x(a, bc),$$

that is, x is a functor. Likewise, that x is a section of F follows from its agreement with $x_{n,n}$ for n sufficiently large on each object and morphism.

It remains only to show that $x = \lim_{n\to\infty} x_{n,n}$ in the topology on F^{∞}. However, the basic opens containing x are given by $\{Z(x(b_i)) : i = 0, 1, 2, \dots\}$, and, by construction, $x_{n,n} \in Z(x(b_i))$ for all $n \geq i$. Thus, the proof is complete. $\square$

Now, having equipped F^∞ with a topology, we consider the continuity of our two families of maps res_μ and ind_μ.

Proposition 4.8. *Let $F : \mathcal{E} \to \mathcal{B}$ be a row-finite* KPf *and $\mu \in \mathcal{E}$. Then,* $\mathrm{res}_\mu : Z(\mu) \to Z(s(\mu))$ *and* $\mathrm{ind}_\mu : Z(s(\mu)) \to Z(\mu)$ *are continuous.*

Proof. In order to see that res_μ is continuous, it suffices to show that $\mathrm{res}_\mu^{-1}(Z(\nu) \cap Z(s(\mu)))$. The inverse image of generic basic open sets for the subspace topology, is open.

However, $Z(\nu) \cap Z(s(\mu)) = \emptyset$, unless $r(\nu) = s(\mu)$, in which case, $Z(\nu) \cap Z(s(\mu)) = Z(\nu)$. Now, $\mathrm{res}_\mu^{-1}(Z(\nu)) = \mathrm{ind}_\mu(Z(\nu)) = Z(\mu\nu)$ by Proposition 3.17, and is thus open.

In order to see that ind_μ is continuous, it again suffices to show that $\mathrm{ind}_\mu^{-1}(Z(\mu) \cap Z(\gamma))$ is open for any $\gamma \in \mathcal{E}$. Now, $Z(\mu) \cap Z(\gamma)$ is either empty or a disjoint union of sets of the form $Z(\mu\nu)$ by Proposition 4.2. It suffices to show that $\mathrm{ind}_\mu^{-1}(Z(\mu\nu))$ is open. Thus, by Proposition 3.17, $\mathrm{ind}_\mu^{-1}(Z(\mu\nu)) = \mathrm{res}_\mu(Z(\mu\nu)) = Z(\nu)$ is open, as desired. $\square$

With this result, we begin our construction of the groupoid. Let

$$\mathcal{G}_F := \{(\mu, \nu, x) \in \mathcal{E} \times \mathcal{E} \times F^\infty : x \in Z(\nu) \text{ and } s(\mu) = s(\nu)\}.$$

We think of (μ, ν, x) as a map taking x to $\mathrm{ind}_\mu(\mathrm{res}_\nu)(x)$. We define a relation on $\mathcal{G}_F$ by $(\mu, \nu, x) \sim (\mu', \nu', x')$, if

(i) $x = x'$;
(ii) there exists a $\lambda \in \mathcal{E}$ with $x \in Z(\lambda) \subset Z(\nu) \cap Z(\nu')$ and $\mathrm{ind}_\mu \circ \mathrm{res}_\nu |_{Z(\lambda)} = \mathrm{ind}_{\mu'} \circ \mathrm{res}_{\nu'} |_{Z(\lambda)}$;
(iii) there exist $a, b \in \mathcal{B}$ with $F(\mu)a = F(\mu')b$ and $F(\nu)a = F(\nu')b$.

A quick check shows that $\sim$ is reflexive, symmetric and transitive, thus, it is an equivalence relation. Define $G_F = \mathcal{G}_F/\sim$, and denote the image of (μ, ν, x) in G_F by $[\mu, \nu, x]$.

We define the composition of morphisms in G_F. As we are thinking of $[\mu, \nu, x]$ as a morphism taking x to $\mathrm{ind}_\mu(\mathrm{res}_\nu)(x)$, we would like to compose $[\mu, \nu, x], [\sigma, \tau, y] \in G_F \times G_F$ if $x = \mathrm{ind}_\sigma \circ \mathrm{res}_\tau(y)$, and the map should be $y \to \mathrm{ind}_\mu \circ \mathrm{res}_\nu \circ \mathrm{ind}_\sigma \circ \mathrm{res}_\tau(y)$. However, to define composition in this way, we need to find $(\xi, \zeta) \in \mathcal{E} \times \mathcal{E}$ with the map

$\operatorname{ind}_\xi \circ \operatorname{res}_\zeta = \operatorname{ind}_\mu \circ \operatorname{res}_\nu \circ \operatorname{ind}_\sigma \circ \operatorname{res}_\tau$ on a neighborhood of y and then define $[\mu, \nu, x] \cdot [\sigma, \tau, y] = [\xi, \zeta, y]$.

Consider $[\mu, \nu, x], [\sigma, \tau, y] \in G_F \times G_F$ with $x = \operatorname{ind}_\sigma \circ \operatorname{res}_\tau(y)$. We construct a candidate for $[\xi, \zeta, y]$. Since $x = \operatorname{ind}_\sigma \circ \operatorname{res}_\tau(y)$, $x(F(\nu)) = \nu$ and $x(F(\sigma)) = \sigma$. Thus, $F(\nu), F(\sigma)$ is a cospan in the right Ore category $\mathcal{B}$, and there exist $a, b \in \mathcal{B}$ with $F(\nu)a = F(\sigma)b$. Therefore,

$$\nu x_2(F(\nu), a) = x(F(\nu)a) = x(F(\sigma)b) = \sigma x_2(F(\sigma), b).$$

Take $\gamma = x_2(F(\nu, a))$ and $\eta = x_2(F(\sigma), b)$ so that $\nu\gamma = \sigma\eta$.

Note that, on $Z(\tau\eta)$, we have

$$\begin{aligned}\operatorname{ind}_\mu \circ \operatorname{res}_\nu \circ \operatorname{ind}_\sigma \circ \operatorname{res}_\tau &= \operatorname{ind}_\mu \circ \operatorname{ind}_\gamma \circ \operatorname{res}_\gamma \circ \operatorname{res}_\nu \circ \operatorname{ind}_\sigma \circ \operatorname{ind}_\eta \circ \operatorname{res}_\eta \circ \operatorname{res}_\tau \\ &= \operatorname{ind}_{\mu\gamma} \circ \operatorname{res}_{\nu\gamma} \circ \operatorname{ind}_{\sigma\eta} \circ \operatorname{res}_{\tau\eta} \\ &= \operatorname{ind}_{\mu\gamma} \circ \operatorname{res}_{\tau\eta}.\end{aligned}$$

Lemma 4.9. *The formula*

$$[\mu, \nu, \operatorname{ind}_\sigma \circ \operatorname{res}_\tau(y)][\sigma, \tau, y] := [\mu\gamma, \tau\eta, y],$$

for η and γ, as in the discussion above, with $x = \operatorname{ind}_\sigma \circ \operatorname{res}_\tau(y)$, gives a well-defined composition in G_F.

Proof. Suppose that $[\sigma, \tau, y] = [\sigma', \tau', y]$ and $[\mu, \nu, \operatorname{ind}_\sigma \circ \operatorname{res}_\tau(y)] = [\mu', \nu', \operatorname{ind}_{\sigma'} \circ \operatorname{res}_{\tau'}(y)]$. Choose γ' and η' such that $\nu'\gamma' = \sigma'\eta'$. Then,

$$[\mu, \nu, \operatorname{ind}_\sigma \circ \operatorname{res}_\tau(y)][\sigma, \tau, y] = [\mu\gamma, \tau\eta, y]$$

and

$$[\mu', \nu', \operatorname{ind}_{\sigma'} \circ \operatorname{res}_{\tau'}(y)][\sigma', \tau', y] = [\mu'\gamma', \tau'\eta', y].$$

We must show that $[\mu\gamma, \tau\eta, y] = [\mu'\gamma', \tau'\eta', y]$. We have $y = y$ and, since composition of germs is well defined, a $\lambda \in \mathcal{E}$ exists so that, on $Z(\lambda)$,

$$\begin{aligned}\operatorname{ind}_{\mu\gamma} \circ \operatorname{res}_{\tau\eta} &= \operatorname{ind}_\mu \circ \operatorname{res}_\nu \circ \operatorname{ind}_\sigma \circ \operatorname{res}_\tau \\ &= \operatorname{ind}_{\mu'} \circ \operatorname{res}_{\nu'} \circ \operatorname{ind}_{\sigma'} \circ \operatorname{res}_{\tau'} \\ &= \operatorname{ind}_{\mu'\gamma'} \circ \operatorname{res}_{\tau'\eta'}.\end{aligned}$$

It remains to show that R and R' exist such that $F(\mu\gamma)R = F(\mu'\gamma')R'$ and $F(\tau\eta)R = F(\tau'\eta')R'$. Let

$$\begin{aligned} m &= F(\mu), & m' &= F(\mu') \\ n &= F(\nu), & n' &= F(\nu') \\ w &= F(\sigma), & w' &= F(\sigma') \\ t &= F(\tau), & t' &= F(\tau') \\ g &= F(\gamma), & g' &= F(\gamma') \\ h &= F(\eta), & h' &= F(\eta'), \end{aligned}$$

so that

$$ng = wh, \tag{4.1}$$

$$n'g' = w'h'. \tag{4.2}$$

Since $[\mu, \nu, \mathrm{ind}_\sigma \circ \mathrm{res}_\tau(y)] = [\mu', \nu', \mathrm{ind}_{\sigma'} \circ \mathrm{res}_{\tau'}(y)]$ and $[\sigma, \tau, y] = [\sigma', \tau', y]$, there exist $p, p', q, q' \in \mathcal{B}$ such that

$$mp = m'p', \qquad np = n'p', \tag{4.3}$$

$$wq = w'q', \qquad tq = t'q'. \tag{4.4}$$

Since $(p, g), (p', g'), (q, b), (q', b')$ are cospans, by the right Ore condition,

$$a_1,\ a_2,\ b_1,\ b_2,\ a_1'a_2',\ b_1',\ b_2' \in \mathcal{B}$$

exist such that

$$pa_1 = gb_1, \qquad qa_2 = hb_2, \tag{4.5}$$

$$p'a_1' = g'b_1', \qquad q'a_2' = h'b_2'. \tag{4.6}$$

Since $s(p) = s(p')$ and $s(q) = s(q')$, we have that (a_1, a_1') and (a_2, a_2') are cospans. Using the right Ore condition again, $c_1, c_1', d_1, d_1' \in \mathcal{B}$ exist such that

$$a_1c_1 = a_1'c_1', \qquad a_2d_1 = a_2'd_1'. \tag{4.7}$$

Since $r(b_1) = s(g) = s(h) = r(b_2)$ and $r(b_1') = s(g') = s(h') = r(b_2')$, we have that (b_1c_1, b_2d_1) and $(b_1'c_1', b_2d_1')$ are cospans. Using the right Ore condition, $c_2, c_2', d_2, d_2' \in \mathcal{B}$ exist such that

$$b_1c_1c_2 = b_2d_1d_2, \qquad b_1'c_1'c_2' = b_2'd_1'd_2'. \tag{4.8}$$

Define

$$M = ngb_1c_1 = npa_1c_1 = n'p'a'_1c'_1 = n'g'b'_1c'_1,$$

and

$$N = whb_2d_1 = wqa_2d_1 = w'q'a'_2d'_1 = w'h'b'_2d'_1.$$

Now, M, N is a cospan and $Mc_2 = Nd_2$ and $Mc'_2 = Nd'_2$. Thus, by strong right Ore, $k, \ell \in \mathcal{B}$ exist with

$$c_2k = c'_2\ell \qquad d_2k = d'_2\ell. \tag{4.9}$$

Take $R = b_1c_1c_2k$ and $R' = b'_1c'_1c'_2\ell$. Then,

$$\begin{aligned} mgR &= mgb_1c_1c_2k = mpa_1c_1c_2k && \text{by (4.5)} \\ &= mpa_1c_1c'_2\ell && \text{by (4.9)} \\ &= mpa'_1c'_1c'_2\ell && \text{by (4.7)} \\ &= m'p'a'_1c'_1c'_2\ell && \text{by (4.3)} \\ &= m'g'b'_1c'_1c'_2\ell && \text{by (4.6)} \\ &= m'g'R', \end{aligned}$$

and

$$\begin{aligned} thR &= thb_1c_1c_2k = thb_2d_1d_2k && \text{by (4.8)} \\ &= tqa_2d_1d_2k && \text{by (4.5)} \\ &= tqa_2d_1d'_2\ell && \text{by (4.9)} \\ &= tqa'_2d'_1d'_2\ell && \text{by (4.7)} \\ &= t'q'a'_2d'_1d'_2\ell && \text{by (4.4)} \\ &= t'h'b'_2d'_1d'_2\ell && \text{by (4.6)} \\ &= t'h'b'_1c'_1c'_2\ell && \text{by (4.8)} \\ &= t'h'R'. \end{aligned}$$

We obtain that multiplication is well defined on G_F. □

Lemma 4.10. *Under the multiplication defined in Lemma 4.9, G_F is a groupoid.*

Proof. Note that, for $[\mu, \nu, x] \in G_F$,

$$[r(\mu), r(\mu), \mathrm{ind}_\mu(\mathrm{res}_\nu(x))][\mu, \nu, x] = [r(\mu)\mu, \nu s(\mu), x] = [\mu, \nu, x]$$

and

$$[\mu, \nu, x][r(\nu), r(\nu), x] = [\mu s(\nu), r(\nu)\nu, x] = [\mu, \nu, x],$$

that is, elements of the form $[r(x), r(x), x]$ act as units in G_F, and the map $x \mapsto [r(x), r(x), x]$ identifies F^∞ with the unit space of G_F. Also, $[r(x), r(x), x] = [x(a), x(a), x]$ for any object a in $\mathcal{B}/F(r(x))$.

Now, for $[\mu, \nu, x] \in G_F$, consider $[\nu, \mu, \mathrm{ind}_\mu(\mathrm{res}_\nu(x))]$. Then

$$[\mu, \nu, x][\nu, \mu, \mathrm{ind}_\mu(\mathrm{res}_\nu(x))] = [\mu, \mu, \mathrm{ind}_\mu(\mathrm{res}_\nu(x))]$$

and

$$[\nu, \mu, \mathrm{ind}_\mu(\mathrm{res}_\nu(x))][\mu, \nu, x] = [\nu, \nu, x],$$

that is, the inverse of $[\mu, \nu, x]$ is $[\nu, \mu, \mathrm{ind}_\mu(\mathrm{res}_\nu(x))]$. Thus, G_F is a groupoid. $\square$

Remark 4.11. Let $F : \mathcal{E} \to \mathcal{B}$ be a row finite Kumjian-Pask fibration where $\mathcal{B}$ (and hence, $\mathcal{E}$) is left and right cancellative. Suppose that $\mathcal{E}$ has no inverses. Then $\mathcal{E}$ is a finitely aligned category of paths in the sense of **[15,** Definition 3.1]. Let X be an object in $\mathcal{E}$. Define A_X to be the set of finite disjoint unions of sets of the form

$$\left(\bigcup_{j=1}^{N} \alpha_j \mathcal{E}\right) - \left(\bigcup_{k=1}^{M} \beta_k \mathcal{E}\right)$$

and Ω_X the set of ultra filters on A_X, that is, $\omega \in \Omega_X$ is a subset of the power set on A_X which does not contain the empty set with the property that, for every $E \in A_X$, either $E \in \omega$ or there is an $F \in \omega$ such that $F \cap E = \emptyset$. Take $\Omega = \cup_{X \in \mathrm{Obj}(\mathcal{E})} \Omega_X$. For $\beta \in X\mathcal{E}$, we can define a set $\widehat{\beta\mathcal{E}} = \{\omega \in \Omega_X : \beta\mathcal{E} \in \omega\}$. The set $\{\widehat{\beta\mathcal{E}}\}_{\beta \in \mathcal{E}}$ forms a subbasis for a topology on Ω. For $\alpha \in \mathcal{E}$, there is a map $\widetilde{\alpha} : A_{s(\alpha)} \to A_{r(\alpha)}$ characterized by $\widetilde{\alpha}(\beta\mathcal{E}) = (\alpha\beta)\mathcal{E}$; $\widetilde{\alpha}$ then induces a continuous map $\widehat{\alpha} : \Omega_{s(\alpha)} \to \Omega_{r(\alpha)}$. In **[15]**, Spielberg defines a groupoid $G(\mathcal{E})$ to be the groupoid generated by the germs of the maps $\widetilde{\alpha}$ for all $\alpha \in \mathcal{E}$. The unit space of $G(\mathcal{E})$ can then be identified with Ω.

We can view $G_F^{(0)} = F^\infty \subset \Omega = G(\mathcal{E})^{(0)}$ by taking $x \in F^\infty$ to

$$\omega_x = \{E \in A_{r(x)} : x(a)\mathcal{E} \subset E \text{ for some } a \in \mathcal{B}, r(a) = F(r(x))\}.$$

In order to see that ω_x is an ultrafilter on $A_{r(x)}$, we need to see that, if $E \in A_{r(x)}$, then either there exists an a with $x(a)\mathcal{E} \subset E$ or an $x(a)\mathcal{E} \cap E = \emptyset$. Suppose that $x(b)\mathcal{E} \cap E \neq \emptyset$ for all $b \in \mathcal{B}$ with $r(b) = F(r(x))$. Now,

$$E = \bigcup_{i=1}^{Q} \left(\left(\bigcup_{j=1}^{N_i} \alpha_{i,j}\mathcal{E} \right) - \left(\bigcup_{k=1}^{M_i} \beta_{i,k}\mathcal{E} \right) \right).$$

Since $\mathcal{B}$ is right Ore, $a \in \mathcal{B}$ exists that extend $F(\alpha_{i,j})$ and $F(\beta_{i,k})$ for all choices of i, j and k. We claim that $x(a)\mathcal{E} \subset E$. We know that $x(a)\mathcal{E} \cap E \neq \emptyset$. Thus, i_0 and j_0 exist such that

$$x(a)\mathcal{E} \cap \left(\alpha_{i_0,j_0}\mathcal{E} - \left(\bigcup_{k=1}^{M_{i_0}} \beta_{i_0,k}\mathcal{E} \right) \right) \neq \emptyset.$$

Let

$$\gamma \in x(a)\mathcal{E} \cap \left(\alpha_{i_0,j_0}\mathcal{E} - \left(\bigcup_{k=1}^{M_{i_0}} \beta_{i_0,k}\mathcal{E} \right) \right),$$

so that $\gamma = x(a)\gamma' = \alpha_{i_0,j_0}\gamma''$. Since a extends $F(\alpha_{i_0,j_0})$, by unique factorization, we obtain $x(a) = \alpha_{i_0,j_0}\eta$ for some η. Thus, $x(a)\mathcal{E} \subset \alpha_{i_0,j_0}\mathcal{E}$. Now, $\gamma \notin \beta_{i_0,k}\mathcal{E}$ for all k. Since a extends $F(\beta_{i_0,k})$ for all k, we must have $x(F(\beta_{i_0,k})) \neq \beta_{i_0,k}$ for all k so that $x(a)\mathcal{E} \cap \beta_{i_0,k}\mathcal{E} = \emptyset$ for all k, that is, $x(a)\mathcal{E} \subset \alpha_{i_0,j_0}\mathcal{E} - (\bigcup_{k=1}^{M_{i_0}} \beta_{i_0,k}\mathcal{E}) \subset E$, as desired. Thus, ω_x is an ultra filter, and we can view $F^\infty \subset \Omega$ as claimed.

We now turn to the problem of defining a topology on G_F. For $\mu, \nu \in \mathcal{E}$, consider the set

$$Z(\mu, \nu) := \{[\alpha, \beta, x] \in G_F : [\alpha, \beta, x] = [\mu, \nu, x]\},$$

that is, $[\alpha, \beta, x] \in Z(\mu, \nu)$ if $x \in Z(\nu)$ and $(\alpha, \beta, x) \sim (\mu, \nu, x)$.

Given $(\alpha, \beta), (\mu, \nu) \in \mathcal{E} \times \mathcal{E}$, note that, if an a exists such that $F(\alpha)a = F(\mu)$ and $F(\beta)a = F(\nu)$, then $Z(\mu, \nu) \subset Z(\alpha, \beta)$ if there exists γ with $F(\gamma) = a$ and $(\alpha\gamma, \beta\gamma) = (\mu, \nu)$, and $Z(\mu, \nu) \cap Z(\alpha, \beta) = \emptyset$, otherwise.

Indeed, if $Z(\mu,\nu)\cap Z(\alpha,\beta)\neq\emptyset$, then $[\mu,\nu,x]\in Z(\alpha,\beta)$ exists. Therefore, $x(F(\nu))=x(F(\beta)a)=x(F(\beta))x_2(F(\beta),a)$ and, by unique factorization, we have $x(F(\beta))=\beta$ and $\nu=\beta\gamma$ with $\gamma=x_2(F(\beta),a)$. Now,

$$\mu=\mathrm{ind}_\alpha\circ\mathrm{res}_\beta(x)(F(\alpha)a)=\alpha\,\mathrm{res}_\beta(x)(a)=\alpha\gamma.$$

Proposition 4.12. *Suppose that $Z(\alpha,\beta)\cap Z(\sigma,\tau)\neq\emptyset$. Then $a,b\in\mathcal{B}$ and $I=\{(\alpha\gamma,\beta\gamma)\in\mathcal{E}\times\mathcal{E}:F(\gamma)=a\}\cap\{(\sigma\eta,\tau\eta)\in\mathcal{E}\times\mathcal{E}:F(\eta)=b\}\neq\emptyset$ exist such that*

$$Z(\alpha,\beta)\cap Z(\sigma,\tau)=\bigcup_{(\mu,\nu)\in I}Z(\mu,\nu),$$

and the union on the right hand side is disjoint.

Proof. Since $Z(\alpha,\beta)\cap Z(\sigma,\tau)\neq\emptyset$, $x\in F^\infty$ exists with $[\alpha,\beta,x]=[\sigma,\tau,x]$. This occurs if and only if $\mathrm{ind}_\alpha\circ\mathrm{res}_\beta=\mathrm{ind}_\sigma\circ\mathrm{res}_\tau$ on some neighborhood of x and $a,b\in\mathcal{B}$ exist such that

$$F(\alpha)a=F(\sigma)b\quad\text{and}\quad F(\beta)a=F(\tau)b.$$

Take I for this $a,b\in\mathcal{B}$. By the definition of I, if $(\mu,\nu)\in I$, then $Z(\mu,\nu)\subset Z(\alpha,\beta)\cap Z(\sigma,\tau)$. Thus,

$$\bigcup_{(\mu,\nu)\in I}Z(\mu,\nu)\subset Z(\alpha,\beta)\cap Z(\sigma,\tau).$$

Now, we assume $[\alpha,\beta,x]=[\sigma,\tau,x]\in Z(\alpha,\beta)\cap Z(\sigma,\tau)\neq\emptyset$. Take $\gamma=x_2(F(\beta),a)$ and $\eta=x_2(F(\tau),b)$. By definition, we then have $\beta\gamma=x(F(\beta)a)=x(F(\tau)b)=\tau\eta$, and since $\mathrm{ind}_\alpha\circ\mathrm{res}_\beta=\mathrm{ind}_\sigma\circ\mathrm{res}_\tau$ on a neighborhood of x, we also have

$$\begin{aligned}\alpha\gamma&=\alpha x_2(F(\beta),a)=\alpha\,\mathrm{res}_\beta(x)(a)\\&=\mathrm{ind}_\alpha\circ\mathrm{res}_\beta(x)(F(\alpha)a)\\&=\mathrm{ind}_\sigma\circ\mathrm{res}_\tau(x)(F(\sigma)b)\\&=\sigma\eta,\end{aligned}$$

that is, $(\alpha\gamma,\beta\gamma)=(\sigma\eta,\tau\eta)\in I$, and thus, $[\alpha,\beta,x]\in\bigcup_{(\mu,\nu)\in I}Z(\mu,\nu)$. Thus,

$$Z(\alpha,\beta)\cap Z(\sigma,\tau)=\bigcup_{(\mu,\nu)\in I}Z(\mu,\nu),$$

as desired. Now, for $(\mu, \nu), (\mu', \nu') \in I$, $F(\mu) = F(\mu')$ and $F(\nu) = F(\nu')$ so that $Z(\mu, \nu) \cap Z(\mu', \nu') = \emptyset$ if $(\mu, \nu) \neq (\mu', \nu')$ by unique factorization. $\square$

Corollary 4.13. *The set $\{Z(\mu, \nu)\}$ is a basis for a topology on G_F.*

Lemma 4.14. *With respect to this topology, composition is continuous and inversion is a homeomorphism, that is, G_F is a topological groupoid.*

Proof. In order to see that inversion is continuous, note that $Z(\mu, \nu)^{-1} = Z(\nu, \mu)$; since inversion is an involution, it is a homeomorphism.

In order to see that composition is continuous, let $Z(\mu, \nu)$ be a basic open set and $[\alpha, \beta, \mathrm{ind}_\sigma(\mathrm{res}_\tau(x))][\sigma, \tau, x] = [\alpha\gamma, \tau\eta, x] \in Z(\mu, \nu)$; in particular, $\beta\gamma = \sigma\eta$. We need to find neighborhoods $Z(\alpha', \beta')$ and $Z(\sigma', \tau')$ of $[\alpha, \beta, \mathrm{ind}_\sigma(\mathrm{res}_\tau(x))]$ and $[\sigma, \tau, x]$, respectively, so that $Z(\alpha', \beta')Z(\sigma', \tau') \subset Z(\mu, \nu)$.

Since $[\alpha\gamma, \tau\eta, x] \in Z(\mu, \nu)$, $d, e \in \mathcal{B}$ exist such that

$$F(\alpha\gamma)\, d = F(\mu)e \quad \text{and} \quad F(\tau\eta)\, d = F(\nu)e.$$

Further, $k \in \mathcal{B}$ exists such that

$$\mathrm{ind}_\mu \circ \mathrm{res}_\nu |_{Z(x(F(\nu)ek))} = \mathrm{ind}_{\alpha\gamma} \circ \mathrm{res}_{\tau\eta} |_{Z(x(F(\tau\eta)dk))}.$$

Choose $\delta, \epsilon, \kappa \in \mathcal{E}$ such that

$$\nu\epsilon\kappa = x(F(\nu)ek) = x(F(\tau\eta)\, dk) = \tau\eta\delta\kappa$$

and $F(\delta) = d$, $F(\epsilon) = e$, $F(\kappa) = k$. Since $\beta\gamma = \sigma\eta$, $s(\gamma) = s(\eta) = r(\delta)$. Take

$$\alpha' = \alpha\gamma\delta\kappa, \quad \beta' = \beta\gamma\delta\kappa, \quad \sigma' = \sigma\eta\delta\kappa \quad \text{and} \quad \tau' = \tau\eta\delta\kappa.$$

By construction, $\beta' = \sigma'$. Thus, by definition,

$$Z(\alpha', \beta')Z(\sigma', \tau') = Z(\alpha', \tau') = Z(\alpha\gamma\delta\kappa, \tau\eta\delta\kappa) = Z(\alpha\gamma\delta\kappa, \nu\epsilon\kappa).$$

We want to show that $Z(\alpha', \beta')Z(\sigma', \tau') \subset Z(\mu, \nu)$, and, by the above computation, it suffices to show that $\alpha\gamma\delta\kappa = \mu\epsilon\kappa$. However,

$$\begin{aligned}\alpha\gamma\delta\kappa &= \mathrm{ind}_{\alpha\gamma} \circ \mathrm{res}_{\tau\eta}(x)(F(\alpha\gamma)\,dk)\\ &= \mathrm{ind}_{\mu} \circ \mathrm{res}_{\nu}(x)(F(\alpha\gamma)\,dk)\\ &= \mathrm{ind}_{\mu} \circ \mathrm{res}_{\nu}(x)(F(\mu)ek) = \mu\epsilon\kappa,\end{aligned}$$

as desired. □

Lemma 4.15. *For μ, $\nu \in \mathcal{E}$. The maps $r|_{Z(\mu,\nu)} : Z(\mu, \nu) \to Z(\mu)$ and $s|_{Z(\mu,\nu)} : Z(\mu, \nu) \to Z(\nu)$ are homeomorphisms, and G_F is étale.*

Proof. We begin by showing that $s|_{Z(\mu,\nu)} : Z(\mu, \nu) \to Z(\nu)$ is a homeomorphism. First, note that $s|_{Z(\mu,\nu)}$ is injective. Now, since $Z(\alpha, \beta)$ forms a basis for G_F (and therefore, for $Z(\mu, \nu)$) and $Z(\beta)$ forms a basis for F^∞ (and hence, for $Z(\nu)$) it suffices to show that $s(Z(\alpha, \beta)) = Z(\beta)$. Now, if $x \in Z(\beta)$, then $[\alpha, \beta, x] \in Z(\alpha, \beta)$, and thus, $Z(\beta) \subset s(Z(\alpha, \beta))$. By definition, $s(Z(\alpha, \beta)) \subset Z(\beta)$, so that $s(Z(\alpha, \beta)) = Z(\beta)$, as desired. Thus, $s|_{Z(\mu,\nu)} : Z(\mu, \nu) \to Z(\nu)$ is a homeomorphism. Since $r([\alpha, \beta, x]) = s([\alpha, \beta, x]^{-1})$, and inversion is a homeomorphism, we obtain that $r|_{Z(\mu,\nu)} : Z(\mu, \nu) \to Z(\mu)$ is a homeomorphism as well. By definition, G_F is now étale. □

Theorem 4.16. *Let $F : \mathcal{E} \to \mathcal{B}$ be a row finite Kumjian-Pask fibration with every slice category $\mathcal{B}/B$ countable. Then, G_F with the topology induced by the basis $\{Z(\mu, \nu) : (\mu, \nu)$ is a span in $\mathcal{E}\}$ is totally disconnected locally compact Hausdorff.*

Proof. Note that $Z(\mu, \nu)$ is homeomrphic to $Z(\nu)$ by Lemma 4.15, and $Z(\nu)$ is compact from Theorem 4.7. Thus, G_F is locally compact; it will also follow that G_F is totally disconnected once we show that G_F is Hausdorff. In order to show that G_F is Hausdorff, first note that, if two morphisms $[\alpha, \beta, x]$ and $[\gamma, \delta, x]$ both lie in a basic open $Z(\mu, \nu)$, then they are equal since both are $[\mu, \nu, x]$.

For brevity, having chosen to denote morphisms in $\mathcal{E}$ with Greek letters, we denote their image under F with the corresponding Latin letter (thus, for instance $F(\alpha)$ will be denoted a).

Now, suppose that $[\alpha, \beta, x] \neq [\gamma, \delta, y]$. First, suppose that $x \neq y$. If $r(\beta) \neq r(\delta)$ or $r(\alpha) \neq r(\gamma)$, it is immediate that $Z(\alpha, \beta)$ and $Z(\gamma, \delta)$ are disjoint. Thus, suppose that α, β and γ, δ are two spans between a pair of objects $\Gamma = r(\gamma) = r(\alpha)$ and $\Delta = r(\delta) = r(\beta)$ in $\mathcal{E}$. Since $x \neq y$, a morphism $f : X \to D$ in $\mathcal{B}$ exists for which $x(f) \neq y(f)$ are distinct lifts.

By the right Ore condition, we can complete the cospan b, d to a commutative square $bk = d\ell$ and the cospan $bk = d\ell, f$ to a commutative square $fn = bkm$ $(= d\ell m)$. By the unique factorization lifting property it follows that $x(f) \neq y(f)$ implies $x(bkm) = x(fn) \neq y(fn) = y(d\ell m)$. However, $[\alpha, \beta, x] = [\alpha x(km), x(fn), x] \in Z(\alpha x(km), x(fn))$ and $[\gamma, \delta, y] = [\gamma y(\ell m), y(fn), y] \in Z(\gamma y(\ell m), y(fn))$, and these two open sets are disjoint since all elements of the first have infinite paths lifting fn to $x(fn)$, while all elements of the second have infinite paths lifting fn to $y(fn)$.

Now, assume that $x = y$. If $Z(\alpha, \beta)$ and $Z(\gamma, \delta)$ are disjoint, we have separated the two morphisms with disjoint opens. If not, $[\epsilon, \zeta, z] \in Z(\alpha, \beta) \cap Z(\gamma, \delta)$ exists, in which case,

$$[\alpha, \beta, z] = [\gamma, \delta, z] = [\epsilon, \zeta, z].$$

From the equality of $[\alpha, \beta, z]$ and $[\gamma, \delta, z]$, we have a span in $\mathcal{B}$ from $s(a) = s(b)$ to $s(c) = s(d)$, m, n such that $am = cn$ and $bm = dn$.

The square $bm = dn$ admits a lift by x (as a span in the slice category $\mathcal{B}/r(m)$). Let $\omega = x(bm) = x(dm)$. Now, $(\mathrm{ind}_\alpha \circ \mathrm{res}_\beta(x))(am) = \alpha x_2(b, m)$, $(\mathrm{ind}_\gamma \circ \delta(x))(cn) = \gamma x_2(d, n)$ and $\alpha x_2(b, m) \neq \gamma x_2(d, n)$; otherwise, $\mathrm{ind}_\alpha \mathrm{res}_\beta = \mathrm{ind}_\gamma \mathrm{res}_\delta$ on $Z(\omega)$, and we obtain $[\alpha, \beta, x] = [\gamma, \delta, x]$. Note that $\alpha x_2(b, m)$ and $\gamma x_2(d, n)$ are then two unequal lifts of $am = cn$. It is clear that $[\alpha, \beta, x] = [\alpha x_2(b, m), \omega, x] \in Z(\alpha x_2(b, m), \omega)$ and $[\gamma, \delta, x] = [\gamma x_2(d, n), \omega, x] \in Z(\gamma x_2(d, n), \omega)$. However, $Z(\alpha x_2(b, m), \omega)$ and $Z(\gamma x_2(d, n), \omega)$ are disjoint since the image of any element of $Z(\omega)$ under any element of the first lifts $am = cn$ to $\alpha x_2(b, m)$, while the images of elements of $Z(\omega)$ under any element of the latter lifts $am = cn$ to $\gamma x_2(d, n)$ by Proposition 3.17. $\square$

Example 4.17. If $d : \Lambda \to \mathbb{N}^k$ is a k-graph, Kumjian and Pask [**7**] define

$$G_\Lambda := \{(x, m - n, y) \in d^\infty \times \mathbb{Z} \times d^\infty : \mathrm{res}_{x(m)}(x) = \mathrm{res}_{y(n)}(y)\},$$

and the topology on G_Λ is generated by

$$\mathcal{Z}(\alpha,\beta) := \{(\mathrm{ind}_\alpha z, d(\alpha)-d(\beta), \mathrm{ind}_\beta y) : r(y)=s(\beta)\}.$$

On the surface, this is different from G_d as defined above. However, the map

$$[\alpha,\beta,x] \longmapsto (\mathrm{ind}_\alpha \circ \mathrm{res}_\beta(x), d(\alpha)-d(\beta), x)$$

defines an isomorphism of G_d to G_Λ in **[7]**. This map is well defined since $[\alpha,\beta,x] = [\mu,\nu,x]$ gives that $\mathrm{ind}_\alpha \circ \mathrm{res}_\beta = \mathrm{ind}_\mu \circ \mathrm{res}_\nu$ on a neighborhood of x, and m,n exist such that $d(\alpha)+m = d(\mu)+n$ and $d(\beta)+m = d(\nu)+n$. In particular, $\mathrm{ind}_\alpha \circ \mathrm{res}_\beta(x) = \mathrm{ind}_\mu \circ \mathrm{res}_\nu(x)$ and $d(\alpha)-d(\beta) = d(\mu)-d(\nu)$, that is, $(\mathrm{ind}_\alpha \circ \mathrm{res}_\beta(x), d(\alpha)-d(\beta), x) = (\mathrm{ind}_\mu \circ \mathrm{res}_\nu(x), d(\mu)-d(\nu), x)$. A quick computation shows that this map is a homomorphism and takes the sets $Z(\alpha,\beta)$ to $\mathcal{Z}(\alpha,\beta)$ so that it is also a homeomorphism.

Example 4.18. Let G be a groupoid. Then G, in particular, is a small category with objects $G^{(0)}$. Further, $\mathrm{Id}_G : G \to G$ is a KPf. For each $x \in G^{(0)}$, there is exactly one section $x : G/\mathrm{Id}_G(x) \to G/x$. Thus, we can identify $G^{(0)}$ with Id_G^∞ as sets. For $\gamma \in G$, the map $\gamma \mapsto [\gamma, s(\gamma), s(\gamma)]$ is a bijective homomorphism of G to G_{Id_G}. Indeed, since $[\alpha,\beta,r(\beta)] = [\alpha\beta^{-1}, s(\alpha\beta^{-1}), s(\alpha\beta^{-1})]$, we obtain that the map is onto. In order to see that it is injective, note that, for $[\gamma, s(\gamma), s(\gamma)] = [\eta, s(\eta), s(\eta)]$, $\sigma,\tau \in G$ exist such that $\mathrm{Id}_G(\gamma)\sigma = \mathrm{Id}_G(\eta)\tau$ and $s(\gamma)\sigma = s(\eta)\tau$, that is, $\sigma=\tau$, and thus, $\gamma=\eta$. It is a homomorphism since

$$\begin{aligned}[]
[\gamma, s(\gamma), s(\gamma)][\eta, s(\eta), s(\eta)] &= [\gamma\eta, \eta, s(\gamma)][\eta, s(\eta), s(\eta)] \\
&= [\gamma\eta, s(\eta), s(\eta)].
\end{aligned}$$

Thus, G is isomorphic to G_{Id_G} as groupoids. However, this isomorphism is not necessarily of topological groupoids. Indeed, we know that G_{Id_G} is totally disconnected; therefore, it can be isomorphic to G only if G is totally disconnected. However, by digging a bit deeper, since each object in G corresponds to exactly one infinite path, we have $Z(\alpha) = \{r(\alpha)\}$ for all elements $\alpha \in G$. Therefore, the unit space in G_{Id_G} has the discrete topology, and therefore, G_{Id_G} also has the discrete topology. Thus, we have that $G \cong G_{\mathrm{Id}_G}$ as topological groupoids if and only if G is discrete, in particular, if H is a discrete group $H \cong G_{\mathrm{Id}_H}$.

5. Isomorphism of C^*-algebras. In this section, we show that $C^*(F) \cong C^*(G_F)$ if $\mathcal{B}$ is left cancellative.

Lemma 5.1. *Let $F : \mathcal{E} \to \mathcal{B}$ be a row finite Kumjian-Pask fibration. Suppose that all the morphisms in $\mathcal{B}$ are monic. Then, the map $s_\alpha \mapsto \mathbb{1}_{Z(\alpha,s(\alpha))}$ extends to a well-defined homomorphism*

$$\Upsilon : C^*(F) \longrightarrow C^*(G_F).$$

Proof. In order to prove the lemma, it suffices to show that $\{\mathbb{1}_{Z(\alpha,s(\alpha))}\}$ is a Cuntz-Krieger F-family in $C^*(G_F)$. First, note that

$$\begin{aligned}\mathbb{1}^*_{Z(\alpha,s(\alpha))}[\mu,\nu,x] &= \mathbb{1}_{Z(\alpha,s(\alpha))}[\nu,\mu,\operatorname{ind}_\mu\circ\operatorname{res}_\nu(x)] \\ &= \begin{cases} 1 & \text{if } [\nu,\mu,\operatorname{ind}_\mu\circ\operatorname{res}_\nu(x)] = [\alpha, s(\alpha)\operatorname{ind}_\mu\circ\operatorname{res}_\nu(x)] \\ 0 & \text{otherwise}\end{cases} \\ &= \begin{cases} 1 & \text{if } [\mu,\nu,x] = [s(\alpha),\alpha,x] \\ 0 & \text{otherwise}\end{cases} \\ &= \mathbb{1}_{Z(s(\alpha),\alpha)}[\mu,\nu,x].\end{aligned}$$

Given $a \in \mathcal{B}$, we want to show that

$$\sum_{F(\alpha)=a} \mathbb{1}_{Z(\alpha,s(\alpha))}\mathbb{1}^*_{Z(\alpha,s(\alpha))} = \mathbb{1}_{Z(r(\alpha),r(\alpha))}.$$

Since $Z(r(\alpha)) = Z(r(\alpha),r(\alpha))$ is the disjoint union $\bigcup_{F(\alpha)=a} Z(\alpha,\alpha)$, it suffices to show that $\mathbb{1}_{Z(\alpha,s(\alpha))}\mathbb{1}_{Z(s(\alpha),\alpha)} = \mathbb{1}_{Z(\alpha,\alpha)}$. We compute

$$\begin{aligned}&\mathbb{1}_{Z(\alpha,s(\alpha))} * \mathbb{1}_{Z(s(\alpha),\alpha)}[\xi,\zeta,x] \\ &\quad= \sum_{r([\mu,\nu,y])=r([\xi,\zeta,x])} \mathbb{1}_{Z(\alpha,s(\alpha))}([\mu,\nu,y])\mathbb{1}_{Z(s(\alpha),\alpha)}([\mu,\nu,y]^{-1}[\xi,\zeta,x]) \\ &\quad= \begin{cases} \mathbb{1}_{Z(s(\alpha),\alpha)}([\alpha,s(\alpha),\operatorname{res}_\alpha\circ\operatorname{ind}_\xi\circ\operatorname{res}_\zeta x]^{-1}[\xi,\zeta,x]), \\ \qquad \text{if } \operatorname{ind}_\xi\circ\operatorname{res}_\zeta x \in Z(\alpha), \\ 0 \quad \text{otherwise}\end{cases} \\ &\quad= \begin{cases} \mathbb{1}_{Z(s(\alpha),\alpha)}([s(\alpha),\alpha,\operatorname{ind}_\xi\circ\operatorname{res}_\zeta x][\xi,\zeta,x]) & \text{if } \operatorname{ind}_\xi\circ\operatorname{res}_\zeta x \in Z(\alpha) \\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

In order for $[s(\alpha), \alpha, \mathrm{ind}_\xi \circ \mathrm{res}_\zeta x], [\xi, \zeta, x]$ to be composable, $\gamma, \eta \in E$ exist such that $\alpha\gamma = \xi\eta$ and

$$= \begin{cases} \mathbb{1}_{Z(s(\alpha),\alpha)}([\gamma, \zeta\eta, x]) & \text{if } \mathrm{ind}_\xi \circ \mathrm{res}_\zeta x \in Z(\alpha) \\ 0 & \text{otherwise.} \end{cases}$$

This is 0 unless $[\gamma, \zeta\eta, x] = [s(\alpha), \alpha, x]$, in particular, $x \in Z(\alpha)$. We want to show that this implies $[\xi, \zeta, x] = [\alpha, \alpha, x]$ so that $\mathbb{1}^*_{Z(\alpha,s(\alpha))} * \mathbb{1}_{Z(\alpha,s(\alpha))} = \mathbb{1}_{Z(\alpha,\alpha)}$. Now, $[\gamma, \zeta\eta, x] = [s(\alpha), \alpha, x]$ implies that there is a suitable neighborhood such that $\mathrm{ind}_\gamma \circ \mathrm{res}_{\zeta\eta} = \mathrm{res}_\alpha$, which occurs if and only if $\mathrm{res}_{\zeta\eta} = \mathrm{res}_{\alpha\gamma}$ on this neighborhood. However, we assumed that $\alpha\gamma = \xi\eta$; thus, $\mathrm{res}_{\zeta\eta} = \mathrm{res}_{\xi\eta}$, which implies $\mathrm{ind}_\xi \circ \mathrm{res}_\zeta = \mathrm{Id}_{Z(\xi)} = \mathrm{Id}_{Z(\alpha)} = \mathrm{ind}_\alpha \circ \mathrm{res}_\alpha$ on this neighborhood. Now, we also know that $a, b \in \mathcal{B}$ exist such that $F(\gamma)a = b$ and $F(\zeta\eta)a = F(\alpha)b = F(\alpha\gamma)a = F(\xi\eta)a$. Therefore, we have $[\xi, \zeta, x] = [\alpha, \alpha, x]$, as desired.

It remains to show that

$$\mathbb{1}^*_{Z(\alpha,s(\alpha))} * \mathbb{1}_{Z(\alpha,s(\alpha))} = \mathbb{1}_{Z(s(\alpha),\alpha)} * \mathbb{1}_{Z(\alpha,s(\alpha))} = \mathbb{1}_{Z(s(\alpha),s(\alpha))}.$$

For this, we compute:

$$\begin{aligned} &\mathbb{1}_{Z(s(\alpha),\alpha)} * \mathbb{1}_{Z(\alpha,s(\alpha))}[\xi, \zeta, x] \\ &\quad = \sum_{r([\mu,\nu,y])=r([\xi,\zeta,x])} \\ &\quad \cdot \mathbb{1}_{Z(s(\alpha),\alpha)}([\mu, \nu, y])\mathbb{1}_{Z(\alpha,s(\alpha))}([\mu, \nu, y]^{-1}[\xi, \zeta, x]) \\ &\quad = \begin{cases} \mathbb{1}_{Z(\alpha,s(\alpha))}([s(\alpha), \alpha, \mathrm{ind}_\xi \circ \mathrm{res}_\zeta x]^{-1}[\xi, \zeta, x]), \\ \qquad \text{if } \mathrm{ind}_\xi \circ \mathrm{res}_\zeta x \in Z(s(\alpha)) \\ 0 \quad \text{otherwise} \end{cases} \\ &\quad = \begin{cases} \mathbb{1}_{Z(\alpha,s(\alpha))}([\alpha, s(\alpha), \mathrm{res}_\alpha \circ \mathrm{ind}_\xi \circ \mathrm{res}_\zeta x][\xi, \zeta, x]) \\ \qquad \text{if } \mathrm{ind}_\xi \circ \mathrm{res}_\zeta x \in Z(s(\alpha)) \\ 0 \quad \text{otherwise.} \end{cases} \end{aligned}$$

In order for $[\alpha, s(\alpha), \mathrm{res}_\alpha \circ \mathrm{ind}_\xi \circ \mathrm{res}_\zeta x], [\xi, \zeta, x]$ to be composable, $r(\xi) = s(\alpha)$, and thus

$$= \begin{cases} \mathbb{1}_{Z(\alpha,s(\alpha))}([\alpha\xi, \zeta, x]) & \text{if } \mathrm{ind}_\xi \circ \mathrm{res}_\zeta x \in Z(s(\alpha)) \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} 1 & \text{if } [\alpha\xi, \zeta, x] = [\alpha, s(\alpha), x] \\ 0 & \text{otherwise.} \end{cases}$$

Now, $[\alpha\xi, \zeta, x] = [\alpha, s(\alpha), x]$ implies $\mathrm{ind}_{\alpha\xi} \circ \mathrm{res}_\zeta = \mathrm{ind}_\alpha$ on some neighborhood; thus, $\mathrm{ind}_\xi \circ \mathrm{res}_\zeta = \mathrm{Id}_{Z(s(\alpha))}$ on that same neighborhood. There exists $a, b \in \mathcal{B}$ such that $F(\alpha\xi)a = F(\alpha)b$ and $F(\zeta)a = b$; thus, $F(\alpha\xi)a = F(\alpha\zeta)a$. Therefore, since morphisms in $\mathcal{B}$ are monic, we have $F(\xi)a = F(\zeta)a$, that is, $[\xi, \zeta, x] = [s(\alpha), s(\alpha), x]$, so that $\mathbb{1}_{Z(s(\alpha),\alpha)} * \mathbb{1}_{Z(\alpha,s(\alpha))} = \mathbb{1}_{Z(s(\alpha),s(\alpha))}$. $\square$

Prior to proving that Υ is an isomorphism, we need Lemma 5.2.

Lemma 5.2. *For $\alpha, \beta \in \mathcal{E}$,*

(i) *if $r(\alpha) = r(\beta) = X$ and $Z(\alpha) = Z(\beta)$, then $S_\alpha S_\alpha^* = S_\beta S_\beta^*$;*
(ii) *if $Z(\alpha) \cap Z(\beta) = \emptyset$, then $S_\alpha^* S_\beta = 0$.*

Proof. For item (i), note that $F(\alpha), F(\beta)$ is a cospan in $\mathcal{B}$; thus, by right Ore, $a, b \in \mathcal{B}$ exist such that $F(\alpha)a = F(\beta)b$. If $\eta \in \mathcal{E}$ such that $r(\eta) = s(\alpha)$ and $F(\eta) = a$, then an infinite path $x \in Z(\alpha) = Z(\beta)$ exists such that $x(F(\alpha)a) = \alpha\eta$. However, then $\alpha\eta = x(F(\alpha)a) = x(F(\beta)b) = \beta x_2(F(\beta), b)$. Thus,

$$\{\alpha\eta : F(\eta) = a, r(\eta) = s(\alpha)\} = \{\beta\gamma : F(\gamma) = b, r(\gamma) = s(\beta)\}.$$

Therefore,

$$s_\alpha s_\alpha^* = \sum_{\substack{F(\eta)=a \\ r(\eta)=s(\alpha)}} s_{\alpha\eta} s_{\alpha\eta}^* = \sum_{\substack{F(\gamma)=b \\ r(\gamma)=s(\beta)}} s_{\beta\gamma} s_{\beta\gamma}^* = s_\beta s_\beta^*.$$

For item (ii), again $F(\alpha), F(\beta)$ is a cospan in $\mathcal{B}$; thus, by right Ore, $a, b \in \mathcal{B}$ exist such that $F(\alpha)a = F(\beta)b$. Therefore,

$$s_\alpha^* s_\beta = \sum_{\substack{F(\eta)=a \\ r(\eta)=s(\alpha) \\ F(\gamma)=b \\ r(\gamma)=s(\beta)}} s_\eta s_{\alpha\eta}^* s_{\beta\gamma} s_\gamma^*.$$

It suffices to show that

$$\{\alpha\eta : F(\eta) = a, r(\eta) = s(\alpha)\} \cap \{\beta\gamma : F(\gamma) = b, r(\gamma) = s(\beta)\} = \emptyset.$$

However, if $\alpha\eta = \beta\gamma$ exist, then, for any $y \in Z(s(\eta))$, $\operatorname{ind}_{\alpha\eta} y = \operatorname{ind}_{\beta\gamma}$, $y \in Z(\alpha) \cap Z(\beta)$, a contradiction. $\square$

The proof of the next theorem is modified from [**9,** Theorem 4.2].

Theorem 5.3. *Let $F : \mathcal{E} \to \mathcal{B}$ be a row finite Kumjian-Pask fibration. If all morphisms of $\mathcal{B}$ are monic, then $\Upsilon : C^*(F) \to C^*(G_F)$ characterized by $s_\alpha \mapsto \mathbb{1}_{Z(\alpha,s(\alpha))}$ is an isomorphism.*

Proof. In order to show that Υ is an isomorphism, we construct an inverse for it. We do this locally by starting with $C(Z(X))$ (for each object X) and then gluing the resulting maps together. Note that the map

$$\phi_{\alpha,\beta} : x \longmapsto [\alpha, \beta, \operatorname{ind}_\beta x]$$

is a homeomorphism of $Z(s(\beta))$ to $Z(\alpha,\beta)$. Also, for an object X in $\mathcal{E}$, $C(Z(X)) = \overline{\operatorname{span}}\{\mathbb{1}_{Z(\alpha)} : r(\alpha) = X\}$ by the Stone-Weierstrass theorem.

By Lemma 5.2 (i), $\theta'_X : \{\mathbb{1}_{Z(\alpha)} : r(\alpha) = X\} \to C^*(F)$ given by $\theta'_X(\mathbb{1}_{Z(\alpha)}) = S_\alpha S_\alpha^*$ is well defined. Extend θ'_X linearly to the span of the $\mathbb{1}_{Z(\alpha)}$s. We want to extend θ'_X to $C(Z(X))$, but, to do this, we need to see that θ'_X is norm decreasing.

Since each $Z(\alpha)$ is compact open and F is row-finite, we can use Lemma 5.2 (ii) to show that any element in $\operatorname{span}\{\mathbb{1}_{Z(\alpha)} : r(\alpha) = v\}$ can be written as $\sum_{\alpha\in I} r_\alpha \mathbb{1}_{Z(\alpha)}$ such that, for $\alpha \neq \beta \in I$, $Z(\alpha) \cap Z(\beta) = \emptyset$. Thus,

$$\theta'_X\left(\sum_{\alpha\in I} r_\alpha \mathbb{1}_{Z(\alpha)}\right) = \sum_{\alpha\in I} r_\alpha s_\alpha s_\alpha^*,$$

where $\{s_\alpha s_\alpha^*\}_{\alpha\in I}$ is a set of mutually orthogonal projections. Therefore,

$$\left\|\theta'_X\left(\sum_{\alpha\in I} r_\alpha \mathbb{1}_{Z(\alpha)}\right)\right\| = \max\{|r_\alpha|\} = \left\|\sum_{\alpha\in I} r_\alpha \mathbb{1}_{Z(\alpha)}\right\|,$$

and thus, θ'_X is norm preserving and extends to a linear map $\theta_X : C(Z(X)) \to C^*(F)$; in fact, it is a $*$-homomorphism.

Define $\rho_{\alpha,\beta} : C(Z(\alpha,\beta)) \to C^*(F)$ by $f \mapsto \theta_{s(\alpha)} \circ \phi_{\alpha,\beta}^{-1}(f)$, such that

$$\rho_{\alpha,\beta}(\mathbb{1}_{Z(\alpha\gamma,\beta\gamma)}) = \theta_{s(\alpha)}(Z(\gamma)) = s_\gamma s_\gamma^*.$$

We claim that, for any c,

$$\rho_{\alpha,\beta}(f) = \sum_{\substack{r(\gamma)=s(\alpha)\\ F(\gamma)=c}} s_\gamma \rho_{\alpha\gamma,\beta\gamma}(f|_{Z(\alpha\gamma,\beta\gamma)})s_\gamma^*. \tag{5.1}$$

Since $\rho_{\alpha\gamma,\beta\gamma}$, $\rho_{\alpha,\beta}$ and Ad_{s_γ} are linear and continuous for all γ and $C(Z(\alpha,\beta)) = \overline{\mathrm{span}}\{\mathbb{1}_{Z(\alpha\eta,\beta\eta)}\}$, it suffices to check (5.1) on characteristic functions of the form $\mathbb{1}_{Z(\alpha\eta,\beta\eta)}$. Choose such an η. By the right Ore condition, a and b exist such that $F(\beta)F(\eta)a = F(\beta)cb$. Since morphisms are monic, we obtain $F(\eta)a = cb$ such that $F(\alpha)F(\eta)a = F(\alpha)cb$ as well. Therefore,

$$\mathbb{1}_{Z(\alpha\eta,\beta\eta)} = \sum_{\substack{F(\xi)=b\\ F(\gamma)=c\\ \eta\zeta=\gamma\xi}} \mathbb{1}_{Z(\alpha\gamma\xi,\beta\gamma\xi)} = \sum_{F(\gamma)=c} \sum_{\substack{F(\xi)=b\\ \eta\zeta=\gamma\xi}} \mathbb{1}_{Z(\alpha\gamma\xi,\beta\gamma\xi)}.$$

Note that

$$\mathbb{1}_{Z(\alpha\eta,\beta\eta)}|_{Z(\alpha\gamma,\beta\gamma)} = \sum_{\substack{F(\xi)=b\\ \eta\zeta=\gamma\xi}} \mathbb{1}_{Z(\alpha\gamma\xi,\beta\gamma\xi)}.$$

Now,

$$\begin{aligned}
\rho_{\alpha,\beta}(\mathbb{1}_{Z(\alpha\eta,\beta\eta)}) &= \rho_{\alpha,\beta}\left(\sum_{F(\gamma)=c}\left(\sum_{\substack{F(\xi)=b\\ \eta\zeta=\gamma\xi}} \mathbb{1}_{Z(\alpha\gamma\xi,\beta\gamma\xi)}\right)\right)\\
&= \sum_{F(\gamma)=c}\left(\sum_{\substack{F(\xi)=b\\ \eta\zeta=\gamma\xi}} s_{\gamma\xi}s_{\gamma\xi}^*\right)\\
&= \sum_{F(\gamma)=c} s_\gamma\left(\sum_{\substack{F(\xi)=b\\ \eta\zeta=\gamma\xi}} s_\xi s_\xi^*\right)s_\gamma^*\\
&= \sum_{F(\gamma)=c} s_\gamma\left(\sum_{\substack{F(\xi)=b\\ \eta\zeta=\gamma\xi}} \rho_{\alpha\gamma,\beta\gamma}(\mathbb{1}_{Z(\alpha\gamma\xi,\beta\gamma\xi)})\right)s_\gamma^*\\
&= \sum_{F(\gamma)=c} s_\gamma(\rho_{\alpha\gamma,\beta\gamma}(\mathbb{1}_{Z(\alpha\eta,\beta\eta)}|_{Z(\alpha\gamma,\beta\gamma)}))s_\gamma^*,
\end{aligned}$$

as desired.

For $f \in C_c(G)$, using Proposition 4.12, we may use $\mathrm{supp}(f)$ as a disjoint union of $Z(\alpha^i, \beta^i)$. We define

$$\begin{aligned}\theta(f) &= \sum s_{\alpha^i}(\rho_{\alpha^i,\beta^i}(f|_{Z(\alpha^i,\beta^i)})))s^*_{\beta^i} \\ &= \sum s_{\alpha^i}(\theta_{s(\alpha_i)}(\phi^{-1}_{\alpha^i,\beta^i}(f|_{Z(\alpha^i,\beta^i)})))s^*_{\beta^i}.\end{aligned}$$

We must show that $\theta(f)$ is well defined, that is, it does not depend on the decomposition of the support of f. Suppose that $\mathrm{supp}(f) = \bigcup Z(\sigma^j, \tau^j)$. Then, $\mathrm{supp}(f) = \cup(Z(\alpha^i, \beta^i) \cap Z(\sigma^j, \tau^j))$. Since each of these intersections is a union of sets of the form $Z(\alpha^i\gamma, \beta^i\gamma)$ with $F(\gamma) = a_i$, it suffices to show that, for f with $\mathrm{supp}(f) \subset Z(\alpha, \beta)$,

$$\begin{aligned}s_\alpha(\rho_{\alpha,\beta}(f))s^*_\beta &= \sum_{\substack{r(\gamma)=s(\alpha)\\ F(\gamma)=b}} s_{\alpha\gamma}(\rho_{\alpha\gamma,\beta\gamma}(f|_{Z(\alpha\gamma,\beta\gamma)})))s^*_{\beta\gamma} \\ &= s_\alpha\left(\sum_{\substack{r(\gamma)=s(\alpha)\\ F(\gamma)=b}} s_\gamma(\theta_{s(\gamma)}(\phi^{-1}_{\alpha\gamma,\beta\gamma}(f|_{Z(\alpha\gamma,\beta\gamma)})))s^*_\gamma\right)s^*_\beta.\end{aligned}$$

However, we know that

$$\rho_{\alpha,\beta}(f)) = \sum_{\substack{r(\gamma)=s(\alpha)\\ F(\gamma)=b}} s_\gamma(\theta_{s(\gamma)}(\phi^{-1}_{\alpha\gamma,\beta\gamma}(f|_{Z(\alpha\gamma,\beta\gamma)})))s^*_\gamma,$$

from above. Thus, θ is well defined.

In order to show that $\theta : C^*(G_F) \to C^*(F)$ is continuous, it is enough to show that $\theta : C_c(G_F) \to C^*(F)$ is continuous in the inductive limit topology. Suppose that $f_i \to f$ uniformly with $\mathrm{supp}(f_i)$ eventually contained in a fixed compact set K. Let $K \subset \cup Z(\alpha^j, \beta^j)$. Since $\mathrm{supp}(f_i) \subset K \subset \cup Z(\alpha^j, \beta^j)$, an N exists such that, for $i \geq N$,

$$\theta(f_i) = \sum s_{\alpha^j}\rho_{\alpha^j,\beta^j}(f_i|_{Z(\alpha^j,\beta^j)})s^*_{\beta^j}.$$

However, the maps ρ_{α^j,β^j} are all norm decreasing for the uniform norm on $C(Z(\alpha^i, \beta^i))$ and the C^*-norm on $C^*(F)$. Thus, $\theta(f_i) \to \theta(f)$, and hence, θ extends to a continuous map $\theta : C^*(G_F) \to C^*(F)$.

It remains to show that θ is a $*$-homomorphism. Since θ is continuous and linear and $C^*(G_F)$ is the closed span of $\{\mathbb{1}_{Z(\alpha,\beta)}\}$, it suffices to show that θ is multiplicative on these functions; however,

$$\begin{aligned}\theta(\mathbb{1}_{Z(\alpha,\beta)} * \mathbb{1}_{Z(\mu,\nu)}) = \theta(\mathbb{1}_{Z(\alpha,\beta)Z(\mu,\nu)}) &= \theta\left(\sum_{\substack{\beta\eta=\mu\lambda \\ F(\eta)=a \\ F(\lambda)=b}} \mathbb{1}_{Z(\alpha\eta,\nu\lambda)}\right) \\ &= \sum_{\substack{\beta\eta=\mu\lambda \\ F(\eta)=a \\ F(\lambda)=b}} \theta(\mathbb{1}_{Z(\alpha\eta,\nu\lambda)}) = \sum_{\substack{\beta\eta=\mu\lambda \\ F(\eta)=a \\ F(\lambda)=b}} s_{\alpha\eta}s_{\nu\lambda}^* \\ &= s_\alpha s_\beta^* s_\mu s_\nu^* = \theta(\mathbb{1}_{Z(\alpha,\beta)})\theta(\mathbb{1}_{Z(\mu,\nu)}).\end{aligned}$$

Also,

$$\theta(\mathbb{1}_{Z(\alpha,\beta)})^* = (s_\alpha s_\beta^*)^* = s_\beta s_\alpha^* = \theta(\mathbb{1}_{Z(\alpha,\beta)}^*).$$

Lastly, by the definition of θ, it is an inverse for Υ. □

It is possible that $Z(\alpha, s(\alpha)) = Z(\beta, s(\beta))$ with $\alpha \neq \beta$. However, in order for this to be true, an a must exist such that $F(\alpha)a = F(\beta)a$. In order for $[\alpha, s(\alpha), x] \in Z(\beta, s(\beta))$, we must have $[\alpha, s(\alpha), x] = [\beta, s(\beta), x]$ such that $\gamma = x(a)$, $\alpha\gamma = \mathrm{ind}_\alpha(x)(F(\alpha)a) = \mathrm{ind}_\beta(x)(F(\beta)a) = \beta\gamma$. Thus,

$$s_\alpha = s_\alpha \sum_{\substack{F(\gamma)=a \\ r(\gamma)=s(\alpha)}} s_\gamma s_\gamma^* = \sum_{\substack{F(\gamma)=a \\ r(\gamma)=s(\alpha)}} s_{\alpha\gamma}s_\gamma^* = \sum_{\substack{F(\gamma)=a \\ r(\gamma)=s(\beta)}} s_{\beta\gamma}s_\gamma^* = s_\beta.$$

Now, if $\mathcal{B}$ is right cancellative, $Z(\alpha, s(\alpha)) = Z(\beta, s(\beta))$ implies that $F(\alpha)a = F(\beta)a$; thus, $F(\alpha) = F(\beta)$ and this gives $\alpha = \beta$.

Proposition 5.4. *Let $F : \mathcal{E} \to \mathcal{B}$ be a row-finite* KPf *with $\mathcal{B}$ left cancellative. The map $\alpha \mapsto s_\alpha$ from $\mathcal{E} \to C^*(F)$ is injective if and only if $\mathcal{B}$ is right cancellative.*

Proof. Let $\Upsilon : C^*(F) \to C^*(G_F)$ be the $*$-homomorphism from Lemma 5.1 which sends $s_\alpha \mapsto \mathbb{1}_{Z(\alpha,s(\alpha))}$. Suppose that $s_\alpha = s_\beta$. Then $\mathbb{1}_{Z(\alpha,s(\alpha))} = \Upsilon(s_\alpha) = \Upsilon(s_\beta) = \mathbb{1}_{Z(\beta,s(\beta))}$. Thus,

$$Z(\alpha, s(\alpha)) = Z(\beta, s(\beta)).$$

Choose $x \in Z(s(\alpha))$. Then, $[\alpha, s(\alpha), x] = [\beta, s(\beta), x]$. In particular,

$$\mathrm{ind}_\alpha(x) = \mathrm{ind}_\beta(x),$$

and $a, b \in \mathcal{B}$ exist with

$$F(\alpha)a = F(\beta)b$$

and

$$F(s(\alpha))a = F(s(\beta))b,$$

which implies

$$F(\alpha)a = F(\beta)a.$$

If $\mathcal{B}$ is right cancellative, we have $F(\alpha) = F(\beta)$, and thus $\alpha = \mathrm{ind}_\alpha(x)(F(\alpha)) = \mathrm{ind}_\beta(x)(F(\beta)) = \beta$, that is, the map $\alpha \mapsto s_\alpha$ is injective.

Now, suppose that $\alpha \neq \beta$ and $s_\alpha = s_\beta$. Then $F(\alpha) \neq F(\beta)$; otherwise, $0 \neq s_\alpha^* s_\alpha = s_\alpha^* s_\beta = 0$. By the above, we have that $a \in \mathcal{B}$ exists with $F(\alpha)a = F(\beta)a$ so $\mathcal{B}$ is not right cancellative. $\square$

Since every C^*-algebra can be faithfully represented on a Hilbert space, the previous result says that a row finite Kumjian-Pask fibration $F : \mathcal{E} \to \mathcal{B}$ with $\mathcal{B}$ left cancellative can be faithfully represented on a Hilbert space if and only if the base category is also right cancellative. In special circumstances, the infinite path representation gives a faithful representation of the Kumjian-Pask fibration.

Definition 5.5. We say that $x \in F^\infty$ is aperiodic if, for all $\alpha, \beta \in \mathcal{E}$, $\mathrm{res}_\alpha(x) = \mathrm{res}_\beta(x)$ implies $\alpha = \beta$. We say that $F : \mathcal{E} \to \mathcal{B}$ is aperiodic if $Z(X)$ contains an aperiodic path for every $X \in \mathrm{Obj}(\mathcal{E})$.

For a groupoid G and $x \in G^{(0)}$, denote $xGx = \{\gamma \in G : r(\gamma) = s(\gamma) = x\}$.

Theorem 5.6. *If $\mathcal{E}$ is aperiodic, then G_F is topologically principal.*

Proof. We need to show that $T = \{x \in F^\infty : xG_Fx = \{x\}\}$ is dense in F^∞. Let $Z(\alpha)$ be a basic open set in F^∞. It suffices to show that $T \cap Z(\alpha) \neq \emptyset$. Now, $Z(\alpha) = Z(\alpha, s(\alpha)) \cdot Z(s(\alpha))$; thus, it suffices to show that $Z(s(\alpha)) \cap T = \emptyset$ since, if $x \in Z(s(\alpha)) \cap T$, then $[\alpha, s(\alpha), x] \cdot x \in Z(\alpha) \cap T$. By assumption, an aperiodic path $x \in Z(s(\alpha))$ exists. We must show that $xG_Fx = \{x\}$. Now, if $[\alpha, \beta, x] \in xG_Ex$, then $\mathrm{ind}_\alpha \circ \mathrm{res}_\beta(x) = x$, that is, $\mathrm{res}_\beta(x) = \mathrm{res}_\alpha(x)$. Since x is aperiodic, we have $\alpha = \beta$; thus, $[\alpha, \beta, x] = [r(x), r(x), x]$, that is, $x \in T \cap Z(s(\alpha))$. Therefore, G_F is topologically principal, as desired. $\square$

Let I_λ be the ideal such that $C^*(G_F)/I_\lambda = C^*_r(G_F)$. If G_F is amenable, then $C^*(G_F) \cong C^*_r(G_F)$. Amenability is extremely complicated for groupoids. For a full discussion, see **[1]**. Fortunately, here we only need that G_F amenable implies $C^*(G_F) \cong C^*_r(G_F)$.

Corollary 5.7. *Let $F : \mathcal{E} \to \mathcal{B}$ be an aperiodic row finite Kumjian-Pask fibration over a right cancellative category $\mathcal{B}$.*

(i) *The kernel of the infinite path representation of $C^*(G_F)$ is contained in I_λ. In particular, $\mathcal{E}$ faithfully represents on $\ell^2(F^\infty)$.*

(ii) *If, additionally, G_F is amenable and $\pi : C^*(F) \to B$ is a $*$-homomorphism, then π is faithful if and only if $\pi(P_X) \neq 0$ for all objects X in $\mathcal{E}$.*

Proof. By **[2,** Theorem 5.5], if G_F is topologically principal, then every nonzero ideal of $C^*_r(G_F)$ intersects $C_0(G_F^0)$ nontrivially.

For item (i), let $q : C^*(G_F) \to C^*_r(G_F)$ be the quotient map and J the kernel of the infinite path representation. Then, $q(J)$ is an ideal in $C^*_r(G_F)$. Now, $C_0(G_F^{(0)}) \cap J = \{0\}$ and, since $q|_{C_0(G_F^{(0)})}$ is the identity, we have $q(J) \cap C_0(G_F^{(0)}) = \{0\}$ which implies $q(J) = \{0\}$; hence, $J \subset I_\lambda$, as desired. For the second statement, $\Upsilon(S_\alpha) = \mathbb{1}_{Z(\alpha, s(\alpha))} \in C_c(G_F)$ and $C_c(G_F) \cap I_\lambda = \{0\}$.

For item (ii), if G_F is amenable $C^*(G_F) \cong C^*_r(G_F)$, then any ideal of $C^*(G_F) \cong C^*(F)$ has nontrivial intersection with $C_0(G_F^{(0)})$. In particular, $\ker(\pi)$ contains $\mathbb{1}_{Z(\alpha)} = \mathbb{1}_{Z(\alpha, s(\alpha))} * \mathbb{1}^*_{Z(\alpha, s(\alpha))} = \Upsilon(S_\alpha S^*_\alpha)$ for some $\alpha \in \mathcal{E}$. Thus, $\pi(S_\alpha) = 0$, which gives $\pi(P_{s(\alpha)}) = \pi(S^*_\alpha S_\alpha) = 0$. $\square$

Corollary 5.7 (ii) is typically referred to as the Cuntz-Krieger uniqueness theorem. Observe that it depends on the amenability of G_F. For a k-graph, $F = d : \Lambda \to \mathbb{N}^k$, the amenability of G_F was shown in [**7,** Theorem 5.5]. On the other hand, if the Kumjian-Pask fibration is $F = \mathrm{Id}_H : H \to H$ for H a nonamenable group (say $\mathbb{F}_2$), then $G_F = H$. Thus, we cannot expect G_F to be amenable in general. Furthermore, it would be desirable to find a sufficient condition, or an ideally necessary and sufficient condition, on F which would ensure the amenability of G_F.

Now, suppose that $F : \mathcal{E} \to \mathcal{B}$ with $\mathcal{B}$ an abelian monoid. Then, F induces a map $c : G_F \to H$ where H is the Grothendieck group constructed from $\mathcal{B}$. In this case, [**15,** Theorem 9.3] shows that G_F is amenable if $c^{-1}(0)$ is an amenable groupoid. In [**15,** Theorem 9.8], it is shown that, under certain conditions for a category of paths, $c^{-1}(0)$ is an AF groupoid, and hence, amenable. Although the presence of multiple objects in $\mathcal{B}$ will complicate the matter, we expect a similar result to hold in the general case.

Besides the investigation of amenability for general Kumjian-Pask fibrations and consideration of the properties of specific examples and classes of examples of C^*-algebras arising from the constructions in the present paper, two other avenues of investigation present themselves: consideration of the construction in the absence of either the countability condition on the slice categories $\mathcal{B}/B$ or row-finiteness. In the latter case, it would appear that the correct generalization of Definition 2.7 (vi) with $6'$, for all X, and for all morphisms $b : B \to F(X)$ in $\mathcal{B}$, as well as all $\alpha \in X\mathcal{E} \cap F^{-1}(b)$

$$S_\alpha S_\alpha^* \leq P_X,$$

and if, moreover, $F^{-1}(b)$ is finite, then

$$\sum_{\{\alpha \in X\mathcal{E} : F(\alpha) = b\}} S_\alpha S_\alpha^* = P_X.$$

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