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NONOSCILLATORY SOLUTIONS TO FORCED HIGHER-ORDER NONLINEAR NEUTRAL DYNAMIC EQUATIONS ON TIME SCALES

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ABSTRACT. By employing Kranoselskii's fixed point theorem, we obtain sufficient conditions for the existence of nonoscillatory solutions of the forced higher-order nonlinear neutral dynamic equation

$$[x(t)+p(t)x(\tau(t))]^{\nabla^m}+\sum_{i=1}^{k}p_i(t)f_i(x(\tau_i(t)))=q(t)$$

on a time scale, where $p_i(t)$, $f_i(t)$ and $q(t)$ may be oscillatory. Then we establish sufficient and necessary conditions for the existence of nonoscillatory solutions to the equation $[x(t)+p(t)x(\tau(t))]^{\nabla^m}+F(t,x(\delta(t)))=q(t)$. Finally, we deal with dynamic equation

$$[x(t)+p(t)x(\tau(t))]^{\nabla^{m-1}\Delta}+\sum_{i=1}^{k}p_i(t)f_i(x(\tau_i(t)))=q(t)$$

with mixed ∇ and Δ derivatives. In particular, some interesting examples are included to illustrate the versatility of our results.

1. Introduction. Following Hilger's breakthrough result [**8**], a rapidly expanding body of literature has sought to unify, extend and generalize ideas from continuous and discrete calculus to arbitrary time-scale calculus, where a time scale is simply any nonempty closed set of real numbers $\mathbb{R}$. Let $\mathbb{T}$ be a time scale which is unbounded above and $t_0\in\mathbb{T}$ a fixed point. For some basic facts on time scale calculus and dynamic equations on time scales, one may consult the excellent texts by Bohner and Peterson [**2, 3**].

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Throughout this work, we investigate the existence of nonoscillatory solutions of the forced higher-order nonlinear neutral dynamic equation with delay and advance terms given by

$$[x(t)+p(t)x(\tau(t))]^{\nabla^m}+\sum_{i=1}^{k}p_i(t)f_i(x(\tau_i(t)))=q(t), \quad t\in[t_0,\infty)_{\mathbb{T}}, \tag{1.1}$$

where $2\le m\in\mathbb{N}$, $t\in\mathbb{T}$, $p,p_i,q\in C_{ld}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$, $\tau,\tau_i\in C([t_0,\infty)_{\mathbb{T}},\mathbb{T})$ with $\lim_{t\to\infty}\tau(t)=\lim_{t\to\infty}\tau_i(t)=+\infty$ and $f_i\in C(\mathbb{R},\mathbb{R})$, $i=1,2,\dots,k$.

We obtain some sufficient conditions for the existence of nonoscillatory solutions of (1.1) without using nondecreasing condition on the functions $f_i(x)$ with $xf_i(x)>0$ $(i=1,2,\dots,k)$ for $x\neq 0$, any sign conditions on the functions $p_i(t)$ $(i=1,2,\dots,k)$ and $q(t)$ via Kranoselskii's fixed point theorem and some new techniques.

After giving our results on the existence of bounded nonoscillatory solution of (1.1) in subsection 3.1, we extend our results to

$$[x(t)+p(t)x(\tau(t))]^{\nabla^m}+F(t,x(\delta(t)))=q(t), \quad t\in[t_0,\infty)_{\mathbb{T}}, \tag{1.2}$$

where $\delta\in C([t_0,\infty)_{\mathbb{T}},\mathbb{T})$ with $\lim_{t\to\infty}\delta(t)=+\infty$ and $F\in C([t_0,\infty)_{\mathbb{T}}\times\mathbb{R},\mathbb{R})$. With some additional assumptions, we establish sufficient and necessary conditions for the existence of nonoscillatory solutions of (1.2). Also, we discuss the existence of unbounded nonoscillatory solutions of (1.1) and (1.2).

Finally, we consider the dynamic equation related to (1.1) with mixed ∇ and Δ derivatives

$$[x(\cdot)+p(\cdot)x(\tau(\cdot))]^{\nabla^{m-1}\Delta}(t)+\sum_{i=1}^{k}p_i(t)f_i(x(\tau_i(t)))=q(t), \quad t\in[t_0,\infty)_{\mathbb{T}}. \tag{1.3}$$

By **[2,** Theorem 8.49(ii)] (see also the following Theorem 5.1), (1.3) can be reduced to a similar form of (1.1).

Related to the above equations is the dynamic equation with Δ derivatives

$$[x(t)+p(t)x(\tau(t))]^{\Delta^m}+F(t,x(\delta(t)))=0, \quad t\in[t_0,\infty)_{\mathbb{T}}. \tag{1.4}$$

In 1992, by employing Kranoselskii's fixed point theorem Chen [**5**] dealt with the existence of nonoscillatory solutions to a special case of (1.4) with $\mathbb{T} = \mathbb{R}$, and some related results are summarized in [**6**]. In 2007, Zhu and Wang [**15**] presented some conditions for the existence of nonoscillatory solutions of (1.4) for $m = 1$. In [**13, 14**], Zhu discussed the existence of unbounded nonoscillatory solutions of (1.4) for $m = 2$ and $2 \leq m \in \mathbb{N}$, respectively. Zhang et al. [**11**] used the contraction principle to obtain sufficient conditions for existence of nonoscillatory solutions of higher-order dynamic equations. Recently, Gao and Wang [**7**] discussed the existence of nonoscillatory solutions of second-order nonlinear neutral dynamic equations of the form

$$[r(t)(x(t) + p(t)x(\tau(t)))^{\Delta}]^{\Delta} + f(t, x(\delta(t))) = 0$$

on a time scale $\mathbb{T}$ under the condition $\int_{t_0}^{\infty} 1/r(s)\Delta s < \infty$. Zhu [**12**] also used Kranoselskii's fixed point theorem to study the existence of bounded nonoscillatory solutions of higher-order dynamic equations with Δ derivative.

This paper is organized as follows. Following this introduction, we prove some basic lemmas in Section 2. Our main results are presented in Section 3, and their applications are given in Section 4. In Section 5, we summarize the basic knowledge on time scales used in this paper just for the convenience of the reader.

2. Preliminaries. Let k be a nonnegative integer and $s, t \in \mathbb{T}$. We define two sequences of functions $\widehat{h}_k(t, s)$ and $\widehat{g}_k(t, s)$ as follows (see [**1**]):

$$\widehat{h}_k(t, s) = \begin{cases} 1, & k = 0, \\ \int_s^t \widehat{h}_{k-1}(\tau, s)\nabla\tau, & k \geq 1, \end{cases}$$
$$\widehat{g}_k(t, s) = \begin{cases} 1, & k = 0, \\ \int_s^t \widehat{g}_{k-1}(\rho(\tau), s)\nabla\tau, & k \geq 1, \end{cases}$$

where the definitions may refer to the following Section 5.

Similar to the Δ derivative, we have that:

$$\widehat{h}_k(t, s) = (-1)^k \widehat{g}_k(s, t), \widehat{h}_k^{\nabla_t}(t, s) = \begin{cases} 0, k = 0, \\ \widehat{h}_{k-1}(t, s), & k \geq 1, \end{cases}$$

$$\widehat{g}_k^{\nabla_t}(t,s) = \begin{cases} 0, k=0, \\ \widehat{g}_{k-1}(\rho(t),s), \quad k \geq 1, \end{cases}$$
$$\widehat{g}_k^{\nabla_s}(t,s) = \begin{cases} 0, k=0, \\ -\widehat{g}_{k-1}(\rho(t),s), \quad k \geq 1, \end{cases}$$

where $\widehat{h}_k^{\nabla_t}(t,s)$ and $\widehat{g}_k^{\nabla_t}(t,s)$ denote for each fixed s the derivative of $\widehat{h}_k(t,s)$ and $\widehat{g}_k(t,s)$ with respect to t, respectively. The definition of $\widehat{g}_k^{\nabla_s}(t,s)$ is similar. From the definition of $\widehat{h}_k(t,s)$, it is easy to obtain the following property of $\widehat{h}_k(t,s)$.

Property 2.1. Using induction and the definition of the function $\widehat{h}_k(t,s)$, it is easy to see that $\widehat{h}_k(t,s) \geq 0$ holds for all $k \in \mathbb{N}_0$ and $s,t \in \mathbb{T}$ with $t \geq s$, and $(-1)^k \widehat{h}_k(t,s) \geq 0$ holds for all $k \in \mathbb{N}_0$ and $s,t \in \mathbb{T}$ with $t \leq s$. In view of the fact that $\widehat{h}_k^{\nabla_t}(t,s) = 0$, $k=0$, and $\widehat{h}_k^{\nabla_t}(t,s) = \widehat{h}_{k-1}(t,s)$, $k \in \mathbb{N}$, $\widehat{h}_n(t,s)$ is increasing in t provided that $t \geq s$, and $(-1)^n \widehat{h}_n(t,s)$ is decreasing in t provided that $t \leq s$. Moreover, $\widehat{h}_n(t,s) \leq (t-s)^{k-l} \widehat{h}_l(t,s)$ holds for all $s,t \in \mathbb{T}$ with $t \geq s$ and for all $k,l \in \mathbb{N}_0$ with $l \leq k$.

Property 2.1 is also true for Δ derivatives. The corresponding result can be found in [**9**].

Similar to [**9,** Lemma 1], we prove the following lemma on the change of order in double (iterated) integrals.

Lemma 2.1. (Change of integration order). *Assume that* $t,s \in \mathbb{T}$ *and* $g \in C_{ld}(\mathbb{T} \times \mathbb{T}, \mathbb{R})$. *Then*

$$\int_s^t \left[\int_\eta^t g(\eta,\xi) \nabla\xi \right] \nabla\eta = \int_s^t \left[\int_s^{\rho(\xi)} g(\eta,\xi)\nabla\eta \right] \nabla\xi. \tag{2.1}$$

Proof. Set

$$A(t) := \int_s^t \left[\int_\eta^t g(\eta,\xi) \nabla\xi \right] \nabla\eta - \int_s^t \left[\int_s^{\rho(\xi)} g(\eta,\xi)\nabla\eta \right] \nabla\xi \tag{2.2}$$

for $t \in \mathbb{T}$. Applying [**2,** Theorem 8.50] (also see Theorem 5.2 (iv)) to (2.2), we have

$$A^\nabla(t) = \int_s^t \frac{\partial}{\nabla t} \left[\int_\eta^t g(\eta,\xi) \nabla\xi \right] \nabla\eta$$

$$
\begin{aligned}
&+\int_t^{\rho(t)} g(t,\xi)\nabla\xi - \int_s^{\rho(t)} g(\eta,t)\nabla\eta \\
&= \int_s^t g(\eta,t)\nabla\eta + \int_t^{\rho(t)} g(t,\xi)\nabla\xi - \int_s^{\rho(t)} g(\eta,t)\nabla\eta \\
&= \int_{\rho(t)}^t g(\eta,t)\nabla\eta + \int_t^{\rho(t)} g(t,\xi)\nabla\xi \\
&= \nu(t)g(t,t) - \nu(t)g(t,t) = 0
\end{aligned}
$$

for all $t \in \mathbb{T}$. Hence, $A(t)$ is a constant function. On the other hand, we see that $A(s) = 0$ holds. Hence, $A(t) \equiv 0$ on $\mathbb{T}$, and this shows that (2.1) is true. □

As an immediate consequence, we can give the following generalization of Lemma 2.1 for n-fold integrals.

Corollary 2.2. *Assume that $n \in \mathbb{N}$, $s,t \in \mathbb{T}$ and $f \in C_{ld}(\mathbb{T}\times\mathbb{R},\mathbb{R})$. Then*

(2.3)

$$\int_s^t \int_{\eta_n}^t \cdots \int_{\eta_2}^t f(\eta_1)\nabla\eta_1\nabla\eta_2\cdots\nabla\eta_n = (-1)^n \int_s^t \widehat{h}_n(s,\rho(\eta))f(\eta)\nabla\eta.$$

Proof. We make use of Lemma 2.1 and the induction principle to complete the proof. From Lemma 2.1, it is clear that (2.3) holds for $n = 2$. Suppose now that (2.3) holds for some $2 \le n \in \mathbb{N}$. Integrating (2.3) over $[s,t)_{\mathbb{T}}$ and using Lemma 2.1, we obtain

$$
\begin{aligned}
&(-1)^n \int_s^t \int_\eta^t \hat{h}_n(\eta,\rho(\xi))f(\xi)\nabla\xi\nabla\eta \\
&= (-1)^n \int_s^t \int_s^{\rho(\xi)} \hat{h}_n(\eta,\rho(\xi))f(\xi)\nabla\eta\nabla\xi \\
&= (-1)^{n+1} \int_s^t \int_{\rho(\xi)}^s \hat{h}_n(\eta,\rho(\xi))f(\xi)\nabla\eta\nabla\xi \\
&= (-1)^{n+1} \int_s^t \hat{h}_{n+1}(s,\rho(\xi))f(\xi)\nabla\eta\nabla\xi,
\end{aligned}
$$

which proves that (2.3) holds for $(n+1)$. The proof is complete. □

Lemma 2.3. *Let $n \in \mathbb{N}_0$, $h \in C_{ld}(\mathbb{T}, [0,\infty))$ and $s \in \mathbb{T}$. Then each of the following is true*:

(i) $\int_s^\infty \widehat{g}_n(\rho(\tau), s)h(\tau)\nabla\tau < \infty$ *implies that* $\int_t^\infty \widehat{g}_n(\rho(\tau), t)h(\tau)\nabla\tau < \infty$ *for all* $t \in \mathbb{T}$;

(ii) $\int_s^\infty \widehat{g}_n(\rho(\tau), s)h(\tau)\nabla\tau = \infty$ *implies that* $\int_t^\infty \widehat{g}_n(\rho(\tau), t)h(\tau)\nabla\tau = \infty$ *for all* $t \in \mathbb{T}$.

Proof. To complete the proof, we shall employ the induction principle. We need to show that

$$\int_s^\infty \hat{h}_n(s, \rho(\tau))h(\tau)\nabla\tau \quad \text{and} \quad \int_t^\infty \hat{h}_n(t, \rho(\tau))h(\tau)\nabla\tau$$

diverge or converge together by the formula $\widehat{h}_n(t,s) = (-1)^n \widehat{g}_n(s,t)$. The proof is trivial for $n = 0$. Suppose that the claim holds for some $n \in \mathbb{N}$. We shall show that it is also true for $n+1$. Without loss of generality, we may suppose that $s \geq t$. From the definition of $\widehat{h}_n(t,s)$ and Lemma 2.1, we have

$$\begin{aligned}
&\int_t^\infty \widehat{h}_{n+1}(t, \rho(\eta))h(\eta)\nabla\eta \\
&= \int_t^\infty \int_{\rho(\eta)}^t \widehat{h}_n(\xi, \rho(\eta))h(\eta)\nabla\xi\nabla\eta \\
&= -\int_t^\infty \int_t^{\rho(\eta)} \widehat{h}_n(\xi, \rho(\eta))h(\eta)\nabla\xi\nabla\eta \\
&= -\int_t^\infty \int_\xi^\infty \widehat{h}_n(\xi, \rho(\eta))h(\eta)\nabla\eta\nabla\xi \\
&= -\int_s^\infty \int_\xi^\infty \widehat{h}_n(\xi, \rho(\eta))h(\eta)\nabla\eta\nabla\xi - \int_t^s \int_\xi^\infty \widehat{h}_n(\xi, \rho(\eta))h(\eta)\nabla\eta\nabla\xi.
\end{aligned}$$

First, consider the case that $(-1)^n \int_r^\infty \widehat{h}_n(r, \rho(\eta))h(\eta)\nabla\eta = \infty$ holds for all $r \in \mathbb{T}$. Clearly, this implies by the above formula that $(-1)^{n+1} \int_s^\infty \widehat{h}_{n+1}(s, \rho(\eta))h(\eta)\nabla\eta = \infty$, and thus

$$(-1)^{n+1} \int_t^\infty \widehat{h}_{n+1}(s, \rho(\eta))h(\eta)\nabla\eta = \infty$$

since $s \geq t$. Next, by property 2.1, we just consider the case that $(-1)^n \int_r^\infty \widehat{h}_n(r, \rho(\eta))h(\eta)\nabla\eta < \infty$ for all $r \in \mathbb{T}$. In view of the definition of $\widehat{h}_n(t,s)$ and Lemma 2.1, we get

$$\begin{aligned}(2.4)\quad \int_s^\infty \widehat{h}_{n+1}(s,\rho(\eta))h(\eta)\nabla\eta &= \int_t^\infty \widehat{h}_{n+1}(t,\rho(\eta))h(\eta)\nabla\eta \\ &\quad + \int_t^s \int_\xi^\infty \widehat{h}_n(\xi,\rho(\eta))h(\eta)\nabla\eta\nabla\xi.\end{aligned}$$

Using the fact that the last term on the right side of (2.4) is finite, we see that $\int_s^\infty \widehat{h}_{n+1}(\rho(\tau),s)h(\tau)\nabla\tau$ and $\int_t^\infty \widehat{h}_{n+1}(t,\rho(\tau))h(\tau)\nabla\tau$ diverge or converge together. This proves that the claim holds for $(n+1)$, and the proof is complete. □

Lemma 2.4. *Let $n \in \mathbb{N}_0$, $h \in C_{ld}(\mathbb{T},[0,\infty))$ and $s \in \mathbb{T}$. Then*

$$(2.5)\quad (-1)^n \int_s^\infty \widehat{h}_n(s,\rho(\tau))h(\tau)\nabla\tau = \int_s^\infty \widehat{g}_n(\rho(\tau),s)h(\tau)\nabla\tau < \infty$$

implies that each of the following is true:

(i) $(-1)^j \int_t^\infty \widehat{h}_j(t,\rho(\tau))h(\tau)\nabla\tau = \int_t^\infty \widehat{g}_j(\rho(\tau),t)\nabla\tau$ *is decreasing for all $t \in \mathbb{T}$ and $0 \le j \le n$;*

(ii) $\lim_{t\to\infty}(-1)^j \int_t^\infty \widehat{h}_j(t,\rho(\tau))h(\tau)\nabla\tau = \lim_{t\to\infty}\int_t^\infty \widehat{g}_j(\rho(\tau),t)\nabla\tau = 0$ *for all $0 \le j \le n$;*

(iii) $(-1)^j \int_t^\infty \widehat{h}_j(t,\rho(\tau))h(\tau)\nabla\tau = \int_t^\infty \widehat{g}_j(\rho(\tau),t)\nabla\tau < \infty$ *for all $t \in \mathbb{T}$ and $0 \le j \le n-1$.*

Proof. The proof for $n = 0$ is trivial. Now, let $n \in \mathbb{N}$. To complete the proof, it suffices to prove (i) and (ii) for $j = n$ and (iii) for $j = n-1$ because the proof can be completed by repeating the emerging pattern. Obviously, (2.5) implies

$$(2.6)\quad \lim_{t\to\infty}(-1)^n \int_t^\infty \hat{h}_n(s,\rho(\tau))h(\tau)\nabla\tau = 0.$$

By Property 2.1, we have

$$0 \le (-1)^n \int_t^\infty \widehat{h}_n(t,\rho(\tau))h(\tau)\nabla\tau \le (-1)^n \int_t^\infty \widehat{h}_n(s,\rho(\tau))h(\tau)\nabla\tau,$$

which proves (ii) for $j=n$ by (2.6). Next, we prove (iii) for $j=n-1$. Suppose, to the contrary, that

$$(-1)^{n-1}\int_s^\infty \widehat{h}_{n-1}(s,\rho(\tau))h(\tau)\nabla\tau=\infty.$$

By Lemmas 2.1 and 2.2 (ii), we obtain

$$\begin{aligned}
&(-1)^n\int_s^\infty \widehat{h}_n(s,\rho(\tau))h(\tau)\nabla\tau\\
&\quad=(-1)^n\int_s^\infty\int_{\rho(\tau)}^s \widehat{h}_{n-1}(\xi,\rho(\tau))h(\tau)\nabla\xi\nabla\tau\\
&\quad=(-1)^{n-1}\int_s^\infty\int_\xi^\infty \widehat{h}_{n-1}(\xi,\rho(\tau))h(\tau)\nabla\tau\nabla\xi=\infty,
\end{aligned}$$

which contradicts (2.5). Therefore, (iii) is true for $j=n-1$. We finally prove (i) for $j=n$; by the property of $\widehat{h}_n(t,s)$ and Property 2.1, we have

$$\left[(-1)^n\int_t^\infty \widehat{h}_n(t,\rho(\tau))h(\tau)\nabla\tau\right]^{\nabla_t}=(-1)^n\int_t^\infty \widehat{h}_{n-1}(t,\rho(\tau))h(\tau)\nabla\tau\le 0$$

for all $t\in\mathbb{T}$. The proof is complete. $\square$

Lemmas 2.2 and 2.3 are analogous to [**9,** Lemma 2] or [**10,** Lemma 2.2] and [**10,** Lemma 2.3], respectively. We would like to point out that the idea to prove Lemmas 2.4 and 2.5 comes from [**4,** Lemmas 2.1 and 2.2].

Lemma 2.5. *Let $u(t)\in C_{ld}^m([t_0,\infty)_{\mathbb{T}},(0,\infty))$. If u^{∇^m} is of constant sign on $[t_0,\infty)_{\mathbb{T}}$ and not identically zero on $[t_1,\infty)_{\mathbb{T}}$ for any $t_1\ge t_0$, then there exist a $t_u\ge t_0$ and integer l, $0\le l\le m$, with $m+l$ even for $u^{\nabla^m}\ge 0$, or $m+l$ odd for $u^{\nabla^m}\le 0$ such that*

$$l>0 \text{ implies that } u^{\nabla^k}(t)>0 \text{ for } t\ge t_u,\quad k=0,1,2,\dots,l-1, \tag{2.7}$$

and

$$l\le m-1 \text{ implies that } (-1)^{l+k}u^{\nabla^k}(t)>0 \text{ for } t\ge t_u,\quad k=l,l+1,l+2,\dots,m-1. \tag{2.8}$$

Proof. We shall consider only the case when $m \geq 3$ is odd and $u^{\nabla^m}(t) \geq 0$ on $[t_0, \infty)_{\mathbb{T}}$ because the proofs of other cases are similar. From the conditions that $u^{\nabla^m}(t) \geq 0$ on $[t_0, \infty)_{\mathbb{T}}$ and is not identically zero on $[t_1, \infty)_{\mathbb{T}}$ for any $t_1 \geq t_0$, we see that $u^{\nabla^{m-1}}(t)$ is increasing on $[t_1, \infty)_{\mathbb{T}}$ for any $t_1 \geq t_0$. This implies that exactly one of the following is true:

(a_1) There exists a $t_2 \geq t_0$ such that $u^{\nabla^{m-1}}(t) > 0$ for $t \in [t_2, \infty)_{\mathbb{T}}$;

(b_1) $u^{\nabla^{m-1}}(t) < 0$ for $t \geq t_0$.

It is easy to see that

$$u^{\nabla^{m-2}}(t) - u^{\nabla^{m-2}}(t_2) = \int_{t_2}^{t} u^{\nabla^{m-1}}(s)\nabla s \geq u^{\nabla^{m-1}}(t_2)(t - t_2), \quad t \in [t_2, \infty).$$

If (a_1) holds, then we have $u^{\nabla^{m-1}}(t_2) > 0$ and $\lim_{t\to\infty} u^{\nabla^{m-2}}(t) = \infty$. Analogously, we get

$$\lim_{t\to\infty} u^{\nabla^{m-3}}(t) = \lim_{t\to\infty} u^{\nabla^{m-4}}(t) = \cdots = \lim_{t\to\infty} u^{\nabla}(t) = \infty.$$

Thus, the conclusions of Lemma 2.4 hold.

If (b_1) holds, then $u^{\nabla^{m-2}}(t)$ is strictly decreasing on $[t_0, \infty)_{\mathbb{T}}$ and exactly one of the following possibilities holds true:

(a_2) There exists a $t_3 \geq t_0$ such that $u^{\nabla^{m-2}}(t) < 0$ for $t \in [t_3, \infty)_{\mathbb{T}}$;

(b_2) $u^{\nabla^{m-2}}(t) > 0$ for $t \geq t_0$.

From (b_1), we have $u^{\nabla^{m-2}}(t) \leq u^{\nabla^{m-2}}(t_3)$ for $t \in [t_3, \infty)_{\mathbb{T}}$. By integrating the both sides of the last inequality from t_3 to t, we obtain

$$u^{\nabla^{m-3}}(t) - u^{\nabla^{m-3}}(t_3) \leq u^{\nabla^{m-2}}(t_3)(t - t_3), \quad t \in [t_3, \infty)_{\mathbb{T}}.$$

If (a_2) holds, then we get $u^{\nabla^{m-2}}(t_3) < 0$ and $\lim_{t\to\infty} u^{\nabla^{m-3}}(t) = -\infty$. Similarly, we find

$$\begin{aligned}\lim_{t\to\infty} u^{\nabla^{m-4}}(t) &= \lim_{t\to\infty} u^{\nabla^{m-5}}(t) = \cdots \\ &= \lim_{t\to\infty} u^{\nabla}(t) = \lim_{t\to\infty} u(t) = -\infty,\end{aligned}$$

which contradicts the fact that $u(t) > 0$ for $t \in [t_0, \infty)_{\mathbb{T}}$. Hence, (a_2) is impossible. From (b_2), we see that $u^{\nabla^{m-3}}(t)$ is strictly increasing on $t \in [t_0, \infty)_{\mathbb{T}}$ and exactly one of the following is valid:
(a_3) There exists a $t_4 \geq t_0$ such that $u^{\nabla^{m-3}}(t) > 0$ for $t \in [t_4, \infty)_{\mathbb{T}}$;
(b_3) $u^{\nabla^{m-3}}(t) < 0$ for $t \geq t_0$.

Therefore, we can repeat the above arguments and show that the conclusions of Lemma 2.4 hold. The proof of Lemma 2.4 is complete. □

Lemma 2.6. *Let $u(t) \in C_{ld}^m([t_0, \infty)_{\mathbb{T}}, (0, \infty))$ be bounded on $[t_0, \infty)_{\mathbb{T}}$. Suppose that u^{∇^m} is of constant sign on $[t_0, \infty)_{\mathbb{T}}$ and not identically zero on $[t_1, \infty)_{\mathbb{T}}$ for any $t_1 \geq t_0$. Then there exist a $t_u \geq t_0$ and integer $l = 0$ or $l = 1$, with $m + l$ even for $u^{\nabla^m} \geq 0$, or $m + l$ odd for $u^{\nabla^m} \leq 0$ such that*

$$(2.9) \qquad (-1)^{l+k} u^{\nabla^k}(t) > 0 \quad \textit{for } t \geq t_u, \ k = 1, 2, \ldots, m-1,$$

and

$$(2.10) \qquad \lim_{t \to \infty} u^{\nabla^k}(t) = 0, \quad k = 1, 2, \ldots, m-1.$$

Proof. We shall discuss only the case when $m \geq 2$ is even and $u^{\nabla^m}(t) \leq 0$ on $[t_0, \infty)_{\mathbb{T}}$ because the proof of the other cases are similar. By Lemma 2.4, there exist a $t_u \geq t_0$ and an odd $l, 0 \leq l \leq m$, such that (2.7) and (2.8) hold. We claim that $l = 1$. Otherwise, $l \geq 3$. According to (2.7), we have $u^{\nabla}(t) > 0$ and $u^{\nabla^2}(t) > 0$ for $t_u \geq t_0$. Thus, we get

$$u^{\nabla}(t) \geq u^{\nabla}(t_u), \quad t \in [t_u, \infty)_{\mathbb{T}}.$$

By integrating both sides of the last inequality from t_u to t, we see

$$u(t) - u(t_u) \geq u^{\nabla}(t_u)(t - t_u), \quad t \in [t_u, \infty)_{\mathbb{T}}.$$

In view of the fact that $u^{\nabla}(t_u) > 0$, letting $t \to \infty$, we find that $\lim_{t\to\infty} u(t) = \infty$, which contradicts the boundedness of $u(t)$. Hence, (2.9) holds.

Next, we prove (2.10). From (2.9), we have $u^{\nabla}(t) > 0$ and $u^{\nabla^2}(t) < 0$ for $t_u \geq t_0$. It follows that $\lim_{t\to\infty} u^{\nabla}(t) := L_1 \geq 0$, and

$$u^{\nabla}(t) \geq L_1, \quad t \in [t_u, \infty)_{\mathbb{T}}.$$

By integrating both sides of the last inequality from t_u to t, we see

$$u(t) - u(t_u) \geq L_1(t - t_u), \quad t \in [t_u, \infty)_{\mathbb{T}}.$$

If $L_1 > 0$, then letting $t \to \infty$ will lead to $\lim_{t\to\infty} u(t) = \infty$, which is a contradiction with the boundedness of $u(t)$. Therefore $L_1 = 0$, i.e.,

$$\lim_{t\to\infty} u^{\nabla}(t) = 0.$$

Also, from (2.9), we have $u^{\nabla^2}(t) < 0$ and $u^{\nabla^3}(t) > 0$ for $t_u \geq t_0$. It follows that $\lim_{t\to\infty} u^{\nabla^2}(t) := L_2 \leq 0$ and

$$u^{\nabla^2}(t) \leq L_2, \quad t \in [t_u, \infty)_{\mathbb{T}}.$$

By integrating the both sides of the last inequality from t_u to t, we see

$$u^{\nabla}(t) - u^{\nabla}(t_u) \leq L_2(t - t_u), \quad t \in [t_u, \infty)_{\mathbb{T}}.$$

If $L_2 < 0$, then letting $t \to \infty$ will lead to $\lim_{t\to\infty} u^{\nabla}(t) = -\infty$, which contradicts the fact that $u^{\nabla}(t) > 0$ for $t \in [t_u, \infty)_{\mathbb{T}}$. Therefore, $L_2 = 0$, i.e.,

$$\lim_{t\to\infty} u^{\nabla^2}(t) = 0. \tag{2.11}$$

From (2.9), we have $u^{\nabla^3}(t) > 0$ and $u^{\nabla^4}(t) < 0$ for $t \in [t_u, \infty)_{\mathbb{T}}$. Hence, we obtain $\lim_{t\to\infty} u^{\nabla^3}(t) := L_3 \geq 0$ and $u^{\nabla^3}(t) \geq L_3$ for $t \in [t_u, \infty)_{\mathbb{T}}$. By integrating both sides of the last inequality from t_u to t, we get

$$u^{\nabla^2}(t) - u^{\nabla^2}(t_u) \geq L_3(t - t_u), \quad t \in [t_u, \infty)_{\mathbb{T}}.$$

If $L_3 > 0$, then letting $t \to \infty$ will give $\lim_{t\to\infty} u^{\nabla^2}(t) = \infty$, which contradicts the fact that $u^{\nabla^2}(t) < 0$ for $t \in [t_u, \infty)_{\mathbb{T}}$. Therefore, we obtain $L_3 = 0$, i.e.,

$$\lim_{t\to\infty} u^{\nabla^3}(t) = 0. \tag{2.12}$$

The rest of the proof is similar to that of (2.11) and (2.12) so that we omit it. The proof of Lemma 2.5 is completed. □

Let $BC_{ld}([t_0, \infty)_{\mathbb{T}}, \mathbb{R})$ be the Banach space of all bounded ld-continuous functions on $[t_0, \infty)_{\mathbb{T}}$ with the norm $||x|| = \sup_{t\in[t_0,\infty)_{\mathbb{T}}} |x(t)|$.

The following is an analogue of the Arzelá-Ascoli theorem on time scales.

Lemma 2.7. [**15,** Lemma 4]. *Suppose that* $X \subseteq BC_{ld}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ *is bounded and uniformly Cauchy. Further, suppose that* X *is equicontinuous on* $[T_0,T_1]_{\mathbb{T}}$ *for any* $T_1 \in [T_0,\infty)_{\mathbb{T}}$. *Then* X *is relatively compact.*

In the next section, we will employ Kranoselskii's fixed point theorem (see [**5, 7, 12, 13, 14, 15**] to establish the existence of nonoscillatory solutions for (1.1). For the sake of convenience, we state this theorem here as follows.

Lemma 2.8. (Kranoselskii's fixed point theorem). *Suppose that* X *is a Banach space and* Ω *is a bounded, convex and closed subset of* X. *Suppose further that there exist two operators* $U, S : \Omega \to X$ *such that*
(i) $Ux + Sy \in \Omega$ *for all* $x, y \in \Omega$;
(ii) U *is a contraction mapping*;
(iii) S *is completely continuous.*
Then $U + S$ *has a fixed point in* Ω.

3. Main results. This section is organized as follows. In subsection 3.1, we give sufficient conditions for the existence of bounded nonoscillatory solutions of (1.1); in subsection 3.2, we state necessary and sufficient conditions for the existence of bounded nonoscillatory solutions of (1.2) and (1.1); in subsection 3.3, we will discuss sufficient (and necessary) conditions for the existence of bounded nonoscillatory solutions of (1.3).

We state the following conditions, which are needed in the sequel:
(H_1) there exists a constant $p \in (\frac{1}{2}, 1)$ such that $|p(t)| \le 1 - p$ for all $t \in [t_0,\infty)_{\mathbb{T}}$;
(H_2) there exist constants $p_1, p_2 \in (-\infty, -1)$ such that $p_1 \le p(t) \le p_2$ for all $t \in [t_0,\infty)_{\mathbb{T}}$;
(H_3) there exists a constant $p \in (-1, 0]$ such that $p \le p(t) \le 0$ for all $t \in [t_0,\infty)_{\mathbb{T}}$;
(H_4) there exists a constant $p \in (0, 1)$ such that $0 < p(t) \le p$ for all $t \in [t_0,\infty)_{\mathbb{T}}$;

(H_5) there exists constants $p_1, p_2 \in (1, +\infty)$ such that $p_1 \le p(t) \le p_2$ for all $t \in [t_0, \infty)_{\mathbb{T}}$.

3.1. Sufficient conditions for (1.1). We state the following results in this subsection, which investigate sufficient conditions for the existence of bounded nonoscillatory solutions of (1.1) with $p(t)$ in one of the ranges (H_1)–(H_5).

Theorem 3.1. *Assume that* (H_1) *holds, and that*

$$\int_{t_0}^{\infty} \widehat{g}_{m-1}(\rho(s), t_0)|p_i(s)|\nabla s < \infty, \quad i = 1, 2, \ldots, k \tag{3.1}$$

and

$$\int_{t_0}^{\infty} \widehat{g}_{m-1}(\rho(s), t_0)|q(s)|\nabla s < \infty. \tag{3.2}$$

Then (1.1) *has a bounded nonoscillatory solution* $x(t)$ *with*

$$\liminf_{t\to\infty} |x(t)| > 0.$$

Proof. For some $d \neq 0$, we choose d_1, c_1 such that $0 < d_1 < (2p-1)|d|$ and $d_1+(1-p)|d| < c_1 < p|d|$. Let $c = \min\{c_1-d_1-(1-p)|d|, p|d|-c_1\}$. By (3.1), (3.2) and Lemma 2.2, there exists a sufficiently large number $t_1 \ge t_0$ such that

$$\int_{t_1}^{\infty} \hat{g}_{m-1}(\rho(s), t_1)\left(\sum_{i=1}^{k} M_1|p_i(s)| + |q(s)|\right)\nabla s \le c$$

and $\tau(t), \tau_i(t) \ge t_0$, $i = 1, 2, \ldots, k$ for $t \ge t_1$, where $M_1 = \max_{d_1 \le x \le |d|}\{|f_i(x)| : 1 \le i \le k\}$.

Let

$$\Omega_1 = \{x \in BC_{ld}([t_0, \infty)_{\mathbb{T}}, \mathbb{R}) \mid d_1 \le x(t) \le |d|, t \in [t_0, \infty)_{\mathbb{T}}\}.$$

It is easy to verify that Ω_1 is a bounded, convex and closed subset of $BC_{ld}([t_0, \infty)_{\mathbb{T}}, \mathbb{R})$.

We define two operators $U_1, S_1 : \Omega_1 \to BC_{ld}([t_0, \infty)_{\mathbb{T}}, \mathbb{R})$ as follows:

$$(U_1x)(t) = \begin{cases} -p(t)x(\tau(t)), & t \in [t_1, \infty)_{\mathbb{T}}, \\ (U_1x)(t_1), & t \in [t_0, t_1]_{\mathbb{T}}, \end{cases}$$

$$(S_1x)(t)=\begin{cases} c_1+(-1)^{m-1}\int_t^\infty \widehat{g}_{m-1}(\rho(s),t) \\ \quad\times\left(\displaystyle\sum_{i=1}^k p_i(s)f_i(x(\tau_i(s)))-q(s)\right)\nabla s, & t\in[t_1,\infty)_{\mathbb{T}}, \\ (S_1x)(t_1), & t\in[t_0,t_1]_{\mathbb{T}}. \end{cases}$$

Next, we show that U_1 and S_1 satisfy the conditions in Lemma 2.7.

(I) We will show that $U_1x+S_1y\in\Omega_1$ for any $x,y\in\Omega_1$. In fact, for any $x,y\in\Omega_1$ and $t\ge t_1$, we have

$$\begin{aligned}
(U_1x)(t)+(S_1y)(t) &\ge c_1-|p(t)|x(\tau(t)) \\
&\quad-\int_t^\infty \hat{g}_{m-1}(\rho(s),t)\left(\sum_{i=1}^k |p_i(s)||f_i(x(\tau_i(s)))|+|q(s)|\right)\nabla s \\
&\ge c_1-(1-p)|d|-\int_{t_1}^\infty \widehat{g}_{m-1}(\rho(s),t_1)\left(\sum_{i=1}^k M_1|p_i(s)|+|q(s)|\right)\nabla s \\
&\ge c_1-(1-p)|d|-[c_1-d_1-(1-p)|d|]=d_1,
\end{aligned}$$

and

$$\begin{aligned}
(U_1x)(t)+(S_1y)(t) &\le c_1+|p(t)|x(\tau(t)) \\
&\quad+\int_t^\infty \widehat{g}_{m-1}(\rho(s),t)\left(\sum_{i=1}^k |p_i(s)||f_i(x(\tau_i(s)))|+|q(s)|\right)\nabla s \\
&\le c_1+(1-p)|d| \\
&\quad+\int_{t_1}^\infty \widehat{g}_{m-1}(\rho(s),t_1)\left(\sum_{i=1}^k M_1|p_i(s)|+|q(s)|\right)\nabla s \\
&\le c_1+(1-p)|d|+p|d|-c_1=|d|,
\end{aligned}$$

which implies that $U_1x+S_1y\in\Omega_1$ for any $x,y\in\Omega_1$.

(II) We will show that U_1 is a contraction mapping. Indeed, for any $x,y\in\Omega_1$, we get

$$\begin{aligned}
||(U_1x)(t)-(U_1y)(t)|| &\le |p(t)||x(\tau(t))-y(\tau(t))| \\
&\le (1-p)||x-y||, \quad t\ge t_1,
\end{aligned}$$

and

$$||(U_1x)(t)-(U_1y)(t)||=0, \quad t_0\le t\le t_1.$$

Hence, U_1 is a contraction mapping.

(III) Finally, we show that S_1 is a completely continuous mapping. According to Lemma 2.6, we need to show that S_1 is continuous and $S_1\Omega_1$ is bounded, uniformly Cauchy and equi-continuous.

(i) Similar to the proof of (I), we see that $d_1 \leq (S_1x)(t) \leq |d|$ for $t \in [t_0, \infty)_{\mathbb{T}}$. That is $S_1\Omega_1 \subset \Omega_1$ and $S_1\Omega_1$ is bounded.

(ii) We claim that S_1 is continuous. Let $x_n \in \Omega_1$ and $||x_n - x|| \to 0$ as $n \to \infty$. Then $x \in \Omega_1$ and $|x_n - x| \to 0$ as $n \to \infty$ for all $t \in [t_0, \infty)_{\mathbb{T}}$. For $t \geq t_1$, we have

$$\begin{aligned}
&||S_1x_n - S_1x|| \\
&\leq \int_{t_1}^{\infty} \widehat{g}_{m-1}(\rho(s), t_1)\left(\sum_{i=1}^{k} |p_i(s)||f_i(x_n(\tau_i(s))) - f_i(x(\tau_i(s)))|\right)\nabla s.
\end{aligned}$$

Since

$$\begin{aligned}
&\widehat{g}_{m-1}(\rho(s), t_1)\left(\sum_{i=1}^{k} |p_i(s)||f_i(x_n(\tau_i(s))) - f_i(x(\tau_i(s)))|\right) \\
&\leq \widehat{g}_{m-1}(\rho(s), t_1)\left(\sum_{i=1}^{k} |p_i(s)||f_i(x_n(\tau_i(s)))| + |f_i(x(\tau_i(s)))|\right) \\
&\leq 2M_1\hat{g}_{m-1}(\rho(s), t_1)\sum_{i=1}^{k} |p_i(s)|
\end{aligned}$$

and

$$|f_i(x_n(\tau_i(s))) - f_i(x(\tau_i(s)))| \longrightarrow 0 \quad (n \to \infty), \text{ for } i = 1, 2, \ldots, k.$$

In view of (3.1) and applying the Lebesgue dominated convergence theorem, we conclude that

$$\lim_{n\to\infty} ||S_1x_n - S_1x|| = 0,$$

which implies that S_1 is continuous on Ω_1.

(iii) Next, we show that $S_1\Omega_1$ is uniformly Cauchy. In fact, for any $\varepsilon \geq 0$, take $t_2 > t_1$ such that

$$\int_{t_2}^{\infty} \widehat{g}_{m-1}(\rho(s), t_2)\left(\sum_{i=1}^{k} M_1|p_i(s)| + |q(s)|\right)\nabla s \leq \varepsilon/2.$$

Then, for any $x \in \Omega_1$ and $t, r \in [t_2, \infty)_{\mathbb{T}}$, we have

$$\begin{aligned}
&||S_1x)(t) - S_1x)(r)|| \\
&\leq \left| \int_t^{\infty} \widehat{g}_{m-1}(\rho(s), t)\left(\sum_{i=1}^{k} |p_i(s)||f_i(x(\tau_i(s)))| + |q(s)|\right)\nabla s\right| \\
&\quad + \left| \int_r^{\infty} \hat{g}_{m-1}(\rho(s), r)\left(\sum_{i=1}^{k} |p_i(s)||f_i(x(\tau_i(s)))| + |q(s)|\right)\nabla s\right| \\
&\leq 2\int_{t_2}^{\infty} \widehat{g}_{m-1}(\rho(s), t_2)\left(\sum_{i=1}^{k} M_1|p_i(s)| + |q(s)|\right)\nabla s \leq \varepsilon.
\end{aligned}$$

Therefore, $S_1\Omega_1$ is uniformly Cauchy.

We also have another method to prove $S_1\Omega_1$ is uniformly Cauchy. To do so, we only check that $S_1^{\nabla}(t)$ is bounded. Here we leave it to the readers.

(iv) We show that $S_1\Omega_1$ is equicontinuous on $[t_0, t_2]_{\mathbb{T}}$ for any $t_2 \in [t_0, \infty)_{\mathbb{T}}$. Without loss of generality, we assume that $t_2 \geq t_1$. For any $\varepsilon > 0$, choose

$$\delta = \varepsilon / \int_{t_0}^{\infty} \widehat{g}_{m-1}(\rho(s), t_0)\left(\sum_{i=1}^{k} M_1|p_i(s)| + |q(s)|\right)\nabla s.$$

Then, for any $x \in \Omega_1$, when $t, r \in [t_0, t_2]_{\mathbb{T}}$ with $|t - r| < \delta$, by Lemmas 2.1 and 2.3, we have

$$\begin{aligned}
&||(S_1x)(t) - (S_1x)(r)|| \\
&= \left| \int_t^{\infty} \widehat{g}_{m-1}(\rho(s), t)\left(\sum_{i=1}^{k} p_i(s)f_i(x(\tau_i(s))) - q(s)\right)\nabla s \right. \\
&\quad \left. - \int_r^{\infty} \widehat{g}_{m-1}(\rho(s), r)\left(\sum_{i=1}^{k} p_i(s)f_i(x(\tau_i(s))) - q(s)\right)\nabla s\right| \\
&= \left| \int_t^{\infty} \widehat{h}_{m-1}(t, \rho(s))\left(\sum_{i=1}^{k} p_i(s)f_i(x(\tau_i(s))) - q(s)\right)\nabla s \right. \\
&\quad \left. - \int_r^{\infty} \widehat{h}_{m-1}(r, \rho(s))\left(\sum_{i=1}^{k} p_i(s)f_i(x(\tau_i(s))) - q(s)\right)\nabla s\right|
\end{aligned}$$

$$
\begin{aligned}
&= \bigg| \int_t^\infty \bigg[\int_{\rho(s)}^t \widehat{h}_{m-2}(\theta,\rho(s)) \Big(\sum_{i=1}^k p_i(s) f_i(x(\tau_i(s))) - q(s) \Big) \nabla\theta \bigg] \nabla s \\
&\quad - \int_r^\infty \bigg[\int_{\rho(s)}^r \widehat{h}_{m-2}(\theta,\rho(s)) \Big(\sum_{i=1}^k p_i(s) f_i(x(\tau_i(s))) - q(s) \Big) \nabla\theta \bigg] \nabla s \bigg| \\
&= \bigg| \int_t^\infty \bigg[\int_\theta^\infty \widehat{h}_{m-2}(\theta,\rho(s)) \Big(\sum_{i=1}^k p_i(s) f_i(x(\tau_i(s))) - q(s) \Big) \nabla s \bigg] \nabla\theta \\
&\quad - \int_r^\infty \bigg[\int_\theta^\infty \widehat{h}_{m-2}(\theta,\rho(s)) \Big(\sum_{i=1}^k p_i(s) f_i(x(\tau_i(s))) - q(s) \Big) \nabla s \bigg] \nabla\theta \bigg| \\
&= \bigg| \int_t^\infty \bigg[\int_\theta^\infty \widehat{g}_{m-2}(\rho(s),\theta) \Big(\sum_{i=1}^k p_i(s) f_i(x(\tau_i(s))) - q(s) \Big) \nabla s \bigg] \nabla\theta \\
&\quad - \int_r^\infty \bigg[\int_\theta^\infty \widehat{g}_{m-2}(\rho(s),\theta) \Big(\sum_{i=1}^k p_i(s) f_i(x(\tau_i(s))) - q(s) \Big) \nabla s \bigg] \nabla\theta \bigg| \\
&= \bigg| \int_t^r \bigg[\int_\theta^\infty \widehat{g}_{m-2}(\rho(s),\theta) \Big(\sum_{i=1}^k p_i(s) f_i(x(\tau_i(s))) - q(s) \Big) \nabla s \bigg] \nabla\theta \bigg| \\
&\le \bigg| \int_t^r \bigg[\int_{t_0}^\infty \widehat{g}_{m-2}(\rho(s),t_0) \Big(\sum_{i=1}^k p_i(s) f_i(x(\tau_i(s))) - q(s) \Big) \nabla s \bigg] \nabla\theta \bigg| \\
&= |t-r| \int_{t_0}^\infty \widehat{g}_{m-2}(\rho(s),t_0) \bigg| \sum_{i=1}^k p_i(s) f_i(x(\tau_i(s))) - q(s) \bigg| \nabla s \\
&\le \delta \int_{t_0}^\infty \widehat{g}_{m-1}(\rho(s),t_0) \Big(\sum_{i=1}^k M_1 |p_i(s)| + |q(s)| \Big) \nabla s \le \varepsilon.
\end{aligned}
$$

This indicates that $S_1\Omega_1$ is equicontinuous on $[t_0,t_2]_{\mathbb{T}}$ for any $t_2 \in [t_0,\infty)_{\mathbb{T}}$. Hence, by Lemma 2.6, $S_1\Omega_1$ is a completely continuous mapping.

It follows from Lemma 2.7 that there exists an $x \in \Omega_1$ such that $(U_1+S_1)x = x$, which is the desired bounded solution of (1.1) with $\liminf_{t\to\infty} |x(t)| > 0$. The proof of Theorem 3.1 is complete. □

Theorem 3.2. *Assume that* (H_2), (3.1) *and* (3.2) *hold, and* τ *has the inverse function* $\tau^{-1} \in C(\mathbb{T},\mathbb{T})$. *Then* (1.1) *has a bounded nonoscilla-*

tory solution $x(t)$ *with* $\liminf_{t\to\infty} |x(t)| > 0$.

Proof. We choose positive constants $0 < a < b$ and $\beta > 0$ such that $-ap_1 < \beta < -b(p_2+1)$. Let $c = \min\{\frac{(\beta+ap_1)p_2}{p_1}, -b(p_2+1)-\beta\}$. By (3.1), (3.2) and Lemma 2.2, there exists a sufficiently large number $t_1 \geq t_0$ such that

$$\int_{\tau^{-1}(t)}^{\infty} \widehat{g}_{m-1}(\rho(s), \tau^{-1}(t)) \left(\sum_{i=1}^{k} M_2 |p_i(s)| + |q(s)| \right) \nabla s \leq c,$$

and $\tau^{-1}(t), \tau_i(\tau^{-1}(t)) \geq t_0$, $i = 1, 2, \ldots, k$ for $t \geq t_1$, where $M_2 = \max_{a \leq x \leq b} \{|f_i(x)| : 1 \leq i \leq k\}$.

Let

$$\Omega_2 = \{x \in BC_{ld}([t_0,\infty)_{\mathbb{T}}, \mathbb{R}) \mid a \leq x(t) \leq b, t \in [t_0,\infty)_{\mathbb{T}}\}.$$

It is easy to verify that Ω_2 is a bounded, convex and closed subset of $BC_{ld}([t_0,\infty)_{\mathbb{T}}, \mathbb{R})$.

We define two operators U_2 and $S_2 : \Omega_2 \to BC_{ld}([t_0,\infty)_{\mathbb{T}}, \mathbb{R})$ as follows:

$$(U_2x)(t) = \begin{cases} -\frac{\beta}{p(\tau^{-1}(t))} - \frac{x(\tau^{-1}(t))}{p(\tau^{-1}(t))}, & t \in [t_1,\infty)_{\mathbb{T}}, \\ (U_2x)(t_1), & t \in [t_0,t_1]_{\mathbb{T}}, \end{cases}$$

$$(S_2x)(t) = \begin{cases} \frac{(-1)^{m-1}}{p(\tau^{-1}(t))} \int_{\tau^{-1}(t)}^{\infty} \widehat{g}_{m-1}(\rho(s), \tau^{-1}(t)) & \\ \quad \times \left(\sum_{i=1}^{k} p_i(s) f_i(x(\tau_i(s))) - q(s) \right) \nabla s, & t \in [t_1,\infty)_{\mathbb{T}}, \\ (S_2x)(t_1), \quad t \in [t_0,t_1]_{\mathbb{T}}. & \end{cases}$$

Next, we show that U_2 and S_2 satisfy the conditions in Lemma 2.7.

We will show that $U_2x + S_2y \in \Omega_2$ for any $x, y \in \Omega_2$. In fact, for any $x, y \in \Omega_2$ and $t \geq t_1$, we have

$$\begin{aligned}
&(U_2x)(t) + (S_2y)(t) \\
&\geq -\frac{\beta}{p(\tau^{-1}(t))} + \frac{1}{p(\tau^{-1}(t))} \\
&\quad \times \int_{\tau^{-1}(t)}^{\infty} \widehat{g}_{m-1}(\rho(s), \tau^{-1}(t)) \Bigg(\sum_{i=1}^{k} M_2|p_i(s)| + |q(s)| \Bigg) \nabla s \\
&\geq -\frac{\beta}{p_1} + \frac{c}{p_2} \geq -\frac{\beta}{p_1} + \frac{(\beta + ap_1)p_2}{p_1p_2} = a
\end{aligned}$$

and

$$\begin{aligned}
&(U_2x)(t) + (S_2y)(t) \\
&\leq -\frac{\beta}{p(\tau^{-1}(t))} - \frac{x(\tau^{-1}(t))}{p(\tau^{-1}(t))} \\
&\quad - \frac{1}{p(\tau^{-1}(t))} \int_{\tau^{-1}(t)}^{\infty} \widehat{g}_{m-1}(\rho(s), \tau^{-1}(t)) \Bigg(\sum_{i=1}^{k} M_2|p_i(s)| + |q(s)| \Bigg) \nabla s \\
&\leq -\frac{\beta}{p_2} - \frac{b}{p_2} - \frac{c}{p_2} \leq -\frac{\beta + b - b(p_2 + 1) - \beta}{p_2} = b.
\end{aligned}$$

Thus, we have proved that $U_2x + S_2y \in \Omega_2$ for any $x, y \in \Omega_2$. It is easy to verify $||(S_2x)(t)|| \leq -\frac{c}{p_2}$ and $S_2\Omega_2$ is uniformly bounded.

It is clear that the mapping U_2 is a contraction mapping.

Then we show that $S_2\Omega_2$ is equicontinuous on $[t_0, t_2]_{\mathbb{T}}$ for any $t_2 \in [t_0, \infty)_{\mathbb{T}}$. Without loss of generality, we assume that $t_2 \geq t_1$. For any $x \in \Omega_2$, we have $||(S_2x)(t) - (S_2x)(r)|| \equiv 0$ for $t, r \in [t_0, t_1]_{\mathbb{T}}$. Since $\frac{1}{p(\tau^{-1}(t))}$ and $\tau^{-1}(t)$ are continuous on $[t_1, t_2]_{\mathbb{T}}$, so they are uniformly continuous on $[t_1, t_2]_{\mathbb{T}}$. For any $\varepsilon > 0$, choose $\delta > 0$ such that when $t, r \in [t_0, t_1]_{\mathbb{T}}$ with $|t - r| < \delta$, we have

$$\left| \frac{1}{p(\tau^{-1}(t))} - \frac{1}{p(\tau^{-1}(r))} \right| < \frac{\varepsilon}{2k_1}, \quad |\tau^{-1}(t)) - \tau^{-1}(r))| < \frac{\varepsilon}{2k_2},$$

where

$$k_i = 1 + \int_{t_0}^{\infty} \widehat{g}_{m-i}(\rho(s), t_0) \Bigg(\sum_{i=1}^{k} M_2|p_i(s)| + |q(s)| \Bigg) \nabla s, \quad i = 1, 2.$$

For any $x \in \Omega_2$, we have

$$
\begin{aligned}
&|(U_2x)(t) - (S_2x)(r)| \\
&= \bigg| \frac{1}{p(\tau^{-1}(t))} \int_{\tau^{-1}(t)}^{\infty} \widehat{g}_{m-1}(\rho(s), \tau^{-1}(t)) \bigg(\sum_{i=1}^{k} p_i(s) f_i(x(\tau_i(s))) - q(s) \bigg) \nabla s \\
&\quad - \frac{1}{p(\tau^{-1}(r))} \int_{\tau^{-1}(r)}^{\infty} \widehat{g}_{m-1}(\rho(s), \tau^{-1}(t)) \bigg(\sum_{i=1}^{k} p_i(s) f_i(x(\tau_i(s))) - q(s) \bigg) \nabla s \bigg| \\
&\le \bigg| \bigg[\frac{1}{p(\tau^{-1}(t))} - \frac{1}{p(\tau^{-1}(r))} \bigg] \int_{\tau^{-1}(t)}^{\infty} \widehat{g}_{m-1}(\rho(s), \tau^{-1}(t)) \\
&\qquad \times \bigg(\sum_{i=1}^{k} p_i(s) f_i(x(\tau_i(s))) - q(s) \bigg) \nabla s| \\
&\quad + \bigg| \frac{1}{p(\tau^{-1}(r))} \bigg[\int_{\tau^{-1}(t)}^{\infty} \widehat{g}_{m-1}(\rho(s), \tau^{-1}(t)) \\
&\qquad \times \bigg(\sum_{i=1}^{k} p_i(s) f_i(x(\tau_i(s))) - q(s) \bigg) \nabla s \\
&\quad - \int_{\tau^{-1}(r)}^{\infty} \widehat{g}_{m-1}(\rho(s), \tau^{-1}(r)) \bigg(\sum_{i=1}^{k} p_i(s) f_i(x(\tau_i(s))) - q(s) \bigg) \nabla s \bigg] \bigg| \\
&= \bigg| \bigg[\frac{1}{p(\tau^{-1}(t))} - \frac{1}{p(\tau^{-1}(r))} \bigg] \int_{\tau^{-1}(t)}^{\infty} \widehat{g}_{m-1}(\rho(s), \tau^{-1}(t)) \\
&\qquad \times \bigg(\sum_{i=1}^{k} p_i(s) f_i(x(\tau_i(s))) - q(s) \bigg) \nabla s \bigg| \\
&\quad + \bigg| \frac{1}{p(\tau^{-1}(r))} \int_{\tau^{-1}(t)}^{\tau^{-1}(r)} \\
&\qquad \times \bigg[\int_{\theta}^{\infty} \widehat{g}_{m-2}(\rho(s), \theta) \bigg(\sum_{i=1}^{k} p_i(s) f_i(x(\tau_i(s))) - q(s) \bigg) \nabla s \bigg] \nabla \theta \bigg| \\
&\le \bigg| \bigg[\frac{1}{p(\tau^{-1}(t))} - \frac{1}{p(\tau^{-1}(r))} \bigg] \\
&\qquad \times \int_{t_0}^{\infty} \widehat{g}_{m-1}(\rho(s), t_0) \bigg(\sum_{i=1}^{k} M_2 |p_i(s)| + |q(s)| \bigg) \nabla s \bigg| \\
&\quad + \frac{1}{|p_2|} \Big| \tau^{-1}(t) - \tau^{-1}(r) \Big| \int_{t_0}^{\infty} \widehat{g}_{m-2}(\rho(s), t_0) \bigg(\sum_{i=1}^{k} M_2 |p_i(s)| + |q(s)| \bigg) \nabla s \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,
\end{aligned}
$$

which implies that $S_2\Omega_2$ is equicontinuous on $[t_0, t_2]_{\mathbb{T}}$ for any $t_2 \in [t_0, \infty)_{\mathbb{T}}$. The remainder of the proof is similar to that of Theorem 3.1. The proof is complete. $\square$

Theorem 3.3. *Assume that* $(H_3), (3.1)$ *and* (3.2) *hold. Then* (1.1) *has a bounded nonoscillatory solution* $x(t)$ *with* $\liminf_{t\to\infty} |x(t)| > 0$.

Proof. By (3.1), (3.2) and Lemma 2.2, there exists a sufficiently large number $t_1 \geq t_0$ such that

$$\int_{t_1}^{\infty} \widehat{g}_{m-1}(\rho(s), t_1)\left(\sum_{i=1}^{k} M_3|p_i(s)| + |q(s)|\right)\nabla s \leq \frac{1+p}{3},$$

and $\tau(t),\ \tau_i(t) \geq t_0,\ i = 1, 2, \ldots, k$ for $t \geq t_1$, where $M_3 = \max_{\frac{1+p}{3} \leq x \leq \frac{4}{3}}\{|f_i(x)| : 1 \leq i \leq k\}$.

Let

$$\Omega_3 = \left\{x \in BC_{ld}([t_0, \infty)_{\mathbb{T}}, \mathbb{R}) \mid \frac{1+p}{3} \leq x(t) \leq \frac{4}{3}, \quad t \in [t_0, \infty)_{\mathbb{T}}\right\}.$$

It is easy to verify that Ω_3 is a bounded, convex and closed subset of $BC_{ld}([t_0, \infty)_{\mathbb{T}}, \mathbb{R})$.

We define two operators U_3 and $S_3 : \Omega_3 \to BC_{ld}([t_0, \infty)_{\mathbb{T}}, \mathbb{R})$ as follows:

$$(U_3x)(t) = (U_1x)(t), \quad t \in [t_0, \infty)_{\mathbb{T}},$$

$$(S_3x)(t) = \begin{cases} 1 + p + (-1)^{m-1} \int_t^{\infty} \widehat{g}_{m-1}(\rho(s), t) & \\ \quad \times\left(\sum_{i=1}^{k} p_i(s) f_i(x(\tau_i(s))) - q(s)\right)\nabla s, & t \in [t_1, \infty)_{\mathbb{T}}, \\ (S_3x)(t_1), & t \in [t_0, t_1]_{\mathbb{T}}. \end{cases}$$

The rest of the proof is similar to that of Theorem 3.1 so that we omit it. The proof is complete. $\square$

Theorem 3.4. *Assume that* (H_4), (3.1) *and* (3.2) *hold. Then* (1.1) *has a bounded nonoscillatory solution* $x(t)$ *with* $\liminf_{t\to\infty} |x(t)| > 0$.

Proof. By (3.1), (3.2) and Lemma 2.2, there exists a sufficiently large number $t_1 \geq t_0$ such that

$$\int_{t_1}^{\infty} \widehat{g}_{m-1}(\rho(s), t_1)\bigg(\sum_{i=1}^{k} M_4|p_i(s)| + |q(s)|\bigg)\nabla s \leq 1 - p,$$

and $\tau(t), \tau_i(t) \geq t_0$, $i = 1, 2, \ldots, k$ for $t \geq t_1$, where

$$M_4 = \max_{1-p\leq x\leq 3}\{|f_i(x)| : 1 \leq i \leq k\}.$$

Let

$$\Omega_4 = \{x \in BC_{ld}([t_0, \infty)_{\mathbb{T}}, \mathbb{R}) \mid 1 - p \leq x(t) \leq 3,\ t \in [t_0, \infty)_{\mathbb{T}}\}.$$

It is easy to verify that Ω_4 is a bounded, convex and closed subset of $BC_{ld}([t_0, \infty)_{\mathbb{T}}, \mathbb{R})$.

We define two operators U_4 and $S_4 : \ \Omega_4 \to BC_{ld}([t_0, \infty)_{\mathbb{T}}, \mathbb{R})$ as follows:

$$(U_4x)(t) = (U_1x)(t), \quad t \in [t_0, \infty)_{\mathbb{T}},$$

$$(S_4x)(t)$$
$$= \begin{cases} 2 + p + (-1)^{m-1} \int_t^{\infty} \widehat{g}_{m-1}(\rho(s), t) & \\ \quad \times\bigg(\displaystyle\sum_{i=1}^{k} p_i(s) f_i(x(\tau_i(s))) - q(s)\bigg)\nabla s, & t \in [t_1, \infty)_{\mathbb{T}}, \\ (S_4x)(t_1), & t \in [t_0, t_1]_{\mathbb{T}}. \end{cases}$$

The rest of the proof is similar to that of Theorem 3.1 so that we omit it. The proof is complete. $\square$

Theorem 3.5. *Assume that* (H_5), (3.1) *and* (3.2) *hold, and* τ *has the inverse function* $\tau^{-1} \in C(\mathbb{T}, \mathbb{T})$. *Then* (1.1) *has a bounded nonoscillatory solution* $x(t)$ *with* $\liminf_{t\to\infty} |x(t)| > 0$.

Proof. We choose a positive constant $\beta > 0$ such that $1 < \beta < p_1$. Let

$$c = \min\left\{\frac{(p_1 - \beta)}{2}, \beta - 1\right\}.$$

By (3.1), (3.2) and Lemma 2.2, there exists a sufficiently large number $t_1 \geq t_0$ such that

$$\int_{\tau^{-1}(t)}^{\infty} \widehat{g}_{m-1}(\rho(s), \tau^{-1}(t)) \left(\sum_{i=1}^{k} M_5 |p_i(s)| + |q(s)| \right) \nabla s \leq c,$$

and $\tau^{-1}(t), \tau_i(\tau^{-1}(t)) \geq t_0$, $1 \leq i \leq k$, $t \geq t_1$, where

$$M_5 = \max \left\{ |f_i(x)| : \frac{(p_1 - \beta)}{2p_2} \leq x \leq \frac{(p_1 + \beta)}{2p_1},\ 1 \leq i \leq k \right\}.$$

Let

$$\Omega_5 = \left\{ x \in BC_{ld}([t_0, \infty)_{\mathbb{T}}, \mathbb{R}) \;\middle|\; \frac{(p_1 - \beta)}{2p_2} \leq x(t) \leq \frac{(p_1 + \beta)}{2p_1}, t \in [t_0, \infty)_{\mathbb{T}} \right\}.$$

It is easy to verify that Ω_5 is a bounded, convex and closed subset of $BC_{ld}([t_0, \infty)_{\mathbb{T}}, \mathbb{R})$.

We define two operators U_5 and $S_5 : \Omega_5 \to BC_{ld}([t_0, \infty)_{\mathbb{T}}, \mathbb{R})$ as follows:

$$(U_5 x)(t) = \begin{cases} \frac{\beta}{p(\tau^{-1}(t))} - \frac{x(\tau^{-1}(t))}{p(\tau^{-1}(t))}, & t \in [t_1, \infty)_{\mathbb{T}}, \\ (U_5 x)(t_1), & t \in [t_0, t_1]_{\mathbb{T}}, \end{cases}$$

$$(S_5 x)(t) = (S_2 x)(t).$$

The rest of the proof is similar to that of Theorem 3.2 so that we omit it. The proof is complete. □

Remark 3.6. Theorems 3.1–3.5 extend, unify and improve essentially some known results in **[5, 6, 7, 11, 13, 14, 15]** because we do not assume that f_i is Lipschitzian or nondecreasing with $xf_i(x) > 0$ for $x \neq 0$, and allow $p(t)$ and $p_i(t)$ to be oscillatory.

3.2. Necessary and sufficient conditions for (1.2)**.** In this subsection, we will extend the results given for (1.1) to (1.2). We establish sufficient and necessary conditions for the existence of bounded nonoscillatory solutions of (1.2) by some new techniques. For this purpose, we need the following additional hypothesis:

(H_6) $F(t,u)$ is nondecreasing in u with $uF(t,u) > 0$ for all $u \neq 0$ and $t \in [t_0, \infty)_{\mathbb{T}}$.

Theorem 3.7. *Assume that* (H_6) *and* (3.2) *hold, and that* $p(t)$ *satisfies one of the conditions* $(H_1) - (H_5)$. *Then* (1.2) *has a bounded nonoscillatory solution* $x(t)$ *with* $\liminf_{t\to\infty} |x(t)| > 0$ *if and only if there exists some constant* $K \neq 0$ *such that*

$$\int_{t_0}^{\infty} \widehat{g}_{m-1}(\rho(s), t_0)|F(s,K)|\nabla s < \infty, \tag{3.3}$$

provided that, for the conditions (H_2) *and* (H_5), *the function* τ *has the inverse* $\tau^{-1} \in C(\mathbb{T}, \mathbb{T})$.

Proof. Necessity. Assume that (1.2) has a bounded nonoscillatory solution $x(t)$ on $[t_0, \infty)_{\mathbb{T}}$ with $\lim_{t\to\infty} \inf |x(t)| > 0$. Without loss of generality, we assume that there exist a constant $K > 0$ and some $t_1 \geq t_0$ such that $x(t) > K$, $x(\tau(t)) > K$ and $x(\delta(t)) > K$ for $t \in [t_1, \infty)_{\mathbb{T}}$. Set

$$Q(t) = (-1)^m \int_t^{\infty} \widehat{g}_{m-1}(\rho(s), t)q(s)\nabla s.$$

By (3.2) and Lemma 2.3, it is easy to certify that $Q(t)$ is bounded and $Q^{\nabla^m}(t) = q(t)$. Let

$$y(t) = x(t) + p(t)x(\tau(t)), \quad z(t) = y(t) - Q(t).$$

We see that $y(t)$ and $z(t)$ are bounded because $x(t)$, $p(t)$ and $Q(t)$ are bounded. From (1.2), for all $t \in [t_1, \infty)_{\mathbb{T}}$, we have

$$z^{\nabla^m}(t) = -F(t, x(\delta(t))) \leq -F(t,K) < 0.$$

Thus, by Lemma 2.5, we know that there exists $t_2 \geq t_1$ such that $(-1)^{m-j+1}z^{\nabla^j}(t) > 0$ for $t \geq t_2$ and $1 \leq j \leq m$. Therefore, for $t \geq t_2$, we obtain

$$\begin{aligned}
&\int_t^{\infty} \widehat{g}_{m-1}(\rho(s), t)F(s,K)\nabla s \\
&\quad \leq -\int_t^{\infty} \widehat{g}_{m-1}(\rho(s), t)z^{\nabla^m}(s)\nabla s \\
&\quad = -\lim_{s\to\infty}(-1)^{m-(m-1)+1}z^{\nabla^{m-1}}(s)\widehat{g}_{m-1}(s,t)
\end{aligned}$$

$$
\begin{aligned}
&\quad + (-1)^2 \int_t^\infty \widehat{g}_{m-2}(\rho(s), t) z^{\nabla^{m-1}}(s) \nabla s \\
&\le (-1)^2 \int_t^\infty \widehat{g}_{m-2}(\rho(s), t) z^{\nabla^{m-1}}(s) \nabla s \\
&= - \lim_{s\to\infty} (-1)^{m-(m-2)+1} z^{\nabla^{m-2}}(s) \widehat{g}_{m-2}(s, t) \\
&\quad + (-1)^3 \int_t^\infty \widehat{g}_{m-3}(\rho(s), t) z^{\nabla^{m-2}}(s) \nabla s \\
&\le (-1)^3 \int_t^\infty \widehat{g}_{m-3}(\rho(s), t) z^{\nabla^{m-2}}(s) \nabla s \\
&\quad \cdots\cdots \\
&\le (-1)^m \int_t^\infty z^{\nabla}(s) \nabla s = (-1)^m z(s)|_t^\infty < \infty.
\end{aligned}
$$

By Lemma 2.2, we see that (3.3) holds.

The proof of sufficiency is similar to that of Theorem 3.1 or Theorem 3.2. So we omit it. The proof is complete. □

Combining Theorem 3.6 with Theorems 3.1–3.5, we can immediately give sufficient and necessary conditions for the existence of bounded nonoscillatory solutions of (1.1) under the following additional hypothesis:

(H_7) $p_i(t) \ge 0$ for $t \in [t_0, \infty)_{\mathbb{T}}$ and f_i is nondecreasing with $x f_i(x) > 0$ for all $x \ne 0$.

Corollary 3.8. *Assume that* (H_7) *and* (3.2) *hold and that* $p(t)$ *satisfies one of the conditions* (H_1)*–*(H_5)*. Then* (1.1) *has a bounded nonoscillatory solution* $x(t)$ *with* $\liminf_{t\to\infty} |x(t)| > 0$ *if only if* (3.1) *holds, provided that for the conditions* (H_2) *and* (H_5)*, the function* τ *has the inverse* $\tau^{-1} \in C(\mathbb{T}, \mathbb{T})$*.*

Remark 3.9. An open problem is presented. Can we get unbounded nonoscillatory solutions of (1.1) or (1.2) provided that, for the conditions (H_2) and (H_5), the function τ has the inverse $\tau^{-1} \in C(\mathbb{T}, \mathbb{T})$, if integration of (3.1), (3.2), (3.3) are $+\infty$? The answer is affirmative, and we need some additional assumptions. In **[13]**, Zhu discussed the existence of unbounded nonoscillatory solutions of (1.4) for $2 \le m \in \mathbb{N}$. Recall the proofs of Theorems 3.1–3.6 and Corollary 3.1; similar to **[13]** we only choose a suitable Banach space $\Omega = \{x \in C_{ld}([t_0,\infty)_{\mathbb{T}},\mathbb{R}) \mid d_1 \le x(t) \le \varphi(t),\ t \in [t_0,\infty)_{\mathbb{T}}\}$, where $d_1 > 0$ is a constant and $\varphi(t) \in C([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ with $\lim_{t\to\infty}\varphi(t) = +\infty$. The proofs are similar.

3.3. Sufficient conditions for (1.3). Related to (1.1) is the dynamic equation (1.3) with mixed nabla and delta derivatives

$$[x(\cdot)+p(\cdot)x(\tau(\cdot))]^{\nabla^{m-1}\Delta}(t)+\sum_{i=1}^{k}p_i(t)f_i(x(\tau_i(t)))=q(t).$$

By **[2,** Theorem 8.49(ii)] (see also the following Theorem 5.5), for $t \in \mathbb{T}^k$ with $\rho(\sigma(t)) = t$, the dynamic equation (1.3) can be reduced to the form

$$[x(\cdot)+p(\cdot)x(\tau(\cdot))]^{\nabla^{m}}(\sigma(t))+\sum_{i=1}^{k}p_i(t)f_i(x(\tau_i(t)))=q(t),$$

or

$$(3.4)\quad [x(\cdot)+p(\cdot)x(\tau(\cdot))]^{\nabla^{m}}(t)+\sum_{i=1}^{k}p_i(\rho(t))f_i(x(\tau_i(\rho(t))))=q(\rho(t)).$$

Thus, the results of Theorems 3.1–3.5 and Corollary 3.1 can be carried over (3.4) and then over (1.3).

4. Examples. Let us consider the following two examples to better understand our results.

Example 4.1. Consider higher-order dynamic equations of the form

$$(4.1)\quad [x(t)+p(t)x(\tau(t))]^{\nabla^{m}}+\sum_{i=1}^{k}\frac{a_i}{(\rho(t)-t_0)^{\alpha_i}}f_i(x(\tau_i(t)))=\frac{1}{(\rho(t)-t_0)^{\alpha}},$$

where $t \in [t_0,\infty)_{\mathbb{T}}$, $2 \le m \in \mathbb{N}$, $\alpha > m$, a_i, $\alpha_i > m$, $i = 1,2,\ldots,k$, are real numbers, $\tau(t), \tau_i(t) \in [t_0,\infty)_{\mathbb{T}}$, $i = 1,2,\ldots,k$, with $\lim_{t\to\infty}\tau(t) = \lim_{t\to\infty}\tau_i(t) = +\infty$, $i = 1,2,\ldots,k$, and $f \in C(\mathbb{R},\mathbb{R})$. For convenience, take $p(t) = r$ to be a constant satisfying the hypotheses (H_1)–(H_5).

We first consider the case $\mathbb{T} = \mathbb{R}$. In this case, $\rho(t) = t$, the nabla derivative is a usual derivative, $\rho(s) \ge t_0$, by Proposition 2.1

and $(-1)^k\widehat{h}_k(t,s)=\widehat{g}_k(s,t)$, we have $\widehat{g}_k(\rho(s),t_0)\geq 0$, $s\in[t_0,\infty)_{\mathbb{T}}$,

$$\widehat{g}_k(t,s)=\frac{(t-s)^k}{k!},\quad \int_{t_0}^{\infty}\widehat{g}_{m-1}(\rho(s),t_0)\frac{|a_i|}{(\rho(s)-t_0)^{\alpha_i}}\nabla s<\infty,$$

and

$$\int_{t_0}^{\infty}\widehat{g}_{m-1}(\rho(s),t_0)\frac{1}{(\rho(s)-t_0)^{\alpha}}\nabla s<\infty.$$

Therefore, (4.1) satisfies the conditions of Theorems 3.1–3.5 and equation (4.1) has a bounded non-oscillatory solution $x(t)$ with $\liminf_{t\to\infty}\times|x(t)|>0$.

Next, we consider the case $\mathbb{T}=\mathbb{N}$. In this case, $\rho(t)=t-1$, the nabla derivative is the backward difference, $\rho(s)\geq t_0$; by Property 2.1 and $(-1)^k\widehat{h}_k(t,s)=\widehat{g}_k(s,t)$, we have $\widehat{g}_k(\rho(s),t_0)\geq 0$, $s\in[t_0,\infty)_{\mathbb{T}}$,

$$\widehat{g}_k(t,s)=\frac{(t-s)^{\bar{k}}}{k!},$$
$$(t-s)^{\bar{k}}=(t-s)(t-s+1)(t-s+2)\cdots(t-s+k-1),$$

and

$$\int_{t_0}^{\infty}\widehat{g}_{m-1}(\rho(s),t_0)\frac{|a_i|}{(\rho(s)-t_0)^{\alpha_i}}\nabla s<\infty,$$
$$\int_{t_0}^{\infty}\widehat{g}_{m-1}(\rho(s),t_0)\frac{1}{(\rho(s)-t_0)^{\alpha}}\nabla s<\infty.$$

Hence, (4.1) satisfies the conditions of Theorems 3.1–3.5 and equation (4.1) has a bounded nonoscillatory solution $x(t)$ with $\liminf_{t\to\infty}|x(t)|>0$.

Likewise, for general time scales $\mathbb{T}$, it is not difficult for us to check that (4.1) also satisfies the conditions of Theorems 3.1–3.5 and equation (4.1) has a bounded nonoscillatory solution $x(t)$ with $\liminf_{t\to\infty}|x(t)|>0$.

In particular, $x(t)\equiv c$ (a constant) is a solution of equation (4.1), if

$$\alpha_i=\alpha,\quad i=1,2,\ldots,k,$$

and

$$\sum_{i=1}^{k} a_i f_i(x(\tau_i(t))) = \sum_{i=1}^{k} a_i f_i(c) \equiv 1.$$

Example 4.2. Consider the higher-order dynamic equations of the form

$$\begin{aligned}(4.2)\quad \left[x(t) + \left(r + \frac{2}{t+4}\right)x(t-1)\right]^{\nabla^m} &+ \sum_{i=1}^{k} \frac{a_i |x(\tau_i(t))|^{\beta_i} x(\tau_i(t))}{(\sigma(t)-t_0)^{\alpha_i} + g_i(t)} \\ &= \frac{(\rho(t))^n}{(\sigma(t)+t_0)^{\alpha+n} + g(t)},\end{aligned}$$

where $t \in [t_0, \infty)_{\mathbb{T}}$, $2 \le m \in \mathbb{N}$, $n \in \mathbb{N}$, $\alpha > m$, $a_i > 0$, $\beta_i > 0$, $\alpha_i > m$ $(i = 1, 2, \dots, k)$ are real numbers, $\tau_i(t) \in [t_0, \infty)_{\mathbb{T}}$, $i = 1, 2, \dots, k$, with $\lim_{t\to\infty} \tau_i(t) = +\infty$ $(i = 1, 2, \dots, k)$, and $g(t)$, $g_i(t) \in C(\mathbb{T}, (0, \infty)_{\mathbb{T}})$, $f_i(x) = |x|^{\beta_i} x \in C(\mathbb{R}, \mathbb{R})$, $i = 1, 2, \dots, k$, $\sigma(t)$ is the forward jump operator. For convenience, take r to be a constant such that $p(t) = r + 2/(t+4)$ satisfies the hypotheses (H_1)–(H_5).

Now we check that equation (4.2) satisfies hypothesis (H_7). In fact,

$$p_i(t) = \frac{a_i}{(\sigma(t)-t_0)^{\alpha_i} + g_i(t)} > 0,$$
$$i = 1, 2, \dots, k, \quad \text{for all } t \in [t_0, \infty)_{\mathbb{T}},$$

and

$$x f_i(x) > 0, \quad \text{for } x \ne 0, i = 1, 2, \dots, k,$$

which implies that hypothesis (H_7) is satisfied.

On one hand, on time scale $[t_0, \infty)_{\mathbb{T}}$, we have

$$\begin{aligned}\int_{t_0}^{\infty} \widehat{g}_{m-1}(\rho(s), t_0) &\frac{(\rho(t))^n}{(\sigma(t)+t_0)^{\alpha+n} + g(t)} \nabla s \\ &< \int_{t_0}^{\infty} \frac{\widehat{g}_{m-1}(\rho(s), t_0)}{(\rho(t)+t_0)^{\alpha}} \cdot \frac{(\rho(t))^n}{(\rho(t)+t_0)^{+n}} \nabla s < \infty,\end{aligned}$$

which implies that (3.2) holds.

On the other hand, on time scale $[t_0, \infty)_{\mathbb{T}}$, we obtain

$$\int_{t_0}^{\infty} \widehat{g}_{m-1}(\rho(s), t_0) \frac{a_i}{(\sigma(t) - t_0)^{\alpha_i} + g_i(t)} \nabla s < \int_{t_0}^{\infty} \frac{a_i \cdot \hat{g}_{m-1}(\rho(s), t_0)}{(\rho(s) - t_0)^{\alpha_i}} \nabla s < \infty,$$

which implies that (3.1) holds.

It is clear that the function $\tau(t) = t - 1$ has inverse function $\tau^{-1}(t) = t+1$. According to Corollary 3.1, equation (4.2) has a bounded nonoscillatory solution $x(t)$ with $\liminf_{t\to\infty} |x(t)| > 0$.

Besides, we also obtain the same result by using Remark 3.9. We only check that the equation

$$\left[x(t) + \left(r + \frac{2}{t+4}\right)x(t-1)\right]^{\nabla^m} + \sum_{i=1}^{k} \frac{a_i |x(\tau_i(t))|^{\beta_i} x(\tau_i(t))}{(\sigma(t) - t_0)^{\alpha_i}} = \frac{(\rho(t))^n}{(\sigma(t) + t_0)^{\alpha+n}}$$

has a bounded nonoscillatory solution $x(t)$ with $\liminf_{t\to\infty} |x(t)| > 0$. In fact, it is true.

5. Appendix.

5.1. Preliminaries on time scales. For convenience, we recall some concepts related to time scales. More details can be found in **[2, 3]**.

Definition 5.1. A time scale is an arbitrary nonempty closed subset of the set $\mathbb{R}$ of real numbers with the topology and ordering inherited from $\mathbb{R}$. Let $\mathbb{T}$ be a time scale; for $t \in \mathbb{T}$, the forward jump operator is defined by $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$, the backward jump operator by $\rho(t) := \sup\{s \in \mathbb{T} : s < t\}$, and the graininess function by $\mu(t) := \sigma(t) - t$, where $\inf \emptyset := \sup \mathbb{T}$ and $\sup \emptyset := \inf \mathbb{T}$. If $\sigma(t) > t$, t is said to be right-scattered; otherwise, it is right-dense. If $\rho(t) < t$, t is said to be left-scattered; otherwise, it is left-dense. The sets $\mathbb{T}^\kappa$ and $\mathbb{T}_\kappa$ are defined as follows. If $\mathbb{T}$ has a left-scattered maximum m or right-scattered minimum m, then $\mathbb{T}^\kappa = \mathbb{T} - \{m\}$ and $\mathbb{T}_\kappa = \mathbb{T} - \{m\}$; otherwise, $\mathbb{T}^\kappa = \mathbb{T}$ and $\mathbb{T}_\kappa = \mathbb{T}$.

Definition 5.2. For a function $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^\kappa$ ($t \in \mathbb{T}_\kappa$), we define the delta-derivative $f^\Delta(t)$ or the nabla-derivative $f^\nabla(t)$ of $f(t)$ to be the number (provided it exists) with the property that, given any $\varepsilon > 0$, there is a neighborhood U of t (i.e., $U = (t-\delta, t+\delta) \cap \mathbb{T}$ for some δ) such that

$$\big|[f(\sigma(t)) - f(s)] - f^\Delta(t)[\sigma(t) - s]\big| \le \varepsilon|\sigma(t) - s| \quad \text{for all } s \in U$$

or

$$\big|[f(\rho(t)) - f(s)] - f^\nabla(t)[\rho(t) - s]\big| \le \varepsilon|\rho(t) - s| \quad \text{for all } s \in U.$$

We say that f is delta-differentiable (or in short, differentiable) on $\mathbb{T}^\kappa$, provided $f^\Delta(t)$ exists for all $t \in \mathbb{T}^\kappa$ and f is nabla-differentiable (or in short, differentiable) on $\mathbb{T}_\kappa$, provided $f^\nabla(t)$ exists for all $t \in \mathbb{T}_\kappa$.

It is easily seen that, if f is continuous at $t \in \mathbb{T}$ and t is right-scattered or left-scattered, then f is delta-differentiable or nabla-differentiable at t with

$$f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)},$$
$$f^\nabla(t) = \frac{f(t) - f(\rho(t))}{\nu(t)} = \frac{f(t) - f(\rho(t))}{t - \rho(t)}.$$

Moreover, if t is right-dense or left-dense, then f is differential at t if and only if the limit

$$\lim_{s \to t} \frac{f(t) - f(s)}{t - s}$$

exists as a finite number. In this case

$$f^\Delta(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s} \quad \text{or} \quad f^\nabla(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}.$$

In addition, if $f^\Delta \ge 0$ or $f^\nabla \ge 0$, then f is nondecreasing. Two useful formulas are

$$f^\sigma(t) = f(t) + \mu(t) f^\Delta(t), \quad \text{where } f^\sigma(t) := f(\sigma(t))$$

and

$$f^\rho(t) = f(t) - \nu(t) f^\nabla(t), \quad \text{where } f^\rho(t) := f(\rho(t)).$$

We will make use of the following product and quotient rules for the derivative of the product fg and the quotient f/g (where $gg^\sigma \neq 0$ or $gg^\rho \neq 0$) of two differentiable functions f and g:

$$(fg)^\Delta = f^\Delta g + f^\sigma g^\Delta = fg^\Delta + f^\Delta g^\sigma \tag{5.1}$$

$$\left(\frac{f}{g}\right)^\Delta = \frac{f^\Delta g - fg^\Delta}{gg^\sigma}.$$

and

$$(fg)^\nabla = f^\nabla g + f^\rho g^\nabla = fg^\nabla + f^\nabla g^\rho \tag{5.2}$$

$$\left(\frac{f}{g}\right)^\nabla = \frac{f^\nabla g - fg^\nabla}{gg^\rho}.$$

Definition 5.3. Let $f : \mathbb{T} \to \mathbb{R}$ be a function, f is called right-dense continuous (rd-continuous) if it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$. A function $F : \mathbb{T} \to \mathbb{R}$ is called an antiderivative of f provided $F^\Delta(t) = f(t)$ holds for all $t \in \mathbb{T}^k$. By the antiderivative, the Cauchy integral of f is defined as

$$\int_a^b f(s)\Delta s = F(b) - F(a),$$

and

$$\int_a^\infty f(s)\Delta s = \lim_{t\to\infty} \int_a^t f(s)\Delta s.$$

Definition 5.4. Let $f : \mathbb{T} \to \mathbb{R}$ be a function, f is called left-dense continuous (ld-continuous) if it is continuous at left-dense points in $\mathbb{T}$, and its right-sided limits exist (finite) at right-dense points in $\mathbb{T}$. A function $F : \mathbb{T} \to \mathbb{R}$ is called an antiderivative of f, provided $F^\nabla(t) = f(t)$ holds for all $t \in \mathbb{T}_k$. By the antiderivative, the Cauchy integral of f is defined as $\int_a^b f(s)\nabla s = F(b) - F(a)$, and $\int_a^\infty f(s)\nabla s = \lim_{t\to\infty} \int_a^t f(s)\nabla s$.

Let $C_{rd}(\mathbb{T}, \mathbb{R})$ denote the set of all rd-continuous functions and $C_{ld}(\mathbb{T}, \mathbb{R})$ denote the set of all ld-continuous functions mapping $\mathbb{T}$ to $\mathbb{R}$.

It is shown in **[2]** that every rd-continuous (or ld-continuous) function has an antiderivative.

Two integration by parts formulas are

$$\int_a^b f(t)g^{\Delta}(t)\Delta t = [f(t)g(t)]\Big|_a^b - \int_a^b f^{\Delta}(t)g^{\sigma}(t)\Delta t \tag{5.3}$$

and

$$\int_a^b f(t)g^{\nabla}(t)\nabla t = [f(t)g(t)]\Big|_a^b - \int_a^b f^{\nabla}(t)g^{\rho}(t)\nabla t. \tag{5.4}$$

Theorem 5.5. *Assume that $f:\mathbb{T}\to\mathbb{R}$ is nabla differentiable on $\mathbb{T}_\kappa$. Then f is delta differentiable at t and*

$$f^{\Delta}(t) = f^{\nabla}(\sigma(t)) \tag{5.5}$$

for $t\in\mathbb{T}^\kappa$ such that $\rho(\sigma(t))=t$. If, in addition, f^{∇} is continuous on $\mathbb{T}_\kappa$, then f is delta differentiable at t and (5.5) *holds for any $t\in\mathbb{T}^\kappa$.*

Theorem 5.6. *If f, f^{Δ} and f^{∇} are continuous, then*

(i) $[\int_a^t f(t,s)\Delta s]^{\Delta} = \int_a^t f^{\Delta}(t,s)\Delta s + f(\sigma(t),t)$;
(ii) $[\int_a^t f(t,s)\Delta s]^{\nabla} = \int_a^t f^{\nabla}(t,s)\Delta s + f(\rho(t),\rho(t))$;
(iii) $[\int_a^t f(t,s)\nabla s]^{\Delta} = \int_a^t f^{\Delta}(t,s)\nabla s + f(\sigma(t),\sigma(t))$;
(iv) $[\int_a^t f(t,s)\nabla s]^{\nabla} = \int_a^t f^{\nabla}(t,s)\nabla s + f(\rho(t),t)$.

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