

G-FRAMES WITH BOUNDED LINEAR OPERATORS

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ABSTRACT. In this paper, we introduce the more general g -frame which is called a K - g -frame by combining a g -frame with a bounded linear operator K in a Hilbert space. We give several equivalent characterizations for K - g -frames and discuss the stability of perturbation for K - g -frames. We also investigate the relationship between a K - g -frame and the range of the bounded linear operator K . In the end, we give two sufficient conditions for the remainder of a K - g -frame after an erasure to still be a K - g -frame. It turns out that although K - g -frames share some properties similar to g -frames, a large part of K - g -frames behaves completely different from g -frames.

1. Introduction. Ordinary frames for Hilbert spaces were first introduced by Duffin and Schaeffer [9] in 1952 for studying some deep problems in nonharmonic Fourier series. Recall that a sequence $\{f_i\}_{i \in I}$ in a Hilbert space H is called an ordinary frame for H , if there exist two constants $A, B > 0$ such that

$$A\|f\|^2 \leq \sum_{i \in I} |\langle f, f_i \rangle|^2 \leq B\|f\|^2, \quad \text{for all } f \in H.$$

Due to the fundamental work [5] done by Daubechies, Grossmann and Meyer in 1986, ordinary frames were reintroduced, developed, and popularized from then on. Now the theory of ordinary frames plays an important role in theoretics and applications; it has been applied extensively in signal and image processing [2], sampling theory [10, 21], filter bank theory [1], system modeling [8], coding and

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communications [13], etc. For more details on ordinary frames readers can consult [3, 4].

One of the main properties of a frame which is different from an orthonormal basis is that the frames provide reconstruction formulas where the coefficients are not necessarily unique. In order to study the atomic system with an operator (actually a kind of reconstruction), Gavruta [12] recently introduced a frame with respect to a bounded linear operator K in a Hilbert space H ; for convenience, we call it a K -frame, to reconstruct the elements in the range of K . Recall that $\{f_i\}_{i \in I} \subset H$ is called a K -frame for H , if there exist two constants $A, B > 0$ such that

$$A\|K^*f\|^2 \leq \sum_{i \in I} |\langle f, f_i \rangle|^2 \leq B\|f\|^2, \quad \text{for all } f \in H.$$

In fact, a K -frame is a more general version of the ordinary frame; the K -frame is equivalent to the ordinary frame only when $K = I_H$. Due to the bounded linear operator K , there are so many differences between a K -frame and an ordinary frame. $\{f_i\}_{i \in I}$ is an ordinary frame if and only if the corresponding synthesis operator is bounded and surjective, but for the K -frame, it is very different. K -frame $\{f_i\}_{i \in I} \subset H$ equals that the corresponding synthesis operator T is bounded and the range of K belongs to the range of T ([12, Theorem 4]). We also discover that the positions of the two sequences related to a K -frame dual ([12, Theorem 3 (iii)]) are not interchanged in general in [25]. Just because the properties of K -frames are so different from the ordinary frames', this inspires us to combine the bounded linear operator K with the more complicated g -frame, which we call the K - g -frame. Compared with [12, 25], we will first discuss the excess of K - g -frames in this paper.

The g -frame in a Hilbert space was first proposed by Sun [19] using a sequence of bounded linear operators to deal with all the existing frames as a united object. In fact, the g -frame is an extension of ordinary frames, bounded invertible linear operators, as well as many new appeared generalizations of frames, e.g., bounded quasi-projectors and fusion frames, etc. Recall that a sequence $\{\Lambda_j : j \in J\}$ is called a g -frame for \mathcal{U} with respect to $\{\mathcal{V}_j : j \in J\}$, if there exist two positive

constants A, B such that

$$A\|f\|^2 \leq \sum_{j \in J} \|\Lambda_j f\|^2 \leq B\|f\|^2, \quad \text{for all } f \in \mathcal{U},$$

where $\mathcal{U}, \mathcal{V}_j$ are Hilbert spaces and Λ_j , $j \in J$, are bounded linear operators from \mathcal{U} into \mathcal{V}_j . From [16, 17, 19, 20], we know that, though g-frames share many nice properties of the previous frames, not all the results of the previous frames can be generalized to g-frames, e.g., in Hilbert spaces exact g-frames are not the g-Riesz bases [19], a g-Riesz frame does not contain a g-Riesz basis [17], etc. Even though some properties of the frames can be generalized to g-frames, the techniques are more complicated. We refer the reader to the papers [14, 15, 22, 23, 24, 26, 27, 28] for more information about g-frames and their generalizations.

In the rest of this section we introduce the organization of this paper and some basic notation.

This paper is organized as follows. In Section 2, we review some basic contents of a K -g-frame. In Section 3, we use the induced sequence $\{u_{j,k} : j \in J, k \in K_j\}$, the synthesis operator and atomic systems, respectively, to equivalently characterize the K -g-frames. In Section 4, we mainly discuss the stability of perturbations for a K -g-frame, we give two kinds of versions of the stability of perturbations for a K -g-frame. In Section 5, we discuss the relationship among two K -g-frames and the range inclusion of the related two operators. In Section 6, we study the excess of K -g-frames.

Throughout this paper, we adopt such notation: \mathcal{U} and \mathcal{V} are Hilbert spaces, with inner product $\langle \cdot, \cdot \rangle$, and norm $\|\cdot\|$; $L(\mathcal{U}, \mathcal{V})$ is denoted by the collection of all the linear bounded operators from \mathcal{U} to \mathcal{V} ; if $\mathcal{U} = \mathcal{V}$, then $L(\mathcal{U}, \mathcal{V})$ is abbreviated as $L(\mathcal{U})$; $0 \neq K \in L(\mathcal{U})$; if $Q \in L(\mathcal{U}, \mathcal{V})$, $R(Q)$ and $N(Q)$ are denoted by the range and null space of Q , respectively; $\{\mathcal{V}_j\}_{j \in J}$ is a sequence of closed subspaces of \mathcal{V} , where J is a subset of the integer set \mathbb{Z} .

2. Preliminaries of K -g-frames. In this section, we first introduce a more general version of the g-frame with respect to a linear bounded operator K in H , and we call it the K -g-frame. The generality for the K -g-frame is mainly in that only part of the elements in H (in fact, the elements in $R(K)$) play a role in the lower frame

bound; the other elements in $R(K)^\perp \subset H$ are turned into zeros by the bounded linear operator K^* .

Definition 2.1. A sequence $\{\Lambda_j \in L(\mathcal{U}, \mathcal{V}_j) : j \in J\}$ is called a K -g-frame for \mathcal{U} with respect to (w.r.t.) $\{\mathcal{V}_j : j \in J\}$, if there exist $A, B > 0$ such that

$$(2.1) \quad A\|K^*f\|^2 \leq \sum_{j \in J} \|\Lambda_j f\|^2 \leq B\|f\|^2, \quad \text{for all } f \in \mathcal{U}.$$

We call A, B the lower frame bound and upper frame bound for the K -g-frame $\{\Lambda_j : j \in J\}$, respectively. We call $\{\Lambda_j : j \in J\}$ a g-Bessel sequence if only the right-hand of (2.1) holds.

If $A\|K^*f\|^2 = \sum_{j \in J} \|\Lambda_j f\|^2$, for all $f \in \mathcal{U}$, we call $\{\Lambda_j : j \in J\}$ a tight K -g-frame; moreover, if $A = 1$, $\{\Lambda_j : j \in J\}$ is called a Parseval K -g-frame.

Example 2.2. Suppose that $\{e_n\}_{n=1}^\infty$ is an orthonormal basis for \mathcal{U} . Let

$$\mathcal{V}_j = \overline{\text{span}} \{e_{(j-1)m+k} : 1 \leq k \leq m\}, \quad j = 1, 2, 3, \dots,$$

where $m \geq 3$ is a fixed positive integer. Now define the linear bounded operators $K : \mathcal{U} \rightarrow \mathcal{U}$ and $\Lambda_j : \mathcal{U} \rightarrow \mathcal{V}_j$ as follows:

$$Ke_1 = e_1, \quad Ke_2 = e_2, \quad Ke_j = 0, \quad j > 2.$$

$$\Lambda_1 f = \sum_{k=1}^m \langle f, e_k \rangle e_k; \quad \Lambda_j f = 0, \quad j \geq 2.$$

It is easy to calculate $K^*e_1 = e_1$, $K^*e_2 = e_2$, $K^*e_j = 0$, $j > 2$. Next we show that $\{\Lambda_j\}_{j=1}^\infty$ is a K -g-frame. In fact, for any $f \in \mathcal{U}$, we have

$$\begin{aligned} \|K^*f\|^2 &= \left\| \sum_{j=1}^\infty \langle f, e_j \rangle K^*e_j \right\|^2 = |\langle f, e_1 \rangle|^2 + |\langle f, e_2 \rangle|^2, \\ \sum_{j=1}^\infty \|\Lambda_j f\|^2 &= \|\Lambda_1 f\|^2 = \left\| \sum_{k=1}^m \langle f, e_k \rangle e_k \right\|^2 = \sum_{k=1}^m |\langle f, e_k \rangle|^2 \geq \|K^*f\|^2. \end{aligned}$$

So we have, for any $f \in \mathcal{U}$,

$$\|K^*f\|^2 \leq \sum_{j=1}^{\infty} \|\Lambda_j f\|^2 \leq \|f\|^2.$$

Next we can show that $\{\Lambda_j\}_{j=1}^{\infty}$ is not a g -frame. In fact, if we take $f = e_{m+1}$, then $\|f\|^2 = 1$, but $\sum_{j=1}^{\infty} \|\Lambda_j f\|^2 = \|\Lambda_1 e_{m+1}\|^2 = 0$. \square

To proceed with this section we need to define a basic space $l^2(\{\mathcal{V}_j\}_{j \in J})$ as follows:

$$l^2(\{\mathcal{V}_j\}_{j \in J}) = \left\{ \{g_j\}_{j \in J} : g_j \in \mathcal{V}_j, j \in J \text{ and } \sum_{j \in J} \|g_j\|^2 < +\infty \right\},$$

with the inner product

$$\langle \{f_j\}_{j \in J}, \{g_j\}_{j \in J} \rangle = \sum_{j \in J} \langle f_j, g_j \rangle.$$

It is trivial to show that $l^2(\{\mathcal{V}_j\}_{j \in J})$ is a Hilbert space.

Suppose that $\{\tilde{e}_{jk}\}_{k \in K_j}$ is an orthonormal basis for \mathcal{V}_j , where K_j is a subset of \mathbb{Z} . For any $j \in J$, $k \in K_j$, e_{ik} is defined by $e_{ik} = \{\delta_{ij} \tilde{e}_{jk}\}_{j \in J}$, where δ_{ik} is the Kronecker delta. It is easy to check that $\{e_{ik} : i \in J, k \in K_j\}$ is an orthonormal basis for $l^2(\{\mathcal{V}_j\}_{j \in J})$.

We are ready to introduce a sequence induced by a K - g -frame and the analysis operator, synthesis operator and frame operator for a K - g -frame.

Suppose that $\{\Lambda_j \in L(\mathcal{U}, \mathcal{V}_j) : j \in J\}$ is a K - g -frame for \mathcal{U} w.r.t. $\{\mathcal{V}_j : j \in J\}$. A sequence $\{u_{j,k} : j \in J, k \in K_j\}$ induced by a K - g -frame $\{\Lambda_j : j \in J\}$ with respect to the orthonormal basis $\{e_{ik} : i \in J, k \in K_j\}$ for $l^2(\{\mathcal{V}_j\}_{j \in J})$ is defined as follows

$$u_{j,k} = \Lambda_j^* e_{j,k} : j \in J, k \in K_j.$$

Assume that $\{\Lambda_j \in L(\mathcal{U}, \mathcal{V}_j) : j \in J\}$ is a g -Bessel sequence. The analysis operator U , synthesis operator T and frame operator S of $\{\Lambda_j : j \in J\}$ are defined as follow:

$$(2.2) \quad U : \mathcal{U} \rightarrow l^2(\{\mathcal{V}_j\}_{j \in J}), \quad Uf = \{\Lambda_j f\}_{j \in J},$$

$$(2.3) \quad T : l^2(\{\mathcal{V}_j\}_{j \in J}) \longrightarrow \mathcal{U}, \quad T(\{g_j\}_{j \in J}) = \sum_{j \in J} \Lambda_j^* g_j,$$

$$(2.4) \quad S : \mathcal{U} \longrightarrow \mathcal{U}, \quad Sf = \sum_{j \in J} \Lambda_j^* \Lambda_j f,$$

It is trivial to check that $T^* = U, S = TU$.

3. Equivalent characterizations for K -g-frames. In this section, we will give the equivalent characterizations for K -g-frames by using the induced sequence $\{u_{j,k} : j \in J, k \in K_j\}$, the synthesis operator and atomic systems, respectively. To do this, we first need to introduce the concept of an atomic system for a linear bounded operator and cite two lemmas.

Definition 3.1. A sequence $\{\Lambda_j \in L(\mathcal{U}, \mathcal{V}_j) : j \in J\}$ is called an atomic system for K , if the following statements hold:

- (i) $\{\Lambda_j \in L(\mathcal{U}, \mathcal{V}_j) : j \in J\}$ is a g -Bessel sequence;
- (ii) for any $f \in \mathcal{U}$, there exists $\{g_j\}_{j \in J} \in l^2(\{\mathcal{V}_j\}_{j \in J})$ such that $Kf = \sum_{j \in J} \Lambda_j^* g_j$, where $\|\{g_j\}_{j \in J}\|_{l^2} \leq C\|f\|$, and C is a positive constant.

Lemma 3.2 ([7]). Assume that $T_1 \in L(\mathcal{H}_1, \mathcal{H})$, $T_2 \in L(\mathcal{H}_2, \mathcal{H})$, where $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}$ are Hilbert spaces. Then the following statements hold:

- (i) $R(T_1) \subset R(T_2)$;
- (ii) $T_1 T_1^* \leq \alpha T_2 T_2^*$, where $\alpha \geq 0$ is a constant;
- (iii) there exists a linear bounded operator $Q \in L(\mathcal{H}_1, \mathcal{H}_2)$ such that $T_1 = T_2 Q$.

Remark 3.3. If $T_1 \neq 0$, then $\alpha > 0$.

Lemma 3.4 ([22]). A sequence $\{\Lambda_j \in L(\mathcal{U}, \mathcal{V}_j) : j \in J\}$ is a g -Bessel sequence with upper bound B , if and only if the operator T defined by (2.3) is well defined and bounded, and $\|T\| \leq \sqrt{B}$.

In the following, we give an equivalent characterization for the K -g-frame by using the synthesis operator, which is completely different from the corresponding part of g-frames (see [28, Theorem 2.4]).

Theorem 3.5. *A sequence $\{\Lambda_j \in L(\mathcal{U}, \mathcal{V}_j) : j \in J\}$ is a K -g-frame for \mathcal{U} with respect to $\{\mathcal{V}_j : j \in J\}$, if and only if the operator T defined by (2.3) is well defined and bounded, and $R(K) \subset R(T)$.*

Proof. Suppose that $\{\Lambda_j \in L(\mathcal{U}, \mathcal{V}_j) : j \in J\}$ is a K -g-frame for \mathcal{U} w.r.t. $\{\mathcal{V}_j : j \in J\}$ with frame bounds A, B . Obviously, $\{\Lambda_j : j \in J\}$ is a g-Bessel sequence for \mathcal{U} with respect to $\{\mathcal{V}_j : j \in J\}$, so we can define the operator T as in (2.3), and we have

$$(3.1) \quad A\|K^*f\|^2 \leq \|T^*f\|^2 \leq B\|f\|^2, \quad \text{for all } f \in \mathcal{U}.$$

It follows that T is bounded, and $\|T\| \leq \sqrt{B}$. From (3.1), we also have

$$(3.2) \quad \langle AKK^*f, f \rangle = A\|K^*f\|^2 \leq \|T^*f\|^2 = \langle TT^*f, f \rangle$$

for any $f \in \mathcal{U}$, that is, $AKK^* \leq TT^*$, so we conclude that $R(K) \subset R(T)$ by Lemma 3.2.

On the converse, if the operator T as in (2.3) is well defined and bounded, we know that $\{\Lambda_j \in L(\mathcal{U}, \mathcal{V}_j) : j \in J\}$ is a g-Bessel sequence for \mathcal{U} with respect to $\{\mathcal{V}_j : j \in J\}$ by Lemma 3.4. And, since $R(K) \subset R(T)$, from Lemma 3.2 and Remark 3.3, we have $AKK^* \leq TT^*$. Combining this with (3.2), we obtain

$$A\|K^*f\|^2 \leq \|T^*f\|^2 = \sum_{j \in J} \|\Lambda_j f\|^2,$$

which implies that $\{\Lambda_j : j \in J\}$ is a K -g-frame for \mathcal{U} with respect to $\{\mathcal{V}_j : j \in J\}$. \square

Next we use the induced sequence $\{u_{j,k} : j \in J, k \in K_j\}$ to characterize K -g-frame equivalently.

Theorem 3.6. *A sequence $\{\Lambda_j \in L(\mathcal{U}, \mathcal{V}_j) : j \in J\}$ is a (tight) K -g-frame, if and only if $\{u_{j,k} = \Lambda_j^* e_{j,k} : j \in J, k \in K_j\}$ is a (tight) K -frame for \mathcal{U} .*

Proof. Suppose that $\{\tilde{e}_{jk}\}_{k \in K_j}$ is an orthonormal basis for \mathcal{V}_j , $j \in J$. Then, for any $f \in \mathcal{U}$ and $j \in J$, we have

$$(3.3) \quad \Lambda_j f = \sum_{k \in K_j} \langle \Lambda_j f, \tilde{e}_{jk} \rangle \tilde{e}_{jk} = \sum_{k \in K_j} \langle f, \Lambda_j^* \tilde{e}_{jk} \rangle \tilde{e}_{jk} = \sum_{k \in K_j} \langle f, u_{j,k} \rangle \tilde{e}_{jk}.$$

So, for any $f \in \mathcal{U}$, from (3.3), we have

$$\sum_{j \in J} \|\Lambda_j f\|^2 = \sum_{j \in J} \left\| \sum_{k \in K_j} \langle f, u_{j,k} \rangle \tilde{e}_{jk} \right\|^2 = \sum_{j \in J} \sum_{k \in K_j} |\langle f, u_{j,k} \rangle|^2.$$

It follows that $\{\Lambda_j : j \in J\}$ is a (tight) K -g-frame, if and only if $\{u_{j,k} : j \in J, k \in K_j\}$ is a (tight) K -frame for \mathcal{U} . \square

Remark 3.7. From the proof of Theorem 3.6, we know that the frame operators of $\{\Lambda_j : j \in J\}$ and $\{u_{j,k} : j \in J, k \in K_j\}$ are equal.

In the rest of this section we give equivalent characterizations for a K -g-frame by using atomic systems.

Theorem 3.8. Suppose that $\{\Lambda_j \in L(\mathcal{U}, \mathcal{V}_j) : j \in J\}$ is a g -Bessel sequence. Then the following statements are equivalent:

- (i) $\{\Lambda_j : j \in J\}$ is an atomic system for K ;
- (ii) $\{\Lambda_j Q \in L(\mathcal{U}, \mathcal{V}_j) : j \in J\}$ is an atomic system for linear bounded operator Q^*K , where Q is surjective on \mathcal{U} ;
- (iii) $\{\Lambda_j \in L(\mathcal{U}, \mathcal{V}_j) : j \in J\}$ is a K -g-frame;
- (iv) there exists another g -Bessel sequence $\{\Gamma_j \in L(\mathcal{U}, \mathcal{V}_j) : j \in J\}$ such that

$$(3.4) \quad Kf = \sum_{j \in J} \Lambda_j^* \Gamma_j f, \quad \text{for all } f \in \mathcal{U};$$

- (v) there exists another g -Bessel sequence $\{\Gamma_j \in L(\mathcal{U}, \mathcal{V}_j) : j \in J\}$ such that $K^*f = \sum_{j \in J} \Gamma_j^* \Lambda_j f$, for all $f \in \mathcal{U}$;
- (vi) for all $f, g \in \mathcal{U}$, there is $\langle Kf, g \rangle = \sum_{j \in J} \langle \Gamma_j f, \Lambda_j g \rangle$, where $\{\Gamma_j \in L(\mathcal{U}, \mathcal{V}_j) : j \in J\}$ is another g -Bessel sequence.

Proof. (i) \Rightarrow (ii). Assume that (i) holds. So, for any $f \in \mathcal{U}$, there exist $\{g_j\}_{j \in J} \in l^2(\{\mathcal{V}_j\}_{j \in J})$ and $C > 0$ such that

$$(3.5) \quad Kf = \sum_{j \in J} \Lambda_j^* g_j, \quad \|\{g_j\}_{j \in J}\| \leq C\|f\|.$$

Since Q is surjective on \mathcal{U} and $Kf \in \mathcal{U}$, we have

$$(3.6) \quad Q^*Kf = \sum_{j \in J} Q^* \Lambda_j^* g_j = \sum_{j \in J} (\Lambda_j Q)^* g_j.$$

Combining this with (3.5), we obtain that $\{\Lambda_j Q \in L(\mathcal{U}, \mathcal{V}_j) : j \in J\}$ is an atomic system for linear bounded operator Q^*K .

(ii) \Rightarrow (i). Assume that (ii) holds. So, for any $f \in \mathcal{U}$, there exist $\{g_j\}_{j \in J} \in l^2(\{\mathcal{V}_j\}_{j \in J})$ such that (3.6) holds, and $\|\{g_j\}_{j \in J}\| \leq C\|f\|$, where C is a positive constant. From (3.6), we obtain that

$$Q^*\left(Kf - \sum_{j \in J} \Lambda_j^* g_j\right) = 0, \quad \text{for all } f \in \mathcal{U}.$$

And, since Q is surjective on \mathcal{U} , so Q^* is injective on \mathcal{U} ; hence, we have

$$Kf - \sum_{j \in J} \Lambda_j^* g_j = 0, \quad \text{for all } f \in \mathcal{U},$$

it follows that $\{\Lambda_j : j \in J\}$ is an atomic system for K .

(i) \Rightarrow (iii). Assume that (i) holds. So, for any $h \in \mathcal{U}$, there exist $\{g_j\}_{j \in J} \in l^2(\{\mathcal{V}_j\}_{j \in J})$ and $C > 0$ such that (3.5) holds. So we have

$$\begin{aligned} \|K^*f\| &= \sup_{h \in \mathcal{U}, \|h\|=1} |\langle K^*f, h \rangle| = \sup_{h \in \mathcal{U}, \|h\|=1} |\langle f, Kh \rangle| \\ &= \sup_{h \in \mathcal{U}, \|h\|=1} \left| \langle f, \sum_{j \in J} \Lambda_j^* g_j \rangle \right| = \sup_{h \in \mathcal{U}, \|h\|=1} \left| \sum_{j \in J} \langle \Lambda_j f, g_j \rangle \right| \\ &\leq \sup_{h \in \mathcal{U}, \|h\|=1} \left(\sum_{j \in J} \|\Lambda_j f\|^2 \right)^{1/2} \cdot \left(\sum_{j \in J} \|g_j\|^2 \right)^{1/2} \\ &\leq \sup_{h \in \mathcal{U}, \|h\|=1} \left(\sum_{j \in J} \|\Lambda_j f\|^2 \right)^{1/2} \cdot C\|h\| = C \left(\sum_{j \in J} \|\Lambda_j f\|^2 \right)^{1/2}. \end{aligned}$$

That is, for any $f \in \mathcal{U}$, we have

$$\frac{1}{C^2} \|K^*f\|^2 \leq \sum_{j \in J} \|\Lambda_j f\|^2,$$

which implies that $\{\Lambda_j \in L(\mathcal{U}, \mathcal{V}_j) : j \in J\}$ is a K -g-frame.

(iii) \Rightarrow (iv). Assume that (iii) holds. So we can define the operator T as in (2.3). Moreover, we know that (3.1) holds, and it follows that $AKK^* \leq TT^*$. By Lemma 3.2, there exists a linear bounded operator $\Gamma : \mathcal{U} \rightarrow l^2(\{\mathcal{V}_j\}_{j \in J})$ such that $K = T\Gamma$. Define $\Gamma_j : \mathcal{U} \rightarrow \mathcal{V}_j$, $\Gamma_j f = (\Gamma f)_j$. It is easy to show that $\Gamma_j \in L(\mathcal{U}, \mathcal{V}_j)$, $j \in J$, and $\{\Gamma_j\}_{j \in J}$

is a g-Bessel sequence. In fact, for any $f \in \mathcal{U}$, $j \in J$, we have

$$\|\Gamma_j f\| \leq \|\{\Gamma_j f\}_{j \in J}\| = \|\Gamma f\| \leq \|\Gamma\| \cdot \|f\|,$$

and

$$\sum_{j \in J} \|\Gamma_j f\|^2 = \|\Gamma f\|^2 \leq \|\Gamma\|^2 \cdot \|f\|^2.$$

Moreover, we have

$$Kf = T\Gamma f = \sum_{j \in J} \Lambda_j^*(\Gamma f)_j = \sum_{j \in J} \Lambda_j^* \Gamma_j f \quad \text{for all } f \in \mathcal{U}.$$

It is trivial to prove (iv) \Leftrightarrow (v), (iv) \Leftrightarrow (vi) and (iv) \Rightarrow (i). \square

4. The stability of perturbations of K -g-frames. In this section, we give two kinds of versions of the stability of perturbations for a K -g-frame. Note that, in Theorem 4.2, we need $R(K)$ to be closed.

Theorem 4.1. *Let $\{\Lambda_j \in L(\mathcal{U}, \mathcal{V}_j) : j \in J\}$ be a K -g-frame for \mathcal{U} w.r.t. $\{\mathcal{V}_j : j \in J\}$, with frame bounds A, B . Let $\{c_j\}_{j \in J}$ be a sequence of positive numbers such that $\sum_{j \in J} c_j^2 < \infty$. Suppose that $\Gamma_j \in L(\mathcal{U}, \mathcal{V}_j)$, $j \in J$. If there exist $\alpha, \beta \in (-1, 1)$ such that $(1 - \alpha)\sqrt{A} > (\sum_{j \in J} c_j^2)^{1/2}$ and, for any $f \in \mathcal{U}$, $j \in J$,*

$$(4.1) \quad \|\Lambda_j f - \Gamma_j f\| \leq \alpha \|\Lambda_j f\| + \beta \|\Gamma_j f\| + c_j \|K^* f\|,$$

then $\{\Gamma_j \in L(\mathcal{U}, \mathcal{V}_j) : j \in J\}$ is a K -g-frame for \mathcal{U} w.r.t. $\{\mathcal{V}_j : j \in J\}$, with frame bounds

$$\left(\frac{(1 - \alpha)\sqrt{A} - \left(\sum_{j \in J} c_j^2\right)^{1/2}}{1 + \beta} \right)^2, \left(\frac{(1 + \alpha)\sqrt{B} + \|K\| \left(\sum_{j \in J} c_j^2\right)^{1/2}}{1 - \beta} \right)^2.$$

Proof. For any $f \in \mathcal{U}$, $j \in J$, from (4.1) we obtain

$$\|\Gamma_j f\| - \|\Lambda_j f\| \leq \|\Lambda_j f - \Gamma_j f\| \leq \alpha \|\Lambda_j f\| + \beta \|\Gamma_j f\| + c_j \|K^* f\|,$$

so we have

$$\|\Gamma_j f\| \leq \frac{1 + \alpha}{1 - \beta} \|\Lambda_j f\| + \frac{c_j \|K\|}{1 - \beta} \|f\|.$$

Hence, for any $f \in \mathcal{U}$, we get

$$\begin{aligned}
 \left(\sum_{j \in J} \|\Gamma_j f\|^2 \right)^{1/2} &\leq \left(\sum_{j \in J} \left(\frac{1+\alpha}{1-\beta} \|\Lambda_j f\| + \frac{c_j \|K\|}{1-\beta} \|f\| \right)^2 \right)^{1/2} \\
 &= \left\| \left\{ \frac{1+\alpha}{1-\beta} \|\Lambda_j f\| \right\}_{j \in J} + \left\{ \frac{\|K\|}{1-\beta} \|f\| c_j \right\}_{j \in J} \right\|_{l^2(J)} \\
 &\leq \left\| \left\{ \frac{1+\alpha}{1-\beta} \|\Lambda_j f\| \right\}_{j \in J} \right\|_{l^2(J)} \\
 &\quad + \left\| \left\{ \frac{\|K\|}{1-\beta} \|f\| c_j \right\}_{j \in J} \right\|_{l^2(J)} \\
 &= \frac{1+\alpha}{1-\beta} \left(\sum_{j \in J} \|\Lambda_j f\|^2 \right)^{1/2} + \frac{\|K\|}{1-\beta} \left(\sum_{j \in J} c_j^2 \right)^{1/2} \|f\| \\
 &\leq \frac{(1+\alpha)\sqrt{B} + \|K\| \left(\sum_{j \in J} c_j^2 \right)^{1/2}}{1-\beta} \|f\|,
 \end{aligned}$$

which implies that $\{\Gamma_j \in L(\mathcal{U}, \mathcal{V}_j) : j \in J\}$ is a g-Bessel sequence for \mathcal{U} w.r.t. $\{\mathcal{V}_j : j \in J\}$.

Next we show that $\{\Gamma_j \in L(\mathcal{U}, \mathcal{V}_j) : j \in J\}$ has the lower frame bound. In fact, for any $f \in \mathcal{U}$, $j \in J$, from (4.1) we also have

$$\|\Lambda_j f\| \leq \frac{1+\beta}{1-\alpha} \|\Gamma_j f\| + \frac{c_j}{1-\alpha} \|K^* f\|.$$

Similarly, we can obtain

$$\left(\sum_{j \in J} \|\Lambda_j f\|^2 \right)^{1/2} \leq \frac{1+\beta}{1-\alpha} \left(\sum_{j \in J} \|\Gamma_j f\|^2 \right)^{1/2} + \frac{1}{1-\alpha} \left(\sum_{j \in J} c_j^2 \right)^{1/2} \|K^* f\|,$$

it follows that

$$\begin{aligned}
 \left(\sum_{j \in J} \|\Gamma_j f\|^2 \right)^{1/2} &\geq \frac{1-\alpha}{1+\beta} \left(\sum_{j \in J} \|\Lambda_j f\|^2 \right)^{1/2} - \frac{1}{1+\beta} \left(\sum_{j \in J} c_j^2 \right)^{1/2} \|K^* f\| \\
 &\geq \frac{(1-\alpha)\sqrt{A} - \left(\sum_{j \in J} c_j^2 \right)^{1/2}}{1+\beta} \|K^* f\|.
 \end{aligned}$$

□

The next perturbation result in Theorem 4.2 is an extension of Theorem 3.13 in [25] from K -frame to K -g-frame.

Theorem 4.2. *Assume that $R(K)$ is closed. Suppose that $\{\Lambda_j \in L(\mathcal{U}, \mathcal{V}_j) : j \in J\}$ is a K -g-frame with frame bounds A, B , and $\Gamma_j \in L(\mathcal{U}, \mathcal{V}_j)$, $j \in J$. If there exist α, β, γ satisfying $\alpha + \gamma\sqrt{A^{-1}}\|K^+\| \in (-1, 1)$ and $\beta \in (-1, 1)$, such that*

$$(4.2) \quad \left\| \sum_{j \in J_1} (\Lambda_j^* - \Gamma_j^*) g_j \right\| \leq \alpha \left\| \sum_{j \in J_1} \Lambda_j^* g_j \right\| + \beta \left\| \sum_{j \in J_1} \Gamma_j^* g_j \right\| + \gamma \left(\sum_{j \in J_1} \|g_j\|^2 \right)^{1/2},$$

where $J_1 \subset J$ is an any finite subset, $g_j \in \mathcal{V}_j$, $j \in J_1$, then $\{\Gamma_j \in L(\mathcal{U}, \mathcal{V}_j) : j \in J\}$ is a $P_{Q(R(K))}K$ -g-frame for \mathcal{U} w.r.t. $\{\mathcal{V}_j : j \in J\}$, with frame bounds

$$\frac{[\sqrt{A}\|K^+\|^{-1}(1-\alpha)-\gamma]^2}{(1+\beta)^2\|K\|^2}, \quad \frac{[\sqrt{B}(1+\alpha)+\gamma]^2}{(1-\beta)^2},$$

where $P_{Q(R(K))}$ is a bounded projection from \mathcal{U} onto $Q(R(K))$, and $Q = T_1 T^*$, T, T_1 are the synthesis operators for $\{\Lambda_j : j \in J\}$ and $\{\Gamma_j : j \in J\}$, respectively.

Proof. The reader can check [20] for the proof of showing that $\{\Gamma_j \in L(\mathcal{U}, \mathcal{V}_j) : j \in J\}$ is a g-Bessel sequence for \mathcal{U} w.r.t. $\{\mathcal{V}_j\}_{j \in J}$. As for the proof of $\{\Gamma_j\}_{j \in J}$ satisfying the lower frame condition, the reader can check it in light of the proof for Theorem 3.13 in [25], step by step. \square

5. Relationship between K -g-frames and the range of K .

Let K_1 and K_2 be two bounded linear operators in \mathcal{U} , and let $\{\Lambda_j \in L(\mathcal{U}, \mathcal{V}_j) : j \in J\}$ be a g-Bessel sequence on \mathcal{U} w.r.t. $\{\mathcal{V}_j : j \in J\}$. Next we investigate the relationship between $\{\Lambda_j : j \in J\}$ and the range inclusion $R(K_1) \subset R(K_2)$. For this, we first need to introduce some notation. Denote $F_K(\{\Lambda_j\}_{j \in J})$, $F_K^T(\{\Lambda_j\}_{j \in J})$ and $F_K^P(\{\Lambda_j\}_{j \in J})$ as the set of all K -g-frames, tight K -g-frames and Parseval K -g-frames, respectively, for \mathcal{U} w.r.t. $\{\mathcal{V}_j : j \in J\}$.

Theorem 5.1. *Assume that $K_2 \neq 0$. If $R(K_2) \subset R(K_1)$, then $F_{K_1}(\{\Lambda_j\}_{j \in J}) \subset F_{K_2}(\{\Lambda_j\}_{j \in J})$.*

Proof. Assume that $K_2 \neq 0$. Since $R(K_2) \subset R(K_1)$, from Lemma 3.2 and Remark 3.3, we have

$$(5.1) \quad \|K_2^* f\|^2 \leq \alpha^2 \|K_1^* f\|^2, \quad \text{for all } f \in \mathcal{U},$$

where $\alpha > 0$. And, since $\{\Lambda_j : j \in J\}$ is a K_1 -g-frame for \mathcal{U} w.r.t. $\{\mathcal{V}_j : j \in J\}$, we then have

$$(5.2) \quad A \|K_1^* f\|^2 \leq \sum_{j \in J} \|\Lambda_j f\|^2 \leq B \|f\|^2, \quad \text{for all } f \in \mathcal{U}.$$

Combining this with (5.1) and (5.2), we obtain that, for any $f \in \mathcal{U}$,

$$(5.3) \quad \frac{A}{\alpha^2} \|K_2^* f\|^2 \leq \sum_{j \in J} \|\Lambda_j f\|^2 \leq B \|f\|^2,$$

which implies that $\{\Lambda_j \in L(\mathcal{U}, \mathcal{V}_j) : j \in J\}$ is a K_2 -g-frame for \mathcal{U} w.r.t. $\{\mathcal{V}_j : j \in J\}$, with frame bounds $A/\alpha^2, B$. \square

For now, we do not know whether the converse of Theorem 5.1 holds for any fusion frames, but if we restrict the K_1 -g-frame $\{\Lambda_j : j \in J\}$ to be tight, we can derive that the converse of Theorem 5.1 still holds.

Theorem 5.2. *If $F_{K_1}^T(\{\Lambda_j\}_{j \in J}) \subset F_{K_2}(\{\Lambda_j\}_{j \in J})$, then $R(K_2) \subset R(K_1)$.*

Proof. Suppose that $\{\Lambda_j : j \in J\}$ is a tight K_1 -g-frame for \mathcal{U} w.r.t. $\{\mathcal{V}_j : j \in J\}$, with frame bound $A > 0$. So we have

$$(5.4) \quad A \|K_1^* f\|^2 = \sum_{j \in J} \|\Lambda_j f\|^2, \quad \text{for all } f \in \mathcal{U}.$$

And, since $F_{K_1}^T(\{\Lambda_j\}_{j \in J}) \subset F_{K_2}(\{\Lambda_j\}_{j \in J})$, it follows that

$$(5.5) \quad C \|K_2^* f\|^2 \leq \sum_{j \in J} \|\Lambda_j f\|^2 \leq D \|f\|^2, \quad \text{for all } f \in \mathcal{U},$$

where $C, D > 0$ are the frame bounds. Combining with (5.4) and (5.5) we have

$$\|K_2^* f\|^2 \leq \frac{A}{C} \|K_1^* f\|^2, \quad \text{for all } f \in \mathcal{U},$$

it follows that $R(K_2) \subset R(K_1)$ by Lemma 3.2. \square

6. Excess of K -g-frames. In this section, we will study the excess of K -g-frames. We will give two sufficient conditions for the remainder of a K -g-frame after an erasure of some elements to be still a K -g-frame; at the same time, we also provide a sufficient condition for the remainder not to be a K -g-frame. It turns out that the excess of K -g-frames behaves completely different from g-frames.

Lemma 6.1 ([4]). *Suppose that $Q \in L(\mathcal{U}, \mathcal{V})$ and $R(Q)$ is closed. Then, there exists a unique pseudo-inverse $Q^+ : \mathcal{V} \rightarrow \mathcal{U}$ of Q that satisfies*

$$N(Q^+) = R(Q)^\perp, \quad R(Q^+) = N(Q)^\perp, \quad QQ^+f = f, \quad \text{for all } f \in R(Q).$$

If $Q \in L(\mathcal{U}, \mathcal{V})$ is invertible, then $Q^+ = Q^{-1}$.

Theorem 6.2. *Let $\{\Lambda_j \in L(\mathcal{U}, \mathcal{V}_j) : j \in J\}$ be a K -g-frame with frame bounds A, B . Let $I \subset J$ and $R(K)$ be closed. Then the following statements hold:*

(i) *If*

$$(6.1) \quad W_{J \setminus I} \triangleq \overline{\left\{ \sum_{j \in J \setminus I} \Lambda_j^* g_j : \text{for any } j \in J \setminus I, g_j \in \mathcal{V}_j \right\}} \subset R(K)$$

and

$$W_I \triangleq \overline{\left\{ \sum_{j \in I} \Lambda_j^* g_j : \text{for any } j \in I, g_j \in \mathcal{V}_j \right\}} \subset R(K)^\perp,$$

then $\{\Lambda_j \in L(\mathcal{U}, \mathcal{V}_j) : j \in J \setminus I\}$ is a K -g-frame with frame bounds A, B .

(ii) *If (6.1) holds and $A - \|K^+\|^2 \sum_{j \in I} \|\Lambda_j\|^2 > 0$, then $\{\Lambda_j \in L(\mathcal{U}, \mathcal{V}_j) : j \in J \setminus I\}$ is a K -g-frame with frame bounds $A - \|K^+\|^2 \sum_{j \in I} \|\Lambda_j\|^2, B$, where K^+ is the pseudo-inverse of K .*

(iii) *If $\{0\} \neq W_I \subset R(K)$ and $W_{J \setminus I} \perp W_I$, then $\{\Lambda_j \in L(\mathcal{U}, \mathcal{V}_j) : j \in J \setminus I\}$ is not a K -g-frame.*

Proof. (i) Since $W_I \subset R(K)^\perp$, $\Lambda_j f \in \mathcal{V}_j$ for any $j \in J$, $f \in \mathcal{U}$, we then have

$$0 = \left\langle \sum_{j \in I} \Lambda_j^* \Lambda_j g, g \right\rangle = \sum_{j \in I} \langle \Lambda_j g, \Lambda_j g \rangle = \sum_{j \in I} \|\Lambda_j g\|^2,$$

where $g \in R(K) \subset \mathcal{U}$, it follows that

$$(6.2) \quad \Lambda_j g = 0, \quad \text{for all } j \in I, g \in R(K).$$

Similarly, by the condition $W_{J \setminus I} \subset R(K)$, we can obtain

$$(6.3) \quad \Lambda_j h = 0, \quad \text{for all } j \in J \setminus I, h \in R(K)^\perp.$$

From (6.2), we have

$$\begin{aligned} (6.4) \quad \sum_{j \in J} \|\Lambda_j g\|^2 &= \sum_{j \in J \setminus I} \|\Lambda_j g\|^2 + \sum_{j \in I} \|\Lambda_j g\|^2 \\ &= \sum_{j \in J \setminus I} \|\Lambda_j g\|^2, \quad \text{for all } g \in R(K) \subset \mathcal{U}. \end{aligned}$$

For any $g \in R(K) \subset \mathcal{U}$, since $\{\Lambda_j \in L(\mathcal{U}, \mathcal{V}_j) : j \in J\}$ is a K -g-frame with frame bounds A, B , we have

$$(6.5) \quad A\|K^*g\|^2 \leq \sum_{j \in J} \|\Lambda_j g\|^2 \leq B\|g\|^2.$$

Combining (6.4) and (6.5), we obtain

$$(6.6) \quad A\|K^*g\|^2 \leq \sum_{j \in J \setminus I} \|\Lambda_j g\|^2, \quad \text{for all } g \in R(K) \subset \mathcal{U}.$$

On the other hand, for any $h \in R(K)^\perp \subset \mathcal{U}$, since $\langle K^*h, f \rangle = \langle h, Kf \rangle = 0$, for all $f \in \mathcal{U}$, it follows that

$$(6.7) \quad K^*h = 0, \quad \text{for all } h \in R(K)^\perp.$$

Since $R(K) \subset \mathcal{U}$ is closed, we have $\mathcal{U} = R(K) \oplus R(K)^\perp$. So, for any $f \in \mathcal{U}$, there is $f = f_1 + f_2$, where $f_1 \in R(K)$, $f_2 \in R(K)^\perp$. From (6.3), (6.6) and (6.7), we get

$$A\|K^*f\|^2 = A\|K^*(f_1 + f_2)\|^2 = A\|K^*f_1\|^2$$

$$\begin{aligned}
&\leq \sum_{j \in J \setminus I} \|\Lambda_j f_1\|^2 = \sum_{j \in J \setminus I} \|\Lambda_j(f_1 + f_2)\|^2 \\
&= \sum_{j \in J \setminus I} \|\Lambda_j f\|^2.
\end{aligned}$$

It is trivial to show that the upper frame condition holds.

(ii) Since $R(K)$ is closed, by Lemma 6.1 there exists a pseudo-inverse K^+ such that $KK^+f = f$, for all $f \in R(K)$, namely, $KK^+|_{R(K)} = I_{R(K)}$, so we have $I_{R(K)}^* = (K^+|_{R(K)})^*K^*$. Then, for any $f \in R(K)$, we get

$$\begin{aligned}
(6.8) \quad \|f\| &= \|(K^+|_{R(K)})^*K^*f\| \\
&\leq \|(K^+|_{R(K)})^*\| \cdot \|K^*f\| \\
&\leq \|K^+\| \cdot \|K^*f\|.
\end{aligned}$$

By the condition that $R(K) \subset \mathcal{U}$ is closed, we can also get that, for any $f \in \mathcal{U}$, there exist $f_1 \in R(K)$, $f_2 \in R(K)^\perp$ such that $f = f_1 + f_2$. From (6.3), (6.7) and (6.8), we obtain

$$\begin{aligned}
\sum_{j \in J \setminus I} \|\Lambda_j f\|^2 &= \sum_{j \in J \setminus I} \|\Lambda_j f_1\|^2 = \sum_{j \in J} \|\Lambda_j f_1\|^2 - \sum_{j \in I} \|\Lambda_j f_1\|^2 \\
&\geq \sum_{j \in J} \|\Lambda_j f_1\|^2 - \sum_{j \in I} \|\Lambda_j\|^2 \cdot \|f_1\|^2 \\
&\geq \sum_{j \in J} \|\Lambda_j f_1\|^2 - \sum_{j \in I} \|\Lambda_j\|^2 \cdot \|K^+\|^2 \cdot \|K^*f_1\|^2 \\
&\geq A\|K^*f_1\|^2 - \|K^*f_1\|^2 \|K^+\|^2 \sum_{j \in I} \|\Lambda_j\|^2 \\
&= \left(A - \|K^+\|^2 \sum_{j \in I} \|\Lambda_j\|^2 \right) \|K^*f_1\|^2 \\
&= \left(A - \|K^+\|^2 \sum_{j \in I} \|\Lambda_j\|^2 \right) \|K^*f\|^2.
\end{aligned}$$

Obviously, the upper bound of $\{\Lambda_j : j \in J \setminus I\}$ is B .

(iii) Since $\{0\} \neq W_I \subset R(K)$, then for any $0 \neq f \in W_I$, we have $K^*f \neq 0$. In fact, since $0 \neq f \in W_I \subset R(K)$, there exists $g \in \mathcal{U}$ such

that $f = Kg$, it follows that

$$\langle K^*f, g \rangle = \langle K^*Kg, g \rangle = \|Kg\|^2 = \|f\|^2 > 0.$$

Hence, we have $\|K^*f\|^2 > 0$, $0 \neq f \in W_I \subset R(K)$.

On the other hand, for $0 \neq f \in W_I \subset R(K)$, since $W_{J \setminus I} \perp W_I$, we have

$$0 = \left\langle \sum_{j \in J \setminus I} \Lambda_j^* \Lambda_j f, f \right\rangle = \sum_{j \in J \setminus I} \|\Lambda_j f\|^2.$$

Hence, (2.1) does not hold, and $\{\Lambda_j \in L(\mathcal{U}, \mathcal{V}_j) : j \in J \setminus I\}$ is not a K -g-frame. \square

From Theorem 6.2, we can easily have the following corollary for $I = \{j_0\}$.

Corollary 6.3. *Let $\{\Lambda_j \in L(\mathcal{U}, \mathcal{V}_j) : j \in J\}$ be a K -g-frame with frame bounds A, B . Let $j_0 \in J$ and $R(K)$ be closed. Then the following statements hold:*

(i) *If*

$$(6.9) \quad W_{J \setminus \{j_0\}} \triangleq \overline{\left\{ \sum_{j \in J \setminus \{j_0\}} \Lambda_j^* g_j : \text{for any } j \in J \setminus \{j_0\}, g_j \in \mathcal{V}_j \right\}} \subset R(K),$$

$$W_{j_0} \triangleq \overline{\{\Lambda_{j_0}^* g : g \in \mathcal{V}_{j_0}\}} \subset R(K)^\perp,$$

then $\{\Lambda_j \in L(\mathcal{U}, \mathcal{V}_j) : j \in J \setminus \{j_0\}\}$ is a K -g-frame with frame bounds A, B .

(ii) *If (6.9) holds and $A - \|K^+\|^2 \|\Lambda_{j_0}\|^2 > 0$, then $\{\Lambda_j \in L(\mathcal{U}, \mathcal{V}_j) : j \in J \setminus \{j_0\}\}$ is a K -g-frame with frame bounds $A - \|K^+\|^2 \|\Lambda_{j_0}\|^2, B$, where K^+ is the pseudo-inverse of K .*

(iii) *If $\{0\} \neq W_{j_0} \subset R(K)$ and $W_{J \setminus \{j_0\}} \perp W_{j_0}$, then $\{\Lambda_j \in L(\mathcal{U}, \mathcal{V}_j) : j \in J \setminus \{j_0\}\}$ is not a K -g-frame.*

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