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ON THE SUBSTRUCTURES Δ AND ∇

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ABSTRACT. In this paper, we discuss the question of when the substructures, the singular sub-bimodule $\Delta[M,N]$ and the cosingular bi-submodule $\nabla[M,N]$ of $\mathrm{Hom}\,(M,N)$, are equal to zero. Some well-known results of regular rings are obtained. Moreover, the substructures $\Delta[M,N]$ and $\nabla[M,N]$ with M and N that are direct sums of submodules are studied.

1. Introduction. In this paper, R will represent an associative ring with identity, and all modules over R are unitary right modules. We write M_R to indicate that M is a right R-module. Throughout this paper, homomorphisms of modules are written on the left of their arguments. Let M and N be modules. For convenience of the reader, we follow the notation used in **[8, 13]**, let $E_M := \mathrm{End}_R(M)$ and $[M,N] := \mathrm{Hom}_R(M,N)$. Then $[M,N]$ is an (E_N, E_M)-bimodule. We also denote $J(R)$ and $\mathrm{Rad}\,(M)$ for the Jacobson radical of R and module M, respectively. For a submodule N of M, we use $N \leq M$ $(N < M)$ and $N \leq^{\oplus} M$ to mean that N is a submodule of M (respectively, proper submodule), N is a direct summand of M, and we write $N \leq^{e} M$ and $N \ll M$ to indicate that N is an essential, respectively small, submodule of M. For a subset X of R, let $r(X)$ denote the right annihilator of X in R.

The concept of the regularity of $[M,N]$ was introduced by Kasch and Mader in **[4]** to extend the notion of the regularity of a ring to $[M,N]$. Recall that $\alpha \in [M,N]$ is called *regular* if $\alpha = \alpha\beta\alpha$ for some $\beta \in [N,M]$. They showed that $\alpha \in [M,N]$ is regular if and only if $\mathrm{Ker}\,(\alpha)$ is a direct summand of M and $\mathrm{Im}\,(\alpha)$ is a direct summand of

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N ([**4,** Corollary II.1.3]). The module $[M,N]$ is said to be *regular* if each $\alpha \in [M,N]$ is regular. An important line of research in module classes is to investigate relationships of regularity to substructures such as Jacobson radical $J[M,N]$ of $[M,N]$, to the singular $\Delta[M,N]$ and cosingular $\nabla[M,N]$ sub-bimodules of $[M,N]$, and to the notion of lying over or under a direct summand. Beidar and Kasch [**2**] defined and studied the singular sub-bimodule $\Delta[M,N]$ and the co-singular sub-bimodule $\nabla[M,N]$ such as:

$$\Delta[M,N] = \{f \in [M,N] : \ \mathrm{Ker}\,(f) \leq^e M\}$$
$$\nabla[M,N] = \{f \in [M,N] : \ \mathrm{Im}\,(f) \ll N\}.$$

The other substructure, Jacobson radical $J[M,N]$, of $[M,N]$, was introduced and studied by Kasch and Mader [**4**] and Nicholson and Zhou [**8**]. If $M = \oplus_{i=1}^{s} M_i$ and $N = \oplus_{j=1}^{t} N_j$ are left R-modules, then (using canonical injections and projections) $[M,N]$ has a natural matrix representation as follows:

$$[M,N] = \begin{pmatrix} [M_1,N_1] & [M_1,N_2] & \cdots & [M_1,N_t] \\ [M_2,N_1] & [M_2,N_2] & \cdots & [M_2,N_t] \\ \cdots & \cdots & \cdots & \\ [M_s,N_1] & [M_s,N_2] & \cdots & [M_s,N_t] \end{pmatrix} = ([M_i,N_j])$$

where the elements of M and N are written as rows, and the matrix $([M_i,N_j])$ acts by right matrix multiplication. In [**8,** Theorem 10], it is shown that if $M = \oplus_{i=1}^{s} M_i$ and $N = \oplus_{j=1}^{t} N_j$ are modules, then $J[M,N] = (J[M_i,N_j])$. In Theorem 2.3, we prove that $\Delta[M,N] = (\Delta[M_i,N_j])$ and $\nabla[M,N] = (\nabla[M_i,N_j])$.

Furthermore, we are going to characterize when Δ or ∇ is zero. We show that if $M = \oplus_{i\in\mathcal{I}} M_i$ and $N = \oplus_{j\in\mathcal{J}} N_j$, then $\Delta[M,N] = 0$ if and only if $\Delta[M_i,N_j] = 0$ for all $i \in \mathcal{I}$, $j \in \mathcal{J}$, and $\nabla[M,N] = 0$ if and only if $\nabla[M_i,N_j] = 0$ for all $i \in \mathcal{I}$, $j \in \mathcal{J}$ (see Theorem 2.2). In [**8,** Theorem 33], Nicholson and Zhou proved that $[M,N]$ is semiregular and $\Delta[M,N] = J[M,N]$ ($[M,N]$ is called *semiregular* if, for every $\alpha \in [M,N]$, there exists a $\beta \in [N,M]$ such that $\beta = \beta\alpha\beta$ and $\alpha - \alpha\beta\alpha \in J[M,N]$) if and only if $\mathrm{Ker}\,(\alpha)$ lies under a direct summand of M for any $\alpha \in [M,N]$. According to Nicholson and Zhou [**8**], M is called a *direct N-injective module* if $K \cong P \leq^{\oplus} M$ with $K \leq N$ implies that $K \leq^{\oplus} N$. Recently, in [**10,** Theorem 3.4], Quynh, Koşan and Thuyet proved that if the module M is both generalized continuous and direct N-injective, then $[M,N]$ is semiregular and $\Delta[M,N] = J[M,N]$. In Theorem 2.5, we show that, if M is direct N-injective, then $[M,N]$ is regular if and only if $\Delta[M,N] = 0$ and $\mathrm{Ker}\,(\alpha)$ lies under a direct summand of M for any $\alpha \in [M,N]$.

A module M is called a *direct projective* if, whenever a factor module M/K is isomorphic to a summand of M, then K is a summand of M (see **[7]**). According to Nicholson and Zhou **[8]**, N is *direct M-projective* if $M/K \cong P \leq^{\oplus} N$ implies that $K \leq^{\oplus} M$. As a dual version of **[8,** Theorem 33], Nicholson and Zhou showed that, if the direct projective module M is direct N-projective, then $[M,N]$ is semiregular and $\nabla[M,N] = J[M,N]$ if and only if $\alpha(M)$ lies over a direct summand of M for any $\alpha \in [M,N]$ (see **[8,** Theorem 35]). Recently, in **[10,** Theorem 3.6], Quynh, Koşan and Thuyet proved that, if the module N is both generalized discrete and direct M-projective, then $[M,N]$ is semiregular and $\nabla[M,N] = J[M,N]$. In Theorem 2.8, we show that if N is direct M-projective, then $[M,N]$ is regular if and only if $\nabla[M,N] = 0$ and $\mathrm{Im}\,(\alpha)$ lies over a direct summand of M for any $\alpha \in [M,N]$.

2. Some properties of modules with $\Delta = 0$ or $\nabla = 0$. In this section, we are going to characterize when Δ or ∇ is zero. The following key result will be needed.

Lemma 2.1. *Let M and N be modules and A a direct summand of M.*

(i) *If $\Delta[M,N] = 0$, then $\Delta[A,N] = 0$.*
(ii) *If $\nabla[M,N] = 0$, then $\nabla[A,N] = 0$.*

Proof. Assume that $M = A \oplus A'$ for some A' of M.

(i) Let $\varphi \in \Delta[A,N]$. Then $\mathrm{Ker}\,(\varphi) \leq^e A$. Let $\phi = \varphi\pi_A : M \to N$ with the canonical projection $\pi_A : M \to A$. It follows that $\mathrm{Ker}\,(\phi) = \mathrm{Ker}\,(\varphi) \oplus A' \leq^e M$. Therefore, $\phi = 0$ by assumption. Thus, $\varphi = 0$.

(ii) Let $\varphi \in \nabla[A,N]$. Then $\mathrm{Im}\,(\varphi) \ll N$. We consider the homomorphism $\varphi \oplus 0 : A \oplus A' \to N$ defined by $(\varphi \oplus 0)(a + a') = \varphi(a)$. Then $\mathrm{Im}\,(\varphi \oplus 0) = \mathrm{Im}\,(\varphi) \ll N$. It follows that $\varphi \oplus 0 = 0$ or $\varphi = 0$. $\square$

Theorem 2.2. *Let $M = \oplus_{i\in\mathcal{I}} M_i$ and $N = \oplus_{j\in\mathcal{J}} N_j$ be R-modules, where $\mathcal{I}, \mathcal{J}$ are arbitrary non-empty sets. Then*

(i) $\Delta[M, N] = 0$ *if and only if* $\Delta[M_i, N_j] = 0$ *for all* $i \in \mathcal{I}$, $j \in \mathcal{J}$.
(ii) $\nabla[M, N] = 0$ *if and only if* $\nabla[M_i, N_j] = 0$ *for all* $i \in \mathcal{I}$, $j \in \mathcal{J}$.

Proof. (i) Assume that $\Delta[M_i, N_j] = 0$ for all $i \in \mathcal{I}$, $j \in \mathcal{J}$. Let $f \in \Delta[M_i, N]$. We consider the canonical projection $\pi_j : N \to N_j$. Then $\mathrm{Ker}\,(\pi_j f) \leq^e M_i$ for all $j \in \mathcal{J}$, because $\mathrm{Ker}\,(f) \leq^e M_i$. By the hypothesis, we can obtain that $\pi_j f = 0$ for all $j \in \mathcal{J}$. It follows that $f = 0$. Now, let $\phi \in \Delta[M, N]$. Then $\mathrm{Ker}\,(\phi) \leq^e M$. For each $i \in \mathcal{I}$, we consider the restriction homomorphism $\phi_i := \phi|_{M_i} : M_i \to N$. Then $\mathrm{Ker}\,(\phi_i) = \mathrm{Ker}\,(\phi) \cap M_i$, and so $\mathrm{Ker}\,(\phi_i) \leq^e M_i$. By the hypothesis, we can obtain that $\phi_i = 0$. Hence, $\phi = 0$.

The converse is clear by Lemma 2.1.

(ii) Assume that $\nabla[M_i, N_j] = 0$ for all $i \in \mathcal{I}$, $j \in \mathcal{J}$. Let $f \in \nabla[M, N_j]$. We consider the inclusion $\iota_i : M_i \to M$. Then $\mathrm{Im}\,(f\iota_i) \ll N_j$ for all $i \in \mathcal{I}$, because $\mathrm{Im}\,(f) \ll N_j$. By the hypothesis, we can obtain that $f\iota_i = 0$ for all $i \in \mathcal{I}$. It follows that $f = 0$. Now, let $\varphi \in \nabla[M, N]$. Then $\mathrm{Im}\,(\varphi) \ll N$. For each $i \in \mathcal{I}$, we consider the projection $\pi_j : N \to N_j$. Let $\varphi_i := \pi_j \varphi : M \to N_j$ for each $j \in \mathcal{J}$. Then $\mathrm{Im}\,(\varphi_i) \ll N_j$. By the hypothesis, we can obtain that $\varphi_i = 0$. Hence, $\varphi = 0$.

The converse is clear by Lemma 2.1. □

Let $M = \oplus_{i=1}^{s} M_i$ and $N = \oplus_{j=1}^{t} N_j$ be left R-modules. We recall the natural matrix representation of $[M, N]$ as we mentioned in the introduction:

$$[M, N] = \begin{pmatrix} [M_1, N_1] & [M_1, N_2] & \cdots & [M_1, N_t] \\ [M_2, N_1] & [M_2, N_2] & \cdots & [M_2, N_t] \\ \cdots & \cdots & \cdots & \\ [M_s, N_1] & [M_s, N_2] & \cdots & [M_s, N_t] \end{pmatrix} = ([M_i, N_j]),$$

and, for every

$$\begin{pmatrix} \varphi_{11} & \varphi_{12} & \cdots & \varphi_{1t} \\ \varphi_{21} & \varphi_{22} & \cdots & \varphi_{2t} \\ \cdots & \cdots & \cdots & \\ \varphi_{s1} & \varphi_{s2} & \cdots & \varphi_{st} \end{pmatrix} \in (\Delta[M_i, N_j])$$

and $(m_1\ m_2 \cdots m_s) \in M$,

$$(m_1\ m_2 \cdots m_s) \begin{pmatrix} \varphi_{11} & \varphi_{12} & \cdots & \varphi_{1t} \\ \varphi_{21} & \varphi_{22} & \cdots & \varphi_{2t} \\ \cdots & \cdots & \cdots & \\ \varphi_{s1} & \varphi_{s2} & \cdots & \varphi_{st} \end{pmatrix} = \left(\sum_{i=1}^{s} (m_i)\varphi_{i1} \sum_{i=1}^{s} (m_i)\varphi_{i2} \cdots \sum_{i=1}^{s} (m_i)\varphi_{it} \right).$$

Theorem 2.3. *If $M = \oplus_{i=1}^{s} M_i$ and $N = \oplus_{j=1}^{t} N_j$ are left R-modules, then*

(i) $\Delta[M, N] = (\Delta[M_i, N_j])$.
(ii) $\nabla[M, N] = (\nabla[M_i, N_j])$.

Proof. (i) Let $\varphi \in \triangle[M, N]$. Then $\operatorname{Ker}(\varphi) \leq^e M$. We consider the homomorphism $\varphi_{ij} := \iota_i \varphi \pi_j$, where $\pi_j : N \to N_j$ is the canonical projection and $\iota_i : M_i \hookrightarrow M$ is the inclusion for every $i \in I$ and $j \in J$. Then $\operatorname{Ker}(\varphi_{ij}) = M_i \cap \operatorname{Ker}(\varphi\pi_j)$. We have that $\operatorname{Ker}(\varphi) \subseteq \operatorname{Ker}(\varphi\pi_j)$, $\operatorname{Ker}(\varphi) \leq^e M$ and obtain that $\operatorname{Ker}(\varphi\pi_j) \leq^e M$. Hence, $\operatorname{Ker}(\varphi\pi_j) \leq^e M_i$, i.e., $\varphi_{ij} \in \Delta[M_i, N_j]$ for every $i \in \{1, 2, \ldots, s\}$ and $j \in \{1, 2, \ldots, t\}$. It follows that $\varphi \in (\Delta[M_i, N_j])$.

Conversely, assume that

$$\begin{pmatrix} \varphi_{11} & \varphi_{12} & \cdots & \varphi_{1t} \\ \varphi_{21} & \varphi_{22} & \cdots & \varphi_{2t} \\ \cdots & \cdots & \cdots & \\ \varphi_{s1} & \varphi_{s2} & \cdots & \varphi_{st} \end{pmatrix} \in (\Delta[M_i, N_j]),$$

where $\varphi_{ij} \in \Delta[M_i, N_j]$ for every $i \in \{1, 2, \ldots, s\}$ and $j \in \{1, 2, \ldots, t\}$. For every $j \in \{1, 2, \ldots, t\}$, let $\varphi_j = \sum_{i=1}^{s} (p_i \varphi_{ij}) \in [M, N]$ and

$$\varphi = \sum_{j=1}^{t} \varphi_j = \sum_{j=1}^{t} \sum_{i=1}^{s} (p_i \varphi_{ij}) \in [M, N],$$

where $p_i : M \to M_i$ is the canonical projection. Now it is easy to see that

$$\varphi = \begin{pmatrix} \varphi_{11} & \varphi_{12} & \cdots & \varphi_{1t} \\ \varphi_{21} & \varphi_{22} & \cdots & \varphi_{2t} \\ \cdots & \cdots & \cdots & \\ \varphi_{s1} & \varphi_{s2} & \cdots & \varphi_{st} \end{pmatrix}$$

and

$$\mathrm{Ker}\,(\varphi) = \bigcap_{j=1}^{t} \mathrm{Ker}\,(\varphi_j).$$

Now, we claim that $\mathrm{Ker}\,(\varphi_j) \leq^e M$. For $j \in \{1, 2, \ldots, t\}$, we can obtain that $\varphi_j = \sum_{i=1}^{s}(p_i\varphi_{ij})$ and

$$\mathrm{Ker}\,(p_i\varphi_{ij}) = \mathrm{Ker}\,(\varphi_{ij}) \oplus (\oplus_{k\neq i}M_k) \leq^e M$$

for all i. Since $\bigcap \mathrm{Ker}\,(p_i\varphi_{ij}) \subseteq \mathrm{Ker}\,(\varphi_j)$, we have $\mathrm{Ker}\,(\varphi_j) \leq^e M$. Thus, $\mathrm{Ker}\,(\varphi) \leq^e M$, i.e., $\varphi \in \Delta[M, N]$.

(ii) Let $\varphi \in \nabla[M, N]$. Then $\mathrm{Im}\,(\varphi) \ll N$. We consider the homomorphism $\varphi_{ij} := \iota_i\varphi\pi_j$, where $\pi_j : N \to N_j$ is the canonical projection and $\iota_i : M_i \hookrightarrow M$ is the inclusion for every $i \in I$ and $j \in J$. We have that $\mathrm{Im}\,(\varphi) \ll N$ and obtain that $\mathrm{Im}\,(\varphi\pi_j) \ll N_j$ for all $j \in \{1, 2, \ldots, t\}$. But $\mathrm{Im}\,(\varphi_{ij}) = \mathrm{Im}\,(\iota_i\varphi\pi_j) \leq \mathrm{Im}\,(\varphi\pi_j)$. So $\mathrm{Im}\,(\varphi_{ij}) \ll N_j$, i.e., $\varphi_{ij} \in [M, N]$ for all i, j.

Conversely, assume that

$$\varphi := \sum_{i,j} \pi_j\varphi_{ij}\iota_i \in (\nabla[M_i, N_j]),$$

where $\varphi_{ij} \in \nabla[M_i, N_j]$, $\pi_i : M \to M_i$ is the canonical projection and $\iota_i : N_i \hookrightarrow N$ is the inclusion for every $i \in I$ and $j \in J$. Then

$$\mathrm{Im}\,(\varphi) = \sum_{i,j} \mathrm{Im}\,(\pi_j\varphi_{ij}\iota_i) = \sum_{i,j} \mathrm{Im}\,(\varphi_{ij}\iota_i).$$

We have that $\mathrm{Im}\,(\varphi_{ij}) \ll N_j$ and obtain that $\mathrm{Im}\,(\varphi_{ij}\iota_i) \ll N$. Hence, $\mathrm{Im}\,(\varphi) \ll N$, i.e., $\varphi \in \nabla[M, N]$. $\square$

We denote $E(M)$ for the injective hull of R module M.

Proposition 2.4. *Let M and N be modules. Then*:

(i) *If $\Delta[E(M), E(N)] = 0$, then $\Delta[M, N] = 0$.*

(ii) *If* $\nabla[E(M), E(N)] = 0$, *then* $\nabla[M, N] = 0$.

Proof. (i) Assume that $f \in \Delta[M, N]$. Then $\mathrm{Ker}\,(f) \leq^e M$, and there exists $\overline{f} \in [E(M), E(N)]$ such that $\overline{f}|_M = f$. Since $M \leq^e E(M)$, we can obtain that $\mathrm{Ker}\,(\overline{f}) \leq^e E(M)$. By hypothesis, $\overline{f} = 0$ or $f = 0$.

(ii) Assume that $f \in \Delta[M, N]$. Then $\mathrm{Im}\,(f) \ll N$, and there exists $\overline{f} \in [E(M), E(N)]$ such that $\overline{f}|_M = f$. Since $\mathrm{Im}\,(f) \ll N$, we can obtain that $\mathrm{Im}\,(f) \ll E(N)$. It follows that $\mathrm{Im}\,(\overline{f}) \ll E(N)$. By hypothesis, $\overline{f} = 0$ or $f = 0$. □

A submodule A of a module M is said to *lie under a summand* of M if there exists a direct decomposition $M = P \oplus Q$ with $A \leq P$ and $A \leq^e P$.

Theorem 2.5. *Let M and N be R-modules. If M is direct N-injective, then the following conditions are equivalent*:

(i) $[M, N]$ *is regular.*
(ii) $\Delta[M, N] = 0$ *and* $\mathrm{Ker}\,(\alpha)$ *lies under a direct summand of M for any $\alpha \in [M, N]$.*

Proof. (i) ⇒ (ii). Let $\alpha \in [M, N]$. Then $\mathrm{Ker}\,(\alpha) \leq^{\oplus} M$ (because α is regular). Moreover, if $\alpha \in \Delta[M, N]$, then $\mathrm{Ker}\,(\alpha) = M$ or $\alpha = 0$.

(ii) ⇒ (i). Let $\alpha \in [M, N]$. By (ii), there exists $\beta^2 = \beta \in [M, M]$ such that $\mathrm{Ker}\,(\alpha) \leq^e \beta(M) = \mathrm{Ker}\,(1_M - \beta)$. We also notice that $\alpha|_{(1_M - \beta)(M)} : (1_M - \beta)(M) \to N$ is a monomorphism. Since M is direct N-injective and $(1_M - \beta)(M)$ is a direct summand of M, $\alpha|_{(1_M - \beta)(M)}$ is a split monomorphism. There exists a homomorphism $\gamma : N \to M$ such that $\gamma\alpha|_{(1_M - \beta)(M)} = 1_{(1_M - \beta)(M)}$ or $\gamma\alpha(1_M - \beta) = 1_M - \beta$. Let $\xi = (1_M - \beta)\gamma$. Then $\mathrm{Ker}\,(\alpha - \alpha\xi\alpha) = \mathrm{Ker}\,(\alpha) \oplus (1_M - \beta)(M) \leq^e M$. It follows that $\alpha - \alpha\xi\alpha \in \Delta[M, N] = 0$. Thus $\alpha = \alpha\beta\alpha$, and (i) follows. □

Corollary 2.6. *Assume that M is N-injective. The following are equivalent for modules M and N*:

(i) $[M, N]$ *is regular.*
(ii) $\Delta[M, N] = 0$.

Proof. (i) $\Rightarrow$ (ii) is obvious.

(ii) $\Rightarrow$ (i). Let $\alpha \in [M, N]$ and $\alpha \neq 0$. Then $\mathrm{Ker}\,(\alpha)$ is not essential in M. Let L be a complement of $\mathrm{Ker}\,\alpha$ in M. We consider the map $\phi : \alpha(L) \to M$, defined by $\phi(\alpha(x)) = x$ for all $x \in L$. Then ϕ is a homomorphism. Since M is N-injective, there exists a $\theta \in [N, M]$ extension of ϕ. It follows that $\mathrm{Ker}\,(\alpha) + L \leq \mathrm{Ker}\,(\alpha\theta\alpha - \alpha)$, and we know that $\mathrm{Ker}\,(\alpha) \oplus L \leq^e M$. Consequently, $\alpha\theta\alpha - \alpha \in \Delta[M, N] = 0$. Thus, $\alpha = \alpha\theta\alpha$. $\square$

Letting $M = R$, the next result extends Chen and Nicholson [**3,** Theorem 4.2].

Corollary 2.7. *Let R be a right self-injective ring. The following conditions are equivalent for a module N:*

(i) *N is regular.*
(ii) *N is nonsingular.*

Dually, a submodule A of a module M is said to *lie over a summand* of M if there exists a direct decomposition $M = P \oplus Q$ with $P \leq A$ and $Q \cap A \ll M$.

Theorem 2.8. *Assume that N is direct M-projective. The following are equivalent for modules M and N:*

(i) *$[M, N]$ is regular.*
(ii) *$\nabla[M, N] = 0$ and $\mathrm{Im}\,(\alpha)$ lies over a direct summand of N for any $\alpha \in [M, N]$.*

Proof. (i) $\Rightarrow$ (ii). Let $\alpha \in [M, N]$. Then $\mathrm{Im}\,(\alpha) \leq^{\oplus} N$ (because α is regular). Moreover, if $\alpha \in \nabla[M, N]$, then $\mathrm{Im}\,(\alpha) = 0$ or $\alpha = 0$.

(ii) $\Rightarrow$ (i). Let $\alpha \in [M, N]$. By (ii), the module N has a decomposition $N = P \oplus K$ such that $P \leq \alpha(M)$ and $\alpha(M) \cap K \ll K$. Let $\pi : N \to N$ be the homomorphism such that $\pi^2 = \pi$, $\pi(N) = P$ and $(1_N - \pi)(N) = K$. Then $\pi\alpha : M \to P$ is an epimorphism. Since N is a direct module M-projective and P is a direct summand of N, $\pi\alpha$ is a split epimorphism. There exists a homomorphism $\theta : P \to M$ such that $(\pi\alpha)\theta = 1_P$. Let $\gamma = \theta\pi : N \to M$ and $\pi\alpha\gamma = \pi$. Let $\beta = \gamma\pi$. We have $\beta\alpha\beta = \beta$ and

$$\begin{aligned}(\alpha - \alpha\beta\alpha)(M) &= (1_N - \alpha\beta)\alpha(M)\\ &= \alpha(M) \cap (1_N - \alpha\beta)(N)\\ &= \alpha(M) \cap K \ll N.\end{aligned}$$

Thus, $\alpha - \alpha\beta\alpha \in \nabla[M, N] = 0$, and hence $\alpha = \alpha\beta\alpha$. □

We recall the following definitions (see **[7, 12]**).

(1) A submodule V of an R-module M is called a *supplement* of U in M if V is a minimal element in the set of submodules L of M with $U + L = M$. V is a supplement of U if and only if $U + V = M$ and $U \cap V$ is small in V.

(2) An R-module M is *supplemented* if every submodule of M has a supplement in M.

Theorem 2.9. *Assume that N is M-projective. If N is a supplemented module or N satisfies* DCC *on non-small submodules, then the following are equivalent for modules M and N:*

(i) $[M, N]$ *is regular.*

(ii) $\nabla[M, N] = 0$.

Proof. (i) $\Rightarrow$ (ii) is obvious.

(ii) $\Rightarrow$ (i). Assume that N is a supplemented module and $\nabla[M, N] = 0$. Let $\alpha \in [M, N]$ and $\alpha \neq 0$. Then $\mathrm{Im}\,(\alpha) \not\ll N$. Since N is supplemented, there exists a submodule L of N such that $N = \mathrm{Im}\,(\alpha) + L$ and $\mathrm{Im}\,(\alpha) \cap L \ll N$. We consider the canonical projection $\pi : N \to N/L$. Then $\pi\alpha : M \to N/L$ is an epimorphism. On the other hand, since N is M-projective, there exists a homomorphism $\gamma \in [N, M]$ such that $\pi\alpha\gamma = \pi$. It follows that $\mathrm{Im}\,(\alpha - \alpha\gamma\alpha) \leq \mathrm{Im}\,(\alpha) \cap L$, and so $\alpha - \alpha\gamma\alpha \in \nabla[M, N] = 0$. Thus, $\alpha = \alpha\gamma\alpha$.

Assume that N satisfies DCC on non-small submodules and $\nabla[M, N] = 0$. Let $\alpha \in [M, N]$ and $\alpha \neq 0$. Then $\mathrm{Im}\,(\alpha) \not\ll N$. Without loss of generality, we can assume that $\mathrm{Im}\,(\alpha) \neq N$. Hence, there exists an $L \leq N$ such that $L \neq N$ and $\mathrm{Im}\,(\alpha) + L = N$. We consider the set of non-small submodules of N: $\Im = \{L < N | \mathrm{Im}\,(\alpha) + L = N\}$. Since $\mathrm{Im}\,(\alpha) \neq N$, we can obtain that L is not a small submodule of

N. It follows that $\Im$ is a non-empty set. Since N satisfies DCC on non-small submodules, the set $\Im$ has a minimal element, say L. Then $L \neq N$ and $\mathrm{Im}\,(\alpha) + L = N$. Now we claim that $\mathrm{Im}\,(\alpha) \cap L \ll N$. Let $\mathrm{Im}\,(\alpha) \cap L + H = N$ with $H \leq N$. Then $\mathrm{Im}\,(\alpha) + (L \cap H) = N$. By minimality of L, we can obtain that $L = L \cap H$, and so $L \leq H$. It follows that $H = N$. Thus, $\mathrm{Im}\,(\alpha) \cap L \ll N$. We consider the canonical projection $\pi : N \to N/L$. Then $\pi\alpha : M \to N/L$ is an epimorphism. Since N is M-projective, there exists a homomorphism $\gamma \in [N, M]$ such that $\pi\alpha\gamma = \pi$. It is easy to see that $\mathrm{Im}\,(\alpha - \alpha\gamma\alpha) \leq \mathrm{Im}\,(\alpha) \cap L$. Hence, $\alpha - \alpha\gamma\alpha \in \nabla[M, N] = 0$. □

The following is the dual of Corollary 2.7.

Corollary 2.10. *Assume that N is a semiperfect module. The following are equivalent for modules N:*

(i) *N is regular.*
(ii) $\mathrm{Rad}\,(N) = 0$.
(iii) *N is semisimple.*

A module M is said to be *retractable* (respectively, *coretractable*) if $\mathrm{Hom}\,(M, K) \neq 0$ for all $0 \neq K \leq M$ (respectively, $\mathrm{Hom}\,(M/K, M) \neq 0$ for all $K \leq M$ and $K \neq M$).

Proposition 2.11. *Let M and N be modules.*

(i) *If M is retractable and $\Delta[M, N] = 0$, then $[M, N]$ is a nonsingular right E_M-module.*
(ii) *Assume that N is coretractable, M-projective and $\nabla[M, N] = 0$. If $\varphi \in [M, N]$ such that $\varphi[N, M]E_N \ll E_N$, then $\varphi = 0$.*

Proof. (i) Suppose that $\Delta[M, N] = 0$. Let $\varphi \in [M, N]$ with $r_{E_M}(\varphi) \leq^e E_M$. Assume that $\mathrm{Ker}\,(\varphi)$ is not essential in M. Then there exists $0 \neq C \leq M$ such that $\mathrm{Ker}\,(\varphi) \oplus C \leq^e M$. By retractability, we can obtain that there exists $0 \neq f \in E_M$ such that $f(M) \leq C$. We have $\mathrm{Ker}\,(\varphi) \cap f(M) \leq \mathrm{Ker}\,(\varphi) \cap C = 0$, and so $fE_M \cap r_{E_M}(\varphi) = 0$. Since $r_{E_M}(\varphi) \leq^e E_M$, $fE_M = 0$, a contradiction. Thus, $\mathrm{Ker}\,(\varphi) \leq^e M$ or $\varphi \in \Delta[M, N]$, $\varphi = 0$.

(ii) Suppose that $\nabla[M,N] = 0$. Let $\varphi \in [M,N]$ be such that $\varphi[N,M]E_N \ll E_N$. Assume that $\mathrm{Im}\,(\varphi)$ is not small in N. Then there exist $C \leq M$ and $C \neq N$ such that $\mathrm{Im}\,(\varphi)+C = N$. By coretractability, we can obtain that there exists $f \in E_N$, $f \neq 0$, such that $f(C) = 0$. Hence, $f(N) = f\varphi(M)$. Since N is M-projective, there exists an $h \in [N,M]$ such that $f = f\varphi h$. Therefore, $E_N = r_{E_N}(f)+\varphi[N,M]E_N$. It follows that $E_N = r_{E_N}(f)$ or $f = 0$, a contradiction. Thus, $\varphi = 0$ by the hypothesis. □

We will use the following notation, where M and N are R-modules:

$$Z^M(N) = \sum_{\varphi \in \Delta[M,N]} \varphi(M)$$

$$Z_M(N) = \bigcap_{\varphi \in \nabla[M,N]} \mathrm{Ker}\,(\varphi).$$

In **[11]**, Talebi and Vanaja defined $\overline{Z}(M)$ as follows:

$$\overline{Z}(M) = \mathrm{Re}\,(M,\mathcal{S}) = \bigcap\{\mathrm{Ker}\,(g) \mid g \in \mathrm{Hom}\,(M,L), L \in \mathcal{S}\},$$

where $\mathcal{S}$ denotes the class of all small modules. We called M a *cosingular* (*noncosingular*) module if $\overline{Z}(M) = 0$ ($\overline{Z}(M) = M$).

A submodule N of M is said to be *fully invariant* if $f(N)$ is contained in N for every $f \in \mathrm{End}\,(M_R)$. Clearly, 0 and M are fully invariant submodules of M.

Theorem 2.12. *Let M and N be modules. Then*

(i) *$\overline{Z}(M)$ is a submodule of $Z_M(N)$.*
(ii) *$Z^M(N)$ is a fully invariant submodule of N. Moreover, $Z^M(N) \leq Z(N)$.*
(iii) *$\Delta[M,N] = 0$ if and only if $Z^M(N) = 0$.*
(iv) *$Z_M(N)$ is a fully invariant submodule of M.*
(v) *$\nabla[M,N] = 0$ if and only if $M/Z_M(N) = 0$.*

Proof. (i) Clear.

(ii) Let $\varphi \in \Delta[M,N]$. Then $\mathrm{Ker}\,(\varphi) \leq^e M$ and so, for all $f \in E_N$, we can obtain that $\mathrm{Ker}\,(\varphi) \leq \mathrm{Ker}\,(f\varphi)$. Therefore, $f\varphi \in \Delta[M,N]$.

For every $n \in Z^M(N)$ and $n \neq 0$, we have $n = n_1 + n_2 + \cdots n_k$ for some $n_i \in \mathrm{Im}\,(\varphi_i)$ and $\varphi_i \in \Delta[M, N]$, $i = 1, 2, \ldots, k$. Assume that $n_i \neq 0$ for all $i = 1, 2, \ldots, k$. For each n_i, there exist $0 \neq m_i \in M$ and $I_i \leq^e R_R$ such that $n_i = \varphi(m_i)$ and $m_i I_i \leq Ker(\varphi_i)$. Then $\varphi_i(m_i I_i) = n_i I_i = 0$ for all $i = 1, 2, \ldots, k$. Let $I = \bigcap_{i=1}^k I_i$. Clearly, $I \leq^e R_R$ and $nI = 0$, which implies that $n \in Z(N)$.

(iii) $Z^M(N) = 0 \Leftrightarrow \varphi = 0$ for all $\varphi \in [M, N]$ with $\mathrm{Ker}\,(\varphi) \leq^e M$.

(iv) Let $\varphi \in \nabla[M, N]$. Then $\mathrm{Im}\,(\varphi) \ll N$ and so, for all $f \in E_M$, we can obtain that $\mathrm{Im}\,(\varphi f) \leq \mathrm{Im}\,(\varphi)$ and so $\varphi f \in \nabla[M, N]$. Therefore, $Z_M(N)$ is a fully invariant submodule of M.

(v) $M/Z_M(N) = 0$ if and only if $M = Z_M(N) \Leftrightarrow \varphi = 0$ for all $\varphi \in [M, N]$ with $\mathrm{Im}\,(\varphi) \ll N$. □

We finish this study with the following result.

Theorem 2.13. *Let $M = \oplus_{i\in\mathcal{I}} M_i$ and $N = \oplus_{j\in\mathcal{J}} N_j$ be modules. Then*

(i) $Z^M(N) = \oplus_{j\in\mathcal{J}} Z^M(N_j)$.
(ii) $Z_M(N) = \oplus_{i\in\mathcal{I}} Z_{M_i}(N)$.

Proof. (i) By Theorem 2.12, $Z^M(N)$ is a fully invariant submodule of N. Then, by [**9,** Lemma 2.1], we have

$$Z^M(N) = \oplus_{j\in\mathcal{J}}[N_j \cap Z^M(N)].$$

For a fixed $j \in \mathcal{J}$, let $x \in Z^M(N_j)$. Then $x = \alpha_1(m_1) + \cdots + \alpha_1(m_n)$ for some n, where $\alpha_k \in [M, N_j]$, $m_k \in M$ and $\mathrm{Ker}\,(\alpha_k) \leq^e M$, for all $1 \leq k \leq n$. Let $\beta_k = \iota_j \alpha_k$ for all $1 \leq k \leq n$ with $\iota_j : N_j \to N$ the inclusion maps. Then $x = \beta_1(m_1) + \cdots + \beta_1(m_n)$ and $\beta_k \in \Delta[M, N]$ for all $1 \leq k \leq n$. It follows that $x \in N_j \cap Z^M(N)$.

The inclusion $Z^M(N) \subseteq \oplus_{j\in\mathcal{J}} Z^M(N_j)$ is obvious.

(ii) Since $Z_M(N)$ is a fully invariant submodule of M by Theorem 2.12, we can obtain that

$$Z_M(N) = \oplus_{i\in\mathcal{I}}[M_i \cap Z_M(N)].$$

For a fixed $i \in \mathcal{I}$, let $m \in Z_{M_i}(N)$. Then $\varphi(m) = 0$ for all $\varphi \in \nabla[M_i, N]$. Let $\iota_i : M_i \to M$ be the inclusion maps. For any $\alpha \in \nabla[M,N]$, we can obtain that $\mathrm{Im}\,(\alpha\iota_i) \leq Im(\alpha) \ll N$ which implies that $\mathrm{Im}\,(\alpha\iota_i) \ll N$ or $\alpha\iota_i \in \nabla[M_i,N]$. It follows that $\alpha(m) = \alpha\iota_i(m) = 0$, i.e., $m \in M_i \cap Z_M(N)$.

The inclusion $Z_M(N) \leq \oplus_{i\in\mathcal{I}} Z_{M_i}(N)$ is obvious. $\square$

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