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GENERAL MIXED CHORD-INTEGRALS OF STAR BODIES

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ABSTRACT. Mixed chord-integrals of star bodies were first defined by Lu [**19**]. In this paper, the concept of mixed chord-integrals is extended to general mixed chord-integrals, which is motivated by the recent work on general L_p-affine isoperimetric inequalities by Haberl, et al. [**16**]. For this new notion of general mixed chord-integrals, isoperimetric and Aleksandrov-Fenchel type inequalities are established which generalize inequalities obtained by Lu, and a cyclic inequality is also obtained. Furthermore, we prove several Brunn-Minkowski type inequalities for general mixed chord-integrals.

1. Introduction and main results. Let S^{n-1} denote the unit sphere in Euclidean space $\mathbb{R}^n$, and let $V(K)$ denote the n-dimensional volume of a body K. For the standard unit ball B in $\mathbb{R}^n$, we write $\omega_n = V(B)$ for its volume.

If K is a compact star-shaped (about the origin) set in $\mathbb{R}^n$, then its radial function, $\rho_K = \rho(K,\cdot) : \mathbb{R}^n \setminus \{0\} \to [0,\infty)$, is defined by (see, e.g., [**8, 33**])

$$\rho(K,u) = \max\{\lambda \geq 0 : \lambda u \in K\}, \quad u \in S^{n-1}.$$

If ρ_K is positive and continuous, then K will be called a *star body* (about the origin), and $\mathcal{S}^n$ denotes the set of star bodies in $\mathbb{R}^n$. We will use $\mathcal{S}^n_o$ and $\mathcal{S}^n_c$ to denote the subset of star bodies in $\mathcal{S}^n$ containing the origin in their interiors and whose centroids lie at the origin, respectively. Two star bodies, K and L, are said to be dilates of one another if $\rho_K(u)/\rho_L(u)$ is independent of $u \in S^{n-1}$.

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Lutwak **[23]** introduced the notion of mixed width-integrals of convex bodies. Motivated by Lutwak's ideas, the notion of mixed chord-integrals of star bodies was recently defined by Lu (see **[19]**): For $K_1, \ldots, K_n \in \mathcal{S}_o^n$, the mixed chord-integral, $C(K_1, \ldots, K_n)$, of $K_1, \ldots, K_n$ was defined by

$$(1.1) \qquad C(K_1, \ldots, K_n) = \frac{1}{n} \int_{S^{n-1}} c(K_1, u) \cdots c(K_n, u0)\, dS(u),$$

where $dS(u)$ is the $(n-1)$-dimensional volume element on S^{n-1} and $c(K, u)$ denotes the half chord of K in the direction u, namely, $c(K, u) = \rho(K, u)/2 + \rho(K, -u)/2$. Thus, the mixed chord-integral is a map $\underbrace{\mathcal{S}_o^n \times \cdots \times \mathcal{S}_o^n}_{n} \to \mathbb{R}$. It is positive, continuous and multilinear with respect to radial Minkowski linear combinations, positively homogeneous and monotone under set inclusion. Star bodies $K_1, \ldots, K_n$ are said to have a similar chord if there exist constants $\lambda_1, \ldots, \lambda_n > 0$ such that $\lambda_1 c(K_1, u) = \cdots = \lambda_n c(K_n, u)$ for all $u \in S^{n-1}$. Lu **[19]** established the following isoperimetric and Aleksandrov-Fenchel inequalities for mixed chord-integrals.

Theorem 1.A. *If $K_1, \ldots, K_n \in \mathcal{S}_o^n$, then*

$$C(K_1, \ldots, K_n)^n \leq V(K_1) \cdots V(K_n),$$

with equality if and only if $K_1, \ldots, K_n$ are centered at the origin and dilates of each other.

Theorem 1.B. *If $K_1, \ldots, K_n \in \mathcal{S}_o^n$ and $1 < m \leq n$, then*

$$C(K_1, \ldots, K_n)^m \leq \prod_{i=1}^{m} C(K_1, \ldots, K_{n-m}, K_{n-i+1}, \ldots, K_{n-i+1}),$$

with equality if and only if $K_{n-m+1}, \ldots, K_n$ are all of a similar chord.

We now propose the definition of general mixed chord-integrals of star bodies, which generalizes the above definition of mixed chord-integrals.

For $\tau \in (-1,1)$, the general mixed chord-integral, $C^{(\tau)}(K_1,\ldots,K_n)$, of $K_1,\ldots,K_n \in \mathcal{S}_o^n$ is defined by

$$C^{(\tau)}(K_1,\ldots,K_n) = \frac{1}{n}\int_{S^{n-1}} c^{(\tau)}(K_1,u)\cdots c^{(\tau)}(K_n,u)\,dS(u), \tag{1.2}$$

where $c^{(\tau)}(K,u) = f_1(\tau)\rho(K,u) + f_2(\tau)\rho(K,-u)$ and the functions $f_1(\tau)$ and $f_2(\tau)$ are defined as follows:

$$f_1(\tau) = \frac{(1+\tau)^2}{2(1+\tau^2)}, \qquad f_2(\tau) = \frac{(1-\tau)^2}{2(1+\tau^2)}. \tag{1.3}$$

From (1.3), it immediately follows that

$$f_1(\tau) + f_2(\tau) = 1; \tag{1.4}$$

$$f_1(-\tau) = f_2(\tau), \qquad f_2(-\tau) = f_1(\tau). \tag{1.5}$$

By (1.3), if we let $\tau = 0$ in definition (1.2), then $C^{(0)}(K_1,\ldots,K_n)$ is just Lu's mixed chord-integral $C(K_1,\ldots,K_n)$. Similarly, the general mixed chord-integral, $C^{(\tau)}(K_1,\ldots,K_n) : \mathcal{S}_o^n \times\cdots\times \mathcal{S}_o^n \to \mathbb{R}$, is also positive, continuous and multilinear with respect to the radial Minkowski linear combinations, positively homogeneous and monotone under set inclusion. Star bodies $K_1,\ldots,K_n$ are said to have a similar general chord if there exist constants $\lambda_1,\ldots,\lambda_n > 0$ such that $\lambda_1 c^{(\tau)}(K_1,u) = \cdots = \lambda_n c^{(\tau)}(K_n,u)$ for all $u \in S^{n-1}$. They are said to have a joint constant general chord if the product $c^{(\tau)}(K_1,u)\cdots c^{(\tau)}(K_n,u)$ is constant for all $u \in S^{n-1}$.

If we take $K_1 = \cdots = K_{n-i} = K$ and $K_{n-i+1} = \cdots = K_n = B$ in definition (1.2), then the general chord-integral of order i, $C_i^{(\tau)}(K)$, of $K \in \mathcal{S}_o^n$ is defined by

$$C_i^{(\tau)}(K) = \frac{1}{n}\int_{S^{n-1}} c^{(\tau)}(K,u)^{n-i}\,dS(u). \tag{1.6}$$

Taking $K_1 = \cdots = K_n = K$ in (1.2), we write $C^{(\tau)}(K)$ for $C^{(\tau)}(K,\cdots,K)$, and call it the general chord-integral of $K \in \mathcal{S}_o^n$.

The asymmetric (dual) Brunn-Minkowski theory has as its starting point the theory of valuations in connection with isoperimetric and analytic inequalities (see [**1, 6, 13, 14, 15, 16, 20, 21, 22, 27, 28, 29, 30, 31, 34, 36, 37, 38, 39, 40**]).

As our main results, we first establish extended versions of Theorem 1.A and Theorem 1.B, given by Theorem 1.1 and Theorem 1.2.

Theorem 1.1. *If $\tau \in (-1,1)$ and $K_1, \ldots, K_n \in \mathcal{S}_o^n$, then*

$$C^{(\tau)}(K_1, \cdots, K_n)^n \leq V(K_1) \cdots V(K_n), \tag{1.7}$$

with equality if and only if $K_1, \ldots, K_n$ are centered at the origin and dilates of each other.

Theorem 1.2. *If $\tau \in (-1,1)$ and $K_1, \ldots, K_n \in \mathcal{S}_o^n$, $1 < m \leq n$, then*

$$C^{(\tau)}(K_1, \ldots, K_n)^m \leq \prod_{i=1}^{m} C^{(\tau)}(K_1, \ldots, K_{n-m}, K_{n-i+1}, \ldots, K_{n-i+1}), \tag{1.8}$$

with equality if and only if $K_{n-m+1}, \ldots, K_n$ are all of similar general chord.

Moreover, we will prove the following cyclic inequality.

Theorem 1.3. *If $\tau \in (-1,1)$ and $K, L \in \mathcal{S}_o^n$, then, for $i < j < k$,*

$$C_i^{(\tau)}(K,L)^{k-j} C_k^{(\tau)}(K,L)^{j-i} \geq C_j^{(\tau)}(K,L)^{k-i}, \tag{1.9}$$

with equality if and only if K and L have a similar general chord.

Here, $C_i^{(\tau)}(K,L) = C^{(\tau)}(K, n-i; L, i)$ in which K appears $n-i$ times and L appears i times.

The proofs of Theorems 1.1–1.3 will be given in Section 4 of this paper. In Section 3, we establish some properties of general chord-integrals of order i. Moreover, several Brunn-Minkowski type inequalities for general chord-integrals of order i are obtained in Section 5.

2. Preliminaries.

If $E \subseteq \mathbb{R}^n$ is non-empty, the polar set E^* of E is defined by (see **[8]**)

$$E^* = \{x \in \mathbb{R}^n : x \cdot y \leq 1, \text{ for all } y \in E\}.$$

For $K \in \mathcal{S}_c^n$, an extension of the well-known Blaschke-Santaló inequality takes the following form (see **[24]**).

Theorem 2.A. *If $K \in \mathcal{S}_c^n$, then*

$$V(K)V(K^*) \le \omega_n^2, \tag{2.1}$$

with equality if and only if K is an ellipsoid.

If $K_1, \dots, K_m \in \mathcal{S}_o^n$ and $\lambda_1, \dots, \lambda_m \in \mathbb{R}$, then the radial Minkowski linear combination is defined by (see [**25**])

$$\rho(\lambda_1 K_1 \widetilde{+} \cdots \widetilde{+} \lambda_m K_m, \cdot) = \lambda_1 \rho(K_1, \cdot) + \cdots + \lambda_m \rho(K_m, \cdot). \tag{2.2}$$

For $K, L \in \mathcal{S}_o^n$ and $\lambda, \mu \ge 0$ (both nonzero), the radial Blaschke linear combination, $\lambda \cdot K \breve{+} \mu \cdot L$, of K and L is defined by (see [**25**]):

$$\rho(\lambda \cdot K \breve{+} \mu \cdot L, \cdot)^{n-1} = \lambda \rho(K, \cdot)^{n-1} + \mu \rho(L, \cdot)^{n-1}. \tag{2.3}$$

For more information on binary operations between convex or star bodies, we refer to the articles [**9, 10, 11**]. For $K \in \mathcal{S}_o^n$, the intersection body of K, IK, is the star body symmetric with respect to origin whose radial function on S^{n-1} is given by (see [**25**]):

$$\rho(IK, u) = \frac{1}{n-1} \int_{S^{n-1} \cap u^\perp} \rho(K, u)^{n-1} d\lambda_{n-2}(u), \tag{2.4}$$

where $d\lambda_{n-2}(u)$ is $(n-2)$-dimensional spherical Lebesgue measure. For $u \in S^{n-1}$, $K \cap u^\perp$ denotes the intersection of K with the subspace $u^\perp$ that passes through the origin and is orthogonal to u.

From equations (2.2), (2.3) and (2.4), it follows that, if $K, L \in \mathcal{S}_o^n$ and $\lambda, \mu \ge 0$ (both nonzero), then

$$I(\lambda \cdot K \breve{+} \mu \cdot L) = \lambda IK \widetilde{+} \mu IL. \tag{2.5}$$

The polar coordinate formula for volume of a body K in $\mathbb{R}^n$ is

$$V(K) = \frac{1}{n} \int_{S^{n-1}} \rho(K, u)^n dS(u). \tag{2.6}$$

3. General chord-integrals of order i. In this section, we establish some properties and inequalities for general chord-integrals of order i.

From definition (1.2) and the multilinearity of general mixed chord-integrals, we see that the general chord-integral of $\lambda_1 K_1 \widetilde{+} \cdots \widetilde{+} \lambda_m K_m$ is a homogeneous polynomial of degree n in $\lambda_1, \ldots, \lambda_m$.

Theorem 3.1. *For $\tau \in (-1,1)$ and $K_1, \ldots, K_m \in \mathcal{S}_o^n$, let $K = \lambda_1 K_1 \widetilde{+} \cdots \widetilde{+} \lambda_m K_m$. Then*

$$C^{(\tau)}(K) = \sum_{j_1=1}^{m} \cdots \sum_{j_n=1}^{m} \lambda_{j_1} \cdots \lambda_{j_n} C^{(\tau)}(K_{j_1}, \ldots, K_{j_n}). \tag{3.1}$$

As a direct consequence of Theorem 3.1, we have:

Theorem 3.2. *For $\tau \in (-1,1)$ and $K \in \mathcal{S}_o^n$, let $K_\mu = K \widetilde{+} \mu B$ ($\mu > 0$). Then, for $j = 0, 1, \ldots, n$,*

$$C_j^{(\tau)}(K_\mu) = \sum_{i=0}^{n-j} \binom{n-j}{i} C_{j+i}^{(\tau)}(K) \mu^i. \tag{3.2}$$

Lemma 3.3. *If $\tau \in (-1,1)$ and $K \in \mathcal{S}_o^n$, then*

$$C^{(\tau)}(K) \leq V(K), \tag{3.3}$$

with equality if and only if K is centered at the origin.

Proof. From Minkowski's inequality (see **[17]**), we have

$$\begin{aligned}
C^{(\tau)}(K)^{1/n} &= \left[\frac{1}{n}\int_{S^{n-1}} c^{(\tau)}(K,u)^n dS(u)\right]^{1/n} \\
&= \left[\frac{1}{n}\int_{S^{n-1}} \left(f_1(\tau)\rho(K,u) + f_2(\tau)\rho(K,-u)\right)^n dS(u)\right]^{1/n} \\
&\leq \left[\frac{1}{n}\int_{S^{n-1}} \left(f_1(\tau)\rho(K,u)\right)^n dS(u)\right]^{1/n} \\
&\quad + \left[\frac{1}{n}\int_{S^{n-1}} \left(f_2(\tau)\rho(K,-u)\right)^n dS(u)\right]^{1/n} \\
&= \left[\frac{1}{n}\int_{S^{n-1}} \rho(K,u)^n dS(u)\right]^{1/n},
\end{aligned}$$

that is,

$$C^{(\tau)}(K) \leq \frac{1}{n}\int_{S^{n-1}} \rho(K,u)^n dS(u) = V(K).$$

From the equality condition of Minkowski's inequality, we see that equality holds in (3.3) if and only if K and $-K$ are dilates of one another, i.e., K is centered at the origin. $\square$

Theorem 3.4. *If $\tau \in (-1,1)$ and $K \in \mathcal{S}_c^n$, then, for $0 < i < n$,*

$$C_i^{(\tau)}(K)C_i^{(\tau)}(K^*) \leq \omega_n^2, \tag{3.4}$$

for $i > n$, inequality (3.4) *is reversed, with equality in each inequality if and only if K is an ellipsoid centered at the origin.*

Proof. From Lemma 3.3 and Jensen's inequality (see [**17**]), we get for $i > 0$ and $i \neq n$,

$$\left[\frac{1}{\omega_n}C_i^{(\tau)}(K)\right]^{1/(n-i)} \leq \left[\frac{1}{\omega_n}C^{(\tau)}(K)\right]^{1/n} \leq \left[\frac{1}{\omega_n}V(K)\right]^{1/n},$$

i.e.,

$$C_i^{(\tau)}(K)^{1/(n-i)} \leq \omega_n^{i/n(n-i)}V(K)^{1/n}. \tag{3.5}$$

By (3.5), we have

$$C_i^{(\tau)}(K^*)^{1/(n-i)} \leq \omega_n^{i/n(n-i)}V(K^*)^{1/n}. \tag{3.6}$$

Combining equations (3.5) and (3.6), it follows from the extended Blaschke-Santaló inequality (see Theorem 2.A) that

$$\left[C_i^{(\tau)}(K)C_i^{(\tau)}(K^*)\right]^{1/(n-i)} \leq \omega_n^{2/(n-i)}. \tag{3.7}$$

If $0 < i < n$ in inequality (3.7), then we have

$$C_i^{(\tau)}(K)C_i^{(\tau)}(K^*) \leq \omega_n^2;$$

if $i > n$ in inequality (3.7), then the above inequality is reversed.

According to the equality conditions of inequality (2.1), inequality (3.3) and Jensen's inequality, we see that equality holds in every inequality if and only if K is an ellipsoid centered at the origin. $\square$

4. Proofs of Theorems 1.1–1.3.

Proof of Theorem 1.1. By Hölder's inequality (see **[17]**), we have

$$\begin{aligned} C^{(\tau)}(K_1,\dots,K_n)^n &= \left(\frac{1}{n}\int_{S^{n-1}} c^{(\tau)}(K_1,u)\cdots c^{(\tau)}(K_n,u)\,dS(u)\right)^n \\ &\le \left(\frac{1}{n}\int_{S^{n-1}} c^{(\tau)}(K_1,u)^n dS(u)\right)\times\cdots \\ &\quad\times\left(\frac{1}{n}\int_{S^{n-1}} c^{(\tau)}(K_n,u)^n dS(u)\right), \end{aligned} \tag{4.1}$$

with equality if and only if $K_1,\cdots,K_n$ have similar general chord. An application of Minkowski's inequality (see [17]) thus yields

$$\begin{aligned} &C^{(\tau)}(K_1,\dots,K_n) \\ &\quad\le \left(\frac{1}{n}\int_{S^{n-1}} c^{(\tau)}(K_1,u)^n dS(u)\right)^{1/n}\times\cdots \\ &\quad\times \left(\frac{1}{n}\int_{S^{n-1}} c^{(\tau)}(K_n,u)^n dS(u)\right)^{1/n} \\ &\quad= \left(\frac{1}{n}\int_{S^{n-1}} (f_1(\tau)\rho(K_1,u)+f_2(\tau)\rho(K_1,-u))^n dS(u)\right)^{1/n}\times\cdots \\ &\quad\times \left(\frac{1}{n}\int_{S^{n-1}} (f_1(\tau)\rho(K_n,u)+f_2(\tau)\rho(K_n,-u))^n dS(u)\right)^{1/n} \\ &\quad\le \left(\frac{1}{n}\int_{S^{n-1}} \rho(K_1,u)^n dS(u)\right)^{1/n}\times\cdots \\ &\quad\times \left(\frac{1}{n}\int_{S^{n-1}} \rho(K_n,u)^n dS(u)\right)^{1/n} \\ &\quad= V(K_1)^{1/n}\cdots V(K_n)^{1/n}, \end{aligned}$$

with equality in the second inequality if and only if $\tau = 0$. Consequently,

$$C^{(\tau)}(K_1,\cdots,K_n)^n \le V(K_1)\cdots V(K_n).$$

According to the equality conditions of the above inequalities, we see that equality holds in (1.7) if and only if $K_1,\dots,K_n$ are dilates of each other and centered at the origin. □

Lemma 4.1 (**[26]**). *If $f_0, f_1, \dots, f_m$ are (strictly) positive continuous functions defined on S^{n-1} and $\lambda_1, \dots, \lambda_m$ are positive constants, the sum of whose reciprocals is unity, then*

(4.2)
$$\int_{S^{n-1}} f_0(u) f_1(u) \cdots f_m(u)\, dS(u) \le \prod_{i=1}^{m} \left[\int_{S^{n-1}} f_0(u) f_i^{\lambda_i}(u)\, dS(u) \right]^{1/\lambda_i},$$

with equality if and only if there exist positive constants $\alpha_1, \dots, \alpha_m$ such that $\alpha_1 f_1^{\lambda_1}(u) = \cdots = \alpha_m f_m^{\lambda_m}(u)$ for all $u \in S^{n-1}$.

Proof of Theorem 1.2. In Lemma 4.1, we take

$$\begin{aligned} &\lambda_i = m \quad (1 \le i \le m); \\ &f_0 = c^{(\tau)}(K_1, u) \cdots c^{(\tau)}(K_{n-m}), u \quad (f_0 = 1 \text{ if } m = n); \\ &f_i = c^{(\tau)}(K_{n-i+1}, u) \quad (1 \le i \le m). \end{aligned}$$

Then it follows that

$$\begin{aligned} &\int_{S^{n-1}} c^{(\tau)}(K_1, u) \cdots c^{(\tau)}(K_n, u)\, dS(u) \\ &\le \prod_{i=1}^{m} \left[\int_{S^{n-1}} c^{(\tau)}(K_1, u) \cdots c^{(\tau)}(K_{n-m}, u) c^{(\tau)}(K_{n-i+1}, u)^m dS(u) \right]^{1/m}. \end{aligned}$$

By definition (1.2), this yields

$$C^{(\tau)}(K_1, \cdots, K_n)^m \le \prod_{i=1}^{m} C^{(\tau)}(K_1, \dots, K_{n-m}, K_{n-i+1}, \dots, K_{n-i+1}).$$

From the equality condition of inequality (4.2), we see that equality holds in (1.8) if and only if $K_{n-m+1}, \dots, K_n$ are all of a similar general chord. $\square$

Proof of Theorem 1.3. From Hölder's inequality (see **[17]**), it follows that:

$$\begin{aligned} &C_i^{(\tau)}(K, L)^{(k-j)/(k-i)} C_k^{(\tau)}(K, L)^{(j-i)/(k-i)} \\ &\qquad = \left(\frac{1}{n} \int_{S^{n-1}} c^{(\tau)}(K, u)^{n-i} c^{(\tau)}(L, u)^i dS(u) \right)^{(k-j)/(k-i)} \end{aligned}$$

$$\times \left(\frac{1}{n}\int_{S^{n-1}} c^{(\tau)}(K,u)^{n-k}c^{(\tau)}(L,u)^k dS(u)\right)^{(j-i)/(k-i)}$$
$$\geq \frac{1}{n}\int_{S^{n-1}} c^{(\tau)}(K,u)^{n-j}c^{(\tau)}(L,u)^j dS(u) = C_j^{(\tau)}(K,L),$$

i.e.,

$$C_i^{(\tau)}(K,L)^{k-j}C_k^{(\tau)}(K,L)^{j-i} \geq C_j^{(\tau)}(K,L)^{k-i}.$$

From the equality condition of Hölder's inequality, we see that equality holds in (1.9) if and only if K and L have a similar general chord. □

If $i = 0$, $j = i$ and $k = n$ in inequality (1.9), then we have the following fact.

Corollary 4.2. *If $\tau \in (-1,1)$ and $K, L \in \mathcal{S}_o^n$, then, for $0 \leq i \leq n$,*

$$C_i^{(\tau)}(K,L)^n \leq C^{(\tau)}(K)^{n-i}C^{(\tau)}(L)^i, \tag{4.3}$$

for $i < 0$ or $i > n$, inequality (4.3) is reversed, with equality in every inequality if and only if $i = n$ or, when $i \neq n$, K and L have a similar general chord.

If we take $i = 1$ and $i = -1$ in Corollary 4.2, then we get the following versions of the dual Minkowski inequalities for general mixed chord-integrals.

Corollary 4.3. *If $\tau \in (-1,1)$ and $K, L \in \mathcal{S}_o^n$, then*

$$C_1^{(\tau)}(K,L)^n \leq C^{(\tau)}(K)^{n-1}C^{(\tau)}(L),$$

with equality if and only if K and L have a similar general chord.

Corollary 4.4. *If $\tau \in (-1,1)$ and $K, L \in \mathcal{S}_o^n$, then*

$$C_{-1}^{(\tau)}(K,L)^n \geq C^{(\tau)}(K)^{n+1}C^{(\tau)}(L)^{-1},$$

with equality if and only if K and L have a similar general chord.

5. Brunn-Minkowski type inequalities. This section is dedicated to the study of Brunn-Minkowski inequalities for general chord-integrals of order i. Log-concavity properties of geometric functionals have long played an important role in analysis and geometry (see **[2, 4, 32, 35, 41, 42]** for some recent results). We first establish the following Brunn-Minkowski type inequality for general chord-integrals of order i with respect to the radial Minkowski addition.

Theorem 5.1. *If $\tau \in (-1,1)$ and $K, L \in \mathcal{S}_o^n$, then, for $i \leq n-1$,*

$$C_i^{(\tau)}(K \widetilde{+} L)^{1/(n-i)} \leq C_i^{(\tau)}(K)^{1/(n-i)} + C_i^{(\tau)}(L)^{1/(n-i)}; \tag{5.1}$$

and, for $i > n$,

$$C_i^{(\tau)}(K \widetilde{+} L)^{1/(n-i)} \geq C_i^{(\tau)}(K)^{1/(n-i)} + C_i^{(\tau)}(L)^{1/(n-i)}, \tag{5.2}$$

with equality in each inequality if and only if K and L have a similar general chord.

Proof. We first prove inequality (5.1). For $i \leq n-1$, it follows from Minkowski's inequality (see **[17]**) that

$$\begin{aligned}
C_i^{(\tau)}(K \widetilde{+} L)^{1/(n-i)} &= \left(\frac{1}{n}\int_{S^{n-1}} c^{(\tau)}(K \widetilde{+} L, u)^{n-i} dS(u)\right)^{1/(n-i)} \\
&= \left(\frac{1}{n}\int_{S^{n-1}} (c^{(\tau)}(K,u) + c^{(\tau)}(L,u))^{n-i} dS(u)\right)^{1/(n-i)} \\
&\leq \left(\frac{1}{n}\int_{S^{n-1}} c^{(\tau)}(K,u)^{n-i} dS(u)\right)^{1/(n-i)} \\
&\quad + \left(\frac{1}{n}\int_{S^{n-1}} c^{(\tau)}(L,u)^{n-i} dS(u)\right)^{1/(n-i)} \\
&= C_i^{(\tau)}(K)^{1/(n-i)} + C_i^{(\tau)}(L)^{1/(n-i)}.
\end{aligned}$$

This gives inequality (5.1). Similarly, Minkowski's inequality yields inequality (5.2).

From the equality conditions of Minkowski's inequality, we see that equality holds in inequalities (5.1) and (5.2) if and only if K and L have a similar general chord. $\square$

Theorem 5.2. *If $K, L \in \mathcal{S}_o^n$ and $\tau \in (-1,1)$, $i, j \in \mathbb{R}$, then, for $i \le n-1 \le j \le n$,*

(5.3)
$$\left(\frac{C_i^{(\tau)}(I(K\breve{+}L))}{C_j^{(\tau)}(I(K\breve{+}L))}\right)^{1/(j-i)} \le \left(\frac{C_i^{(\tau)}(IK)}{C_j^{(\tau)}(IK)}\right)^{1/(j-i)} + \left(\frac{C_i^{(\tau)}(IL)}{C_j^{(\tau)}(IL)}\right)^{1/(j-i)};$$

for $j \ge n \ge i \ge n-1$,

(5.4)
$$\left(\frac{C_i^{(\tau)}(I(K\breve{+}L))}{C_j^{(\tau)}(I(K\breve{+}L))}\right)^{1/(j-i)} \ge \left(\frac{C_i^{(\tau)}(IK)}{C_j^{(\tau)}(IK)}\right)^{1/(j-i)} + \left(\frac{C_i^{(\tau)}(IL)}{C_j^{(\tau)}(IL)}\right)^{1/(j-i)},$$

with equality in each inequality if and only if IK and IL have a similar general chord.

In order to prove Theorem 5.2, the following lemmas are required. An extension of Beckenbach's inequality (see **[3]**) was obtained by Dresher **[5]** through the means of moment-space techniques.

Lemma 5.3 (The Beckenbach-Dresher inequality)**.** *If $p \ge 1 \ge r \ge 0$, $f, g \ge 0$, and ϕ is a distribution function, then*

(5.5)
$$\left(\frac{\int_{\mathbb{E}}(f+g)^p d\phi}{\int_{\mathbb{E}}(f+g)^r d\phi}\right)^{1/(p-r)} \le \left(\frac{\int_{\mathbb{E}} f^p d\phi}{\int_{\mathbb{E}} f^r d\phi}\right)^{1/(p-r)} + \left(\frac{\int_{\mathbb{E}} g^p d\phi}{\int_{\mathbb{E}} g^r d\phi}\right)^{1/(p-r)},$$

with equality if and only if the functions f and g are positively proportional.

Here $\mathbb{E}$ is a bounded measurable subset in $\mathbb{R}^n$.

Moreover, the inverse Beckenbach-Dresher inequality was obtained by Li and Zhao **[18]**.

Lemma 5.4 (The inverse Beckenbach-Dresher inequality)**.** *If $r \le 0 \le p \le 1$, $f, g \ge 0$, and ϕ is a distribution function, then*

(5.6)
$$\left(\frac{\int_{\mathbb{E}}(f+g)^p d\phi}{\int_{\mathbb{E}}(f+g)^r d\phi}\right)^{1/(p-r)} \ge \left(\frac{\int_{\mathbb{E}} f^p d\phi}{\int_{\mathbb{E}} f^r d\phi}\right)^{1/(p-r)} + \left(\frac{\int_{\mathbb{E}} g^p d\phi}{\int_{\mathbb{E}} g^r d\phi}\right)^{1/(p-r)},$$

with equality if and only if the functions f and g are positively proportional.

Proof of Theorem 5.2. From equations (1.2) and (2.5), it follows that, for $p \geq 1 \geq r \geq 0$,

$$\begin{aligned} C_{n-p}^{(\tau)}(I(K\check{+}L)) &= \frac{1}{n}\int_{S^{n-1}} c^{(\tau)}(I(K\check{+}L),u)^p dS(u) \\ &= \frac{1}{n}\int_{S^{n-1}} c^{(\tau)}(IK\tilde{+}IL,u)^p dS(u) \\ &= \frac{1}{n}\int_{S^{n-1}} (c^{(\tau)}(IK,u)+c^{(\tau)}(IL,u))^p dS(u). \end{aligned} \tag{5.7}$$

Similarly,

$$C_{n-r}^{(\tau)}(I(K\check{+}L)) = \frac{1}{n}\int_{S^{n-1}} (c^{(\tau)}(IK,u)+c^{(\tau)}(IL,u))^r dS(u). \tag{5.8}$$

Combining equations (5.7) and (5.8), we obtain, using Lemma 5.3,

$$\begin{aligned} &\left(\frac{C_{n-p}^{(\tau)}(I(K\check{+}L))}{C_{n-r}^{(\tau)}(I(K\check{+}L))}\right)^{1/(p-r)} \\ &\quad= \left(\frac{\int_{S^{n-1}}(c^{(\tau)}(IK,u)+c^{(\tau)}(IL,u))^p dS(u)}{\int_{S^{n-1}}(c^{(\tau)}(IK,u)+c^{(\tau)}(IL,u))^r dS(u)}\right)^{1/(p-r)} \\ &\quad\leq \left(\frac{\int_{S^{n-1}} c^{(\tau)}(IK,u)^p dS(u)}{\int_{S^{n-1}} c^{(\tau)}(IK,u)^r dS(u)}\right)^{1/(p-r)} \\ &\qquad+ \left(\frac{\int_{S^{n-1}} c^{(\tau)}(IL,u)^p dS(u)}{\int_{S^{n-1}} c^{(\tau)}(IL,u)^r dS(u)}\right)^{1/(p-r)} \\ &\quad= \left(\frac{C_{n-p}^{(\tau)}(IK)}{C_{n-r}^{(\tau)}(IK)}\right)^{1/(p-r)} + \left(\frac{C_{n-p}^{(\tau)}(IL)}{C_{n-r}^{(\tau)}(IL)}\right)^{1/(p-r)}. \end{aligned} \tag{5.9}$$

Suppose $p = n-i$ and $r = n-j$. It follows from $0 \leq r \leq 1 \leq p$ that $i \leq n-1 \leq j \leq n$. Taking $p = n-i$ and $r = n-j$ in (5.9), this yields the desired inequality (5.3). Using the same method, and Lemma 5.4 instead, we obtain inequality (5.4).

From the equality conditions of inequalities (5.5) and (5.6), we see that equality holds in inequalities (5.3) and (5.4) if and only if

$c^{(\tau)}(IK,u)$ and $c^{(\tau)}(IL,u)$ are positively proportional, that is, IK and IL have a similar general chord. $\square$

If $j=n$ in (5.3), then $C_n^{(\tau)}(I(K\check{+}L))=C_n^{(\tau)}(IK)=C_n^{(\tau)}(IL)=\omega_n$ is a constant, and we obtain the following result.

Corollary 5.5. *If $\tau\in(-1,1)$ and $K,L\in\mathcal{S}_o^n$, then, for $i\le n-1$,*

$$C_i^{(\tau)}(I(K\check{+}L))^{1/(n-i)}\le C_i^{(\tau)}(IK)^{1/(n-i)}+C_i^{(\tau)}(IL)^{1/(n-i)},$$

with equality if and only if IK and IL have similar general chords.

An inequality of Giannopoulos et al. **[12]** states that, if K is a convex body and L is an n-ball in $\mathbb{R}^n$, then, for $k=0,\ldots,n-1$,

$$\frac{W_k(K+L)}{W_{k+1}(K+L)}\ge\frac{W_k(K)}{W_{k+1}(K)}+\frac{W_k(L)}{W_{k+1}(L)}. \tag{5.10}$$

However, inequality (5.10) does not hold for an arbitrary pair of nonempty compact convex sets K and L. Also, (5.10) only holds if $k=n-2$ or $k=n-1$ **[7]**.

In the following, we will prove two analogous inequalities for general chord-integrals of order i.

Theorem 5.6. *If $\tau\in(-1,1)$, and K,L is an arbitrary pair of star bodies in $\mathbb{R}^n$, then, for $k=n-2$ or $k=n-1$,*

$$\frac{C_k^{(\tau)}(K\widetilde{+}L)}{C_{k+1}^{(\tau)}(K\widetilde{+}L)}\le\frac{C_k^{(\tau)}(K)}{C_{k+1}^{(\tau)}(K)}+\frac{C_k^{(\tau)}(L)}{C_{k+1}^{(\tau)}(L)}. \tag{5.11}$$

Proof. Let $k=n-2$, and let $\mathcal{B}=(B,\ldots,B)$ be an $(n-2)$-tuple of the unit ball B. It follows from Theorem 1.2 that, for all $t,s\ge 0$,

$$C^{(\tau)}(K\widetilde{+}sB,L\widetilde{+}tB,\mathcal{B})^2-C_{n-2}^{(\tau)}(K\widetilde{+}sB)C_{n-2}^{(\tau)}(L\widetilde{+}tB)\le 0.$$

Since general mixed chord-integrals are multilinear with respect to the radial Minkowski linear combination, we obtain

$$\begin{aligned}s^2\left[C_{n-1}^{(\tau)}(L)^2-\omega_nC_{n-2}^{(\tau)}(L)\right]&+2st\left[\omega_nC^{(\tau)}(K,L,\mathcal{B})-C_{n-1}^{(\tau)}(K)C_{n-1}^{(\tau)}(L)\right]\\&+t^2\left[C_{n-1}^{(\tau)}(K)^2-\omega_nC_{n-2}^{(\tau)}(K)\right]+g(s,t)\le 0,\end{aligned}$$

where $g(s,t)$ is a linear function of s and t. From Theorem 1.2, we know that

$$C_{n-1}^{(\tau)}(K)^2 - \omega_n C_{n-2}^{(\tau)}(K) \le 0 \tag{5.12}$$

and

$$C_{n-1}^{(\tau)}(L)^2 - \omega_n C_{n-2}^{(\tau)}(L) \le 0. \tag{5.13}$$

Together with (5.12), it follows that either

$$\omega_n C^{(\tau)}(K, L, \mathcal{B}) - C_{n-1}^{(\tau)}(K) C_{n-1}^{(\tau)}(L) \le 0, \tag{5.14}$$

or

$$\begin{aligned}\Big[\omega_n C^{(\tau)}(K, L, \mathcal{B}) &- C_{n-1}^{(\tau)}(K) C_{n-1}^{(\tau)}(L)\Big]^2 \\ &\le \Big[C_{n-1}^{(\tau)}(K)^2 - \omega_n C_{n-2}^{(\tau)}(K)\Big]\Big[C_{n-1}^{(\tau)}(L)^2 - \omega_n C_{n-2}^{(\tau)}(L)\Big].\end{aligned} \tag{5.15}$$

Using Theorem 1.2 again, we obtain for $s, t \ge 0$,

$$C_{n-1}^{(\tau)}(sK \widetilde{+} tL)^2 - \omega_n C_{n-2}^{(\tau)}(sK \widetilde{+} tL) \le 0. \tag{5.16}$$

Using the multilinearity of general mixed chord-integrals we obtain from (5.16):

$$\begin{aligned} s^2 &\Big[C_{n-1}^{(\tau)}(L)^2 - \omega_n C_{n-2}^{(\tau)}(L)\Big] \\ &+ 2st \Big[C_{n-1}^{(\tau)}(K) C_{n-1}^{(\tau)}(L) - \omega_n C^{(\tau)}(K, L, \mathcal{B})\Big] \\ &\qquad + t^2 \Big[C_{n-1}^{(\tau)}(K)^2 - \omega_n C_{n-2}^{(\tau)}(K)\Big] \le 0. \end{aligned} \tag{5.17}$$

By equations (5.12) and (5.14), we know that, if (5.17) holds, then the discriminant of the above quadratic form (5.17) is non-positive. Thus, equation (5.15) always holds. Now using equation (5.15), it follows from the arithmetic geometric means inequality that

$$\begin{aligned} \omega_n C^{(\tau)}(K, L, \mathcal{B}) &- C_{n-1}^{(\tau)}(K) C_{n-1}^{(\tau)}(L) \\ &\le \Big[\omega_n C_{n-2}^{(\tau)}(K) - C_{n-1}^{(\tau)}(K)^2\Big]^{1/2} \Big[\omega_n C_{n-2}^{(\tau)}(L) - C_{n-1}^{(\tau)}(L)^2\Big]^{1/2} \\ &\le \frac{1}{2} \frac{C_{n-1}^{(\tau)}(L)}{C_{n-1}^{(\tau)}(K)} \Big[\omega_n C_{n-2}^{(\tau)}(K) - C_{n-1}^{(\tau)}(K)^2\Big] \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2}\frac{C^{(\tau)}_{n-1}(K)}{C^{(\tau)}_{n-1}(L)}\left[\omega_n C^{(\tau)}_{n-2}(L)-C^{(\tau)}_{n-1}(L)^2\right]\\
=&\ \frac{\omega_n}{2}\left[\frac{C^{(\tau)}_{n-1}(L)}{C^{(\tau)}_{n-1}(K)}C^{(\tau)}_{n-2}(K)+\frac{C^{(\tau)}_{n-1}(K)}{C^{(\tau)}_{n-1}(L)}C^{(\tau)}_{n-2}(L)\right]\\
&-C^{(\tau)}_{n-1}(K)C^{(\tau)}_{n-1}(L),
\end{aligned}
$$

that is,

$$
2C^{(\tau)}(K,L,\mathcal{B})\le\frac{C^{(\tau)}_{n-1}(L)}{C^{(\tau)}_{n-1}(K)}C^{(\tau)}_{n-2}(K)+\frac{C^{(\tau)}_{n-1}(K)}{C^{(\tau)}_{n-1}(L)}C^{(\tau)}_{n-2}(L). \tag{5.18}
$$

By the multilinearity of general mixed chord-integrals and inequality (5.18), we infer

$$
\begin{aligned}
C^{(\tau)}_{n-2}(K\widetilde{+}L)&=C^{(\tau)}_{n-2}(K)+C^{(\tau)}_{n-2}(L)+2C^{(\tau)}(K,L,\mathcal{B})\\
&\le C^{(\tau)}_{n-2}(K)+C^{(\tau)}_{n-2}(L)+\frac{C^{(\tau)}_{n-1}(L)}{C^{(\tau)}_{n-1}(K)}C^{(\tau)}_{n-2}(K)\\
&\quad+\frac{C^{(\tau)}_{n-1}(K)}{C^{(\tau)}_{n-1}(L)}C^{(\tau)}_{n-2}(L)\\
&=\left[\frac{C^{(\tau)}_{n-2}(K)}{C^{(\tau)}_{n-1}(K)}+\frac{C^{(\tau)}_{n-2}(L)}{C^{(\tau)}_{n-1}(L)}\right]\left(C^{(\tau)}_{n-1}(K)+C^{(\tau)}_{n-1}(L)\right),
\end{aligned}
$$

that is,

$$
\frac{C^{(\tau)}_{n-2}(K\widetilde{+}L)}{C^{(\tau)}_{n-1}(K\widetilde{+}L)}\le\frac{C^{(\tau)}_{n-2}(K)}{C^{(\tau)}_{n-1}(K)}+\frac{C^{(\tau)}_{n-2}(L)}{C^{(\tau)}_{n-1}(L)}.
$$

For the case $k=n-1$, note that since $C^{(\tau)}_{n}(K\widetilde{+}L)=C^{(\tau)}_{n}(K)=C^{(\tau)}_{n}(L)=\omega_n$, Theorem 5.6 reduces to the inequality

$$
C^{(\tau)}_{n-1}(K\widetilde{+}L)\le C^{(\tau)}_{n-1}(K)+C^{(\tau)}_{n-1}(L),
$$

which holds for every pair of star bodies. □

Theorem 5.7. *Let K be a star body and L an n-ball in $\mathbb{R}^n$ and $\tau \in (-1,1)$. Then, for all $k = 0,\dots,n-1$,*

$$\frac{C_k^{(\tau)}(K\widetilde{+}L)}{C_{k+1}^{(\tau)}(K\widetilde{+}L)} \le \frac{C_k^{(\tau)}(K)}{C_{k+1}^{(\tau)}(K)} + \frac{C_k^{(\tau)}(L)}{C_{k+1}^{(\tau)}(L)}. \tag{5.19}$$

Proof. Let $L = tB$ for $t \ge 0$ and define, for every $k = 0,1,\dots,n$,

$$f_k(s) = C_k^{(\tau)}(K\widetilde{+}sB).$$

From the multilinearity of general mixed chord-integrals, it follows that

$$\begin{aligned} f_k(s+\varepsilon) &= C_k^{(\tau)}(K\widetilde{+}sB\widetilde{+}\varepsilon B) \\ &= C_k^{(\tau)}(K\widetilde{+}sB) + \varepsilon(n-k)C_{k+1}^{(\tau)}(K\widetilde{+}sB) + O(\varepsilon^2) \\ &= f_k(s) + \varepsilon(n-k)f_{k+1}(s) + O(\varepsilon^2). \end{aligned}$$

Therefore,

$$f_k'(s) = (n-k)f_{k+1}(s).$$

Apply Theorem 1.2 to get, for $k = 0,1,\dots,n-2$,

$$C_{k+1}^{(\tau)}(K\widetilde{+}sB)^2 \le C_k^{(\tau)}(K\widetilde{+}sB)C_{k+2}^{(\tau)}(K\widetilde{+}sB),$$

i.e.,

$$f_{k+1}(s)^2 \le f_k(s)f_{k+2}(s).$$

Define

$$F_k(s) = \frac{f_k(s)}{f_{k+1}(s)} \quad (k = 0,1,\dots,n-2). \tag{5.20}$$

It follows that

$$\begin{aligned} F_k'(s) &= \frac{f_k'(s)f_{k+1}(s) - f_k(s)f_{k+1}'(s)}{f_{k+1}(s)^2} \\ &= \frac{f_{k+1}(s)^2 + (n-k-1)(f_{k+1}(s)^2 - f_k(s)f_{k+2}(s))}{f_{k+1}(s)^2} \le 1; \end{aligned} \tag{5.21}$$

thus,

$$F_k(t) \le F_k(0) + t. \tag{5.22}$$

Since

$$\frac{C_k^{(\tau)}(L)}{C_{k+1}^{(\tau)}(L)} = \frac{C_k^{(\tau)}(tB)}{C_{k+1}^{(\tau)}(tB)} = t, \tag{5.23}$$

it follows from equations (5.20), (5.22) and (5.23) that, for $k = 0, 1, \ldots, n-2$,

$$\frac{C_k^{(\tau)}(K \tilde{+} L)}{C_{k+1}^{(\tau)}(K \tilde{+} L)} \leq \frac{C_k^{(\tau)}(K)}{C_{k+1}^{(\tau)}(K)} + \frac{C_k^{(\tau)}(L)}{C_{k+1}^{(\tau)}(L)}.$$

When $k = n-1$, inequality (5.19) always holds as an equality. $\square$

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