ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 45, Number 2, 2015

NONTRIVIAL PERIODIC SOLUTIONS OF SECOND ORDER SINGULAR DAMPED DYNAMICAL SYSTEMS

JIFENG CHU, SHENGJUN LI AND HAILONG ZHU

ABSTRACT. Assuming that the linear equation $x'' + h(t)x' + a(t)x = 0$ has a positive Green's function, we study the existence of nontrivial periodic solutions of second order damped dynamical systems

$$x'' + h(t)x' + a(t)x = f(t,x) + e(t),$$

where $h,\, a \in \mathbb{C}((\mathbb{R}/T\mathbb{Z}), \mathbb{R})$, $e = (e_1, \dots, e_N)^T \in \mathbb{C}((\mathbb{R}/T\mathbb{Z}), \mathbb{R}^N)$, $N \geq 1$, and the nonlinearity $f = (f_1, \dots, f_N)^T \in \mathbb{C}((\mathbb{R} = T\mathbb{Z}) \times \mathbb{R}^N \setminus \{0\}, \mathbb{R}^N)$ has a repulsive singularity at the origin. We consider a very general singularity and do not need any kind of strong force condition. The proof is based on a nonlinear alternative principle of Leray-Schauder. Recent results in the literature are generalized and improved.

1. Introduction. The purpose of this work is to study the existence of nontrivial T-periodic solutions for the following second order damped dynamical system

$$x'' + h(t)x' + a(t)x = f(t,x) + e(t), \tag{1.1}$$

where $h,\, a \in \mathbb{C}((\mathbb{R}/T\mathbb{Z}), \mathbb{R})$, $e = (e_1, \dots, e_N)^T \in \mathbb{C}((\mathbb{R}/T\mathbb{Z}, \mathbb{R}^N)$, $N \geq 1$, and the nonlinearity $f = (f_1, \dots, f_N)^T \in \mathbb{C}((\mathbb{R}/T\mathbb{Z}) \times \mathbb{R}^N \setminus \{0\}, \mathbb{R}^N)$ has a repulsive singularity at the origin. By a nontrivial periodic solution, we mean a function $x = (x_1, \dots, x_N)^T \in \mathbb{C}^2(\mathbb{R}/T\mathbb{Z}, \mathbb{R}^N)$ solving (1.1) and such that $x(t) \neq 0$ for all t. Such a solution is also called *collisionless orbit* in the literature.

2010 AMS *Mathematics subject classification.* Primary 34C25, 34D20.

Keywords and phrases. Nontrivial periodic solutions, singular, damped, dynamical systems, Leray- Schauder alternative principle.

The first author was supported by the National Natural Science Foundation of China (Grant No. 11171090). The second author was supported by the National Natural Science Foundation of China (Grant Nos. 11461016 and 11161017). The third author was supported by the National Natural Science Foundation of China (Grant No. 11301001).

Received by the editors on May 15, 2012, and in revised form on March 24, 2013.

DOI:10.1216/RMJ-2015-45-2-457

We are mainly motivated by the recent nice works [**10, 18, 35**] and focus on the case where (1.1) presents a repulsive singularity at $x = 0$. We say that (1.1) has a repulsive singularity at $x = 0$ if there exists a fixed vector $v \in \mathbb{R}_+^N$ such that

$$\lim_{\substack{x \to 0 \\ x \in \mathbb{R}_+^N}} (v, f(t,x)) = +\infty, \text{ uniformly in } t, \tag{1.2}$$

whereas (1.1) has an attractive singularity at $x = 0$ if

$$\lim_{\substack{x \to 0 \\ x \in \mathbb{R}_+^N}} (v, f(t,x)) = -\infty; \text{ uniformly in } t;$$

here, $\mathbb{R}_+^n$ denotes the set of vectors of $\mathbb{R}^N$ with positive components. Note that in our case the word "singularity" is understood in a more general way than those in [**10, 17, 38**] because we do not require that each component of the nonlinearity $f(t,x)$ have a singularity at $x = 0$, which is necessary in the above-mentioned works.

During the last few decades, singular differential equations or singular dynamical systems have been studied by many researchers. Singular equations appear in many problems of applications such as the Brillouin focusing system [**5**] and nonlinear elasticity [**14**]. It was also found recently that one special singular differential equation which is called "Ermakov-Pinney equation" plays an important role in the study of the stability of periodic solutions of conservative systems with degree of lower freedom (see [**11**] and the references therein). Concerned with singular equations, the question of the existence of periodic solutions is one of the central topics, and therefore has attracted much attention [**2, 3, 4, 7, 12, 15, 16, 21, 22, 23, 26, 27, 31, 32, 33**]. Usually, the proof is based on either the variational approach [**1, 2, 4, 30, 33**] or topological methods. The proof of the main results in this paper is based on topological methods, which started with the pioneering paper of Lazer and Solimini [**26**]. From then on, the method of upper and lower solutions [**6, 24, 29**], degree theory [**15, 37, 38, 39**], some fixed point theorems in cones for completely continuous operators [**18, 34, 35**], Schauder's fixed point theorem [**9, 17, 36**] and a nonlinear alternative principle of Leray-Schauder type [**10, 13, 25**] have been widely applied. In this line of research, a common device is to assume some strong force condition, which was first introduced by Gordon in [**19**]. This condition has been widely used for avoiding collisions. For example, if we consider the system

$$x'' + \nabla\left(\frac{1}{|x|^\alpha}\right) = e(t), \tag{1.3}$$

the strong force condition corresponds to the case $\alpha \geq 2$. Compared with the case of strong singularity, the case of weak singularity was less studied, but there are also rich results **[9, 10, 14, 21, 25, 29, 34, 35]** in the literature during the last few years.

In this paper, we will generalize and improve those results contained in **[10, 18, 35]** in the following three directions. Firstly, we do not require that all components of the nonlinearity $f(t,x)$ have a singularity, and therefore we can deal with some systems which were not covered in the literature. Secondly, we improve those results in **[10, 17, 18]** because we do not require that the mean of each component of γ (see Section 3 for such a notion) is non-negative. Finally, there is the damping term in our case, which was not covered in the above mentioned works and will imply some new technical difficulties in the proof. We remark that the approach used in this paper is similar to those in **[10, 13]**. The remaining part of this paper is organized as follows. In Section 2, some preliminary results are given. In Section 3, by employing a nonlinear alternative principle of Leray-Schauder, we establish the main result. To illustrate the new results, some applications are also given.

2. Preliminaries.

2.1. Green's function of the linear system. We say that the linear equation

$$x'' + h(t)x' + a(t)x = 0 \tag{2.1}$$

is nonresonant if its unique T-periodic solution is the trivial one. When (2.1) is nonresonant, as a consequence of Fredholm's alternative, the nonhomogeneous system

$$x'' + h(t)x' + a(t)x = l(t)$$

admits a unique T-periodic solution which can be written as

$$x(t) = \int_0^T G(t,s)l(s)\,ds,$$

where $G(t, s)$ is the Green's function of (2.1). Throughout this paper, we always assume that

(A) The Green's function $G(t, s)$ of (2.1) is positive for all $(t, s) \in [0, T] \times [0, T]$.

For the general case, it is difficult to verify that condition (A) holds. However, two classes of the functions h, a for (A) to hold have been found recently in **[8, 13]** (which was proved by the anti-maximum principle established by Hakl and Torres in **[23]**). To describe these, let us define the functions

$$\sigma(h)(t) = \exp\left(\int_0^t h(s)\,ds\right),$$

and

$$\sigma_1(h)(t) = \sigma(h)(T)\int_0^t (h)(s)\,ds + \int_t^T \sigma(h)(s)\,ds.$$

Theorem 2.1. **[13,** Corollary 2.6] *Assume that* $a \not\equiv 0$ *and the following two inequalities are satisfied*:

$$\int_0^T a(s)\sigma(h)(s)\sigma_1(-h)(s)\,ds \geq 0,$$

and

$$\sup_{0\leq t\leq T}\left\{\int_t^{t+T}\sigma(-h)(s)\,ds\int_t^{t+T}[a(s)]_+\sigma(h)(s)\,ds \leq 4, \right.$$
$$\left. \textit{where } [a(s)]_+ = \max\{a(s), 0\right\}.$$

Then (A) *holds.*

For the special case $\int_0^T a(t)\sigma(h)(t)\,dt > 0$ and $h \in \widetilde{\mathbb{C}}(\mathbb{R}/T\mathbb{Z}) := \{h \in \mathbb{C}(\mathbb{R}/T\mathbb{Z}) : \overline{h} = 0\}$, another criterion has also been established by Cabada and Cid in **[8]**. To describe these, given an exponent $q \in [1, \infty]$, the best constant in the Sobolev inequality

$$C\|u\|_{q,[0,1]} \leq \|u'\|_{2,[0,1]} \quad \text{for all } u \in H_0^1(0,1)$$

is denoted by $M(q)$. The explicit formula for $M(q)$ is known, that is,

$$M(q) = \begin{cases} \left(\frac{2\pi}{q}\right)^{1/2} \left(\frac{2}{q+2}\right)^{1/2-1/q} \frac{\Gamma(1/q)}{\Gamma(1/2+1/q)} & \text{for } 1 \le q < \infty, \\ 2 & \text{for } q = \infty, \end{cases}$$

where $\Gamma(\cdot)$ is the Gamma function of Euler. The usual L^p-norm is denoted by $\|\cdot\|_p$. The conjugate exponent of p is denoted by $q : 1/p + 1/q = 1$.

Theorem 2.2. **[8,** Theorem 5.1] *Assume that* $h \in \widetilde{\mathbb{C}}(\mathbb{R}/T\mathbb{Z})$ *and* $\int_0^T a(t)\sigma(h)(t)\,dt > 0$. *Suppose further that there exists* $1 \le p \le \infty$ *such that*

$$(B(T))^{1+1/q} \|[\mathcal{A}]_+\|_{p,T} < \mathbf{M}^2(2q),$$

where

$$B(T) = \int_0^T \sigma(-h)(t)\,dt, \qquad [\mathcal{A}(t)]_+ = [a(t)]_+ (\sigma(h)(t))^{2-1/p}.$$

Then (A) *holds.*

As a special case of Theorem 2.2, we can recover the following L^p-criterion proved by Torres in **[34]** (see also **[40]** for a complete discussion).

Corollary 2.3. *Assume that* $h \equiv 0$, $\int_0^T a(t)\,dt > 0$ *and* $a \in L^p[0,T]$ *for some* $1 \le p \le \infty$. *If*

$$T^{1+1/q} \|[a]_+\|_{p,T} < \mathbf{M}^2(2q),$$

then (A) *holds.*

Remark 2.4. For a given function h, from Theorem 2.1 and Theorem 2.2, we can easily obtain that condition (A) holds for the following linear equation:

$$x'' + h(t)x' + k^2 x = 0,$$

if the constant $k > 0$ is small enough. In fact, for the general case, we can take k such that:

$$k^2 \le \frac{4}{\sup_{0 \le t \le T}\{\int_t^{t+T} \sigma(-h)(s)\,ds \int_t^{t+T} \sigma(h)(s)\,ds\}}. \tag{2.2}$$

For $h \in \widetilde{\mathbb{C}}(\mathbb{R}/T\mathbb{Z})$, we can take k such that

$$k^2 < \frac{\mathbf{M}^2(2q)}{(B(T)^{1+1/q}\|(\sigma(h)))^{2-1/p}\|_{p,T}}.$$

In particular, when $h \equiv 0$ and $a(t) = k^2$, condition (A) is equivalent to $0 < k^2 < (\pi/T)^2$.

2.2. Basic tool. The proof of the main result is based on the following nonlinear alternative of Leray-Schauder, which can be found in [**20,** pages 120–130] and has been used in [**13, 28**].

Lemma 2.5. *Assume that Ω is an open subset of a convex set K in a normed linear space X and $p \in \Omega$. Let $\mathcal{A} : \overline{\Omega} \to K$ be a compact and continuous map. Then one of the following two conclusions holds*:

(I) *$\mathcal{A}$ has at least one fixed point in $\overline{\Omega}$.*
(II) *There exists $x \in \partial\Omega$ and $0 < \lambda < 1$ such that $x = \lambda\mathcal{A}x + (1-\lambda)p$.*

2.3. Notation. For a given function $p \in L^1[0,T]$ essentially bounded, we denote the essential supremum and infimum of p by p^* and p_*, respectively. Given $x, y \in \mathbb{R}^N$, the usual scalar product is denoted by (x, y). The usual Euclidean norm is denoted by $|x|$. For a fixed vector $v \in \mathbb{R}^N_+$, we have a well-defined norm

$$|x|_v = \sum_{i=1}^N v_i|x_i|.$$

Let $\|\cdot\|$ denote the supremum norm of $\mathbb{C}_T = \{x : x \in \mathbb{C}(\mathbb{R}/T\mathbb{Z}, \mathbb{R})\}$ and take $X = \mathbb{C}_T \times \cdots \times \mathbb{C}_T$ (N copies). For $x = (x_1, \ldots, x_N) \in X$, the natural norm becomes

$$|x|_v = \sum_{i=1}^N v_i\|x_i\| = \sum_{i=1}^N v_i \cdot \max_t |x_i(t)|.$$

Obviously, X is a Banach space.

Under hypothesis (A), we always denote

$$m = \min_{\substack{0\leq s\\ t\leq T}} G(t,s), \qquad M = \max_{\substack{0\leq s\\ t\leq T}} G(t,s), \quad \sigma = m/M.$$

Obviously, $0 < \sigma < 1$.

3. Main results. Let us denote $\gamma(\mathrm{t})$ by

$$\gamma(t) = \int_0^T G(t,s)e(s)\,ds,$$

which is the unique T-periodic solution of the linear system

$$x'' + h(t)x' + a(t)x = e(t).$$

One important observation is that $y(t) = x(t) + \gamma(t)$ is a T-periodic solution of (1.1) if the system

$$x'' + h(t)x' + a(t)x = f(t, x(t) + \gamma(t)) \tag{3.1}$$

has a T-periodic solution $x(t)$. For a given vector $v \in \mathbb{R}_+^N$, we always denote

$$\Gamma(t) = (v, \gamma(t)), \qquad \Lambda(t) = |\gamma(t)|_v = \sum_{i=1}^N v_i|\gamma(t)|.$$

Using the notation given in Section 2, we denote

$$\Gamma_* = \min_t \Gamma(t), \qquad \Lambda^* = \max_t \Lambda(t).$$

Theorem 3.1. *Suppose that the linear system* (2.1) *satisfies* (A) *and*

$$\int_0^T a(t)\sigma(h)(t)\,dt > 0. \tag{3.2}$$

Assume further that there exists a constant $r > 0$ such that:

(H1) *There exists a continuous function $\phi_{r+\Lambda^*} \succeq 0$ (which means $\phi_{r+\Lambda^*}(t) \geq 0$ for almost every $t \in [0,T]$ and it is positive in a set of positive measure) such that*

$$(v, f(t,x)) \geq \phi_{r+\Lambda^*}(t), \text{ for all } t \text{ and } x \in \mathbb{R}_+^N \text{ with } 0 < |x|_v \leq r + \Lambda^*;$$

(H2) *there exist continuous non-negative functions $g(\cdot)$, $h(\cdot)$ on $(0,\infty)$ such that*

$$0 \leq (v, f(t,x)) \leq g(|x|_v) + h(|x|_v),$$

for all t and $x \in \mathbb{R}_+^N$ with $0 < |x|_v \leq r + \Lambda^*$, here $g(\cdot) > 0$ is non-increasing and $h(\cdot)/g(\cdot)$ is non-decreasing;

(H3) *the following inequality holds*:

$$g(\sigma r + \Gamma_*)\left\{1 + \frac{h(r+\Lambda^*)}{g(r+\Lambda^*)}\right\} < \frac{r}{MT}.$$

If $\Gamma_* \geq 0$, then (1.1) has at least one nontrivial T-periodic solution.

Proof. Since (H3) holds, we can choose $n_0 \in \{1, 2, \ldots\}$ such that

$$\frac{1}{n_0} < \min\{r, \sigma r + \Gamma_*\}$$

and

$$g(\sigma r + \Gamma_*)\left\{1 + \frac{h(r+\Lambda^*)}{g(r+\Lambda^*)}\right\} + \frac{1}{n_0} < \frac{r}{MT}.$$

Let $N_0 = \{n_0, n_0 + 1, \ldots\}$. Fix $n \in N_0$. Consider the family of systems

$$x'' + h(t)x' + a(t)x = \lambda f^n(t, x(t) + \gamma(t)) + \frac{a(t)\hbar}{n}, \tag{3.3}$$

where $\lambda \in [0, 1]$, $\hbar \in \mathbb{R}_+^N$ is chosen such that

$$(v, \hbar) = 1$$

and

$$f^n(t,x) = \begin{cases} f(t,x), & \text{if } |x|_v \geq \frac{1}{n}, \\ \widetilde{f}(t,x), & \text{if } |x|_v < \frac{1}{n}, \end{cases}$$

where $\widetilde{f}$ is chosen such that $f^n(t,x)$ is nonnegative and continuous for all $(t,x) \in [0,T] \times \mathbb{R}^N$. Solving (3.3) is equivalent to the following fixed point problem

$$x(t) = \lambda(\mathcal{A}^n x)(t) + (1-\lambda)p, \tag{3.4}$$

where $p = \hbar/n$ and $\mathcal{A}^n$ is defined by

$$(\mathcal{A}^n x)(t) = \int_0^T G(t,s) f^n(s, x(s) + \gamma(s))\, ds + p,$$

where we have used the fact that

$$\int_0^T G(t,s)a(s)\, ds = 1.$$

First we claim that any fixed point x of (3.4) for any $\lambda \in [0,1]$ must satisfy $|x|_v \neq r$. Otherwise, assume that x is a fixed point of (3.4) for some $\lambda \in [0,1]$ such that $|x|_v = r$. By using (H_1) together with the non-negativeness of $G(t,s)$, we have $(v, x(t)) \geq 0$ for all t. Moreover, we have

$$\begin{aligned}(v,x(t)) - (v,p) &= \lambda \int_0^T G(t,s)(v, f^n(s, x(s)+\gamma(s)))\,ds \\ &\geq \lambda m \int_0^T (v, f^n(s,x(s)+\gamma(s)))\,ds \\ &= \sigma M \lambda \int_0^T (v, f^n(s,x(s)+\gamma(s)))\,ds \\ &\geq \sigma\left(v, \max_t \left\{\lambda \int_0^T G(t,s) f^n(s,x(s)+\gamma(s))\,ds\right\}\right) \\ &= \sigma|x-p|_v.\end{aligned}$$

Hence, for all t, we have

$$\begin{aligned}(v,x(t)) &\geq \sigma|x-p|_v + (v,p) \\ &\geq \sigma(|x|_v - (v,p)) + (v,p) \\ &\geq \sigma r.\end{aligned}$$

Since $1/n \leq 1/n_0 < \sigma r + \Gamma_*$, we have

$$\begin{aligned}|x(t)+\gamma(t)|_v \geq (v, x(t)+\gamma(t)) &\geq \sigma r + (v,\gamma(t)) \\ &\geq \sigma r + \Gamma_* > \frac{1}{n},\end{aligned}$$

which implies that

$$f^n(t, x(t)+\gamma(t)) = f(t, x(t)+\gamma(t)).$$

Thus, we have from condition (H_2), for all $t \in [0,T]$,

$$\begin{aligned}(v,x(t)) &= \lambda \int_0^T G(t,s)(v, f(s,x(s)+\gamma(s)))\,ds + (v,p) \\ &\leq \int_0^T G(t,s)(v, f(s,x(s)+\gamma(s)))\,ds + (v,p) \\ &\leq M \int_0^T (v, f(s,x(s)+\gamma(s)))\,ds + (v,p)\end{aligned}$$

$$\begin{aligned}&\le M\int_0^T g(|x(s)+\gamma(s)|_v)\left\{1+\frac{h(|x(s)+\gamma(s)|v)}{g(|x(s)+\gamma(s)|v)}\right\}ds\\&\quad+(v,p)\\&\le MTg(\sigma r+\Gamma_*)\left\{1+\frac{h(r+\Lambda^*)}{g(r+\Lambda^*)}\right\}+\frac{1}{n_0}.\end{aligned}$$

Therefore,

$$r=|x|_v\le MTg(\sigma r+\Gamma_*)\left\{1+\frac{h(r+\Lambda^*)}{g(r+\Lambda^*)}\right\}+\frac{1}{n_0},$$

which is a contradiction to the choice of n_0, and the claim is proved.

Let X be given as in Section 2. Define

$$K=\{x\in X:(v,x(t))>0\quad\text{for each }t\}$$

and

$$\Omega=\{x\in K:|x|_v<r\}.$$

Then K is a convex set in X and Ω is an open subset of K with $p=\hbar/n\in\Omega$. For each $x\in\overline{\Omega}$, using condition (H_1), we see that $(v,\mathcal{A}^n x(t))>0$ for each t, and thus $\mathcal{A}^n(\overline{\Omega})\subset K$. Moreover, one may easily verify that $\mathcal{A}^n:\overline{\Omega}\to K$ are compact and continuous maps.

Now, using Lemma 2.5, we know that

$$x=\mathcal{A}^n x$$

has a fixed point, denoted by x_n, i.e., the system

$$x''+h(t)x'+a(t)x=f_n(t,x(t)+\gamma(t))+\frac{a(t)\hbar}{n}\tag{3.5}$$

has a periodic solution x_n with $|x_n|_v<r$. Since $(v,x_n(t))\ge(v,p)>0$ for all t, and x_n is actually a nontrivial periodic solution of (3.5).

In order to pass the solutions x_n of the truncation systems (3.5) to that of the original system (3.1), we need the following fact

$$|x_n'|_v\le H\tag{3.6}$$

for some constant $H>0$ and for all $n\ge n_0$. To this end, for each $i=1,\dots,N$, by the periodic boundary conditions, $x_{ni}'(t_i)=0$ for some $t_i\in[0,T]$, here we use the notation x_{ni} to denote the ith component of x_n. Note that (3.5) is equivalent to

$$(3.7)\quad (\sigma(h)(t)x'_{ni})' + a(t)\sigma(h)(t)x_{ni} = \sigma(h)(t)\left(f_i^n(t, x_n(t)+\gamma(t)) + \frac{a(t)\hbar_i}{n}\right),$$

for each $i = 1, \ldots, N$. Integrating (3.7) from 0 to T, we obtain

$$\int_0^T a(t)\sigma(h)(t)x_{ni}(t)\,dt = \int_0^T \sigma(h)(t)\left[f_i^n(t, x_n(t)+\gamma(t)) + \frac{a(t)\hbar_i}{n}\right]dt.$$

Therefore, for each $i = 1, \ldots, N$,

$$\begin{aligned}
|\sigma(h)(t)x'_{ni}(t)| &= \left|\int_{t_i}^t (\sigma(h)(s)x'_{ni}(s))'ds\right| \\
&= \left|\int_{t_i}^t \sigma(h)(s)\Big[f_i^n(s, x_n(s)+\gamma(s))\right. \\
&\qquad \left. + \frac{a(s)\hbar_i}{n} - a(s)x_{ni}(s)\Big]\,ds\right| \\
&\le \int_0^T \sigma(h)(t)\left[f_i^n(t, x_n(t)+\gamma(t)) + \frac{a(t)\hbar_i}{n}\right]dt \\
&\quad + \int_0^T \sigma(h)(t)a(t)x_{ni}(t)\,dt \\
&= 2\int_0^T \sigma(h)(t)a(t)x_{ni}(t)\,dt \\
&< 2\|x_{ni}\| \int_0^T \sigma(h)(t)a(t)\,dt,
\end{aligned}$$

where we have used the assumption (3.2). Therefore,

$$\Big(\min_{0\le t\le T} \sigma(h)(t)\Big)|(x_{ni})'(t)| \le 2\|x_{ni}\| \int_0^T a(t)\sigma(h)(t)\,dt,$$

which is equivalent to

$$|(x_{ni})'(t)| \le 2L\|x_{ni}\|,$$

where

$$L = \frac{\int_0^T a(t)\sigma(h)(t)\,dt}{\min_{0\le t\le T}\sigma(h)(t)}.$$

Therefore,

$$|x_n'|_v = (v, \|x_n'\|) \le 2L \sum_{i=1}^{N} v_i \|x_{ni}\| = 2L|x_n|_v \le 2Lr.$$

Thus, (3.6) holds for $H = 2Lr$.

Next we claim that there exists a constant $\delta > 0$, independent of $n \in N_0$, such that

$$(v, x_n(t) + \gamma(t)) \ge \delta, \tag{3.8}$$

for all t and n large enough. In fact, since (H_1) holds, there exists a continuous function $\phi_{r+\Lambda^*}(t) \succeq 0$ such that

$$(v, f(t, x)) \ge \phi_{r+\Lambda*}(t),$$

for all t and x with $0 < |x|_v \le r + \Lambda^*$. Then we have

$$\begin{aligned}(v, x_n(t) + \gamma(t)) &= \int_0^T G(t,s)(v, f^n(s, x_n(s) + \gamma(s)))\,ds + (v, \gamma(t)) + \frac{1}{n} \\ &\ge m \int_0^T (v, f(s, x_n(s) + \gamma(s)))\,ds + \Gamma(t) \\ &\ge m \int_0^T \phi_{r+\Lambda^*}(s)\,ds + \Gamma_*,\end{aligned}$$

which implies that the inequality (3.8) holds if we take

$$\delta = m \int_0^T \phi_{r+\Lambda^*}(s)\,ds + \Gamma_* \tag{3.9}$$

Note that $\delta > 0$ since we have assumed that $\Gamma_* \ge 0$.

From the above facts (3.6) and (3.8) that $\{x_n\}_{n \in N_0}$ has a subsequence, $\{x_{n_k}\}_{k \in \mathbf{N}}$, converging uniformly on $[0, T]$ to a function $x \in X$. Moreover, we have

$$\delta \le (v, x(t) + \gamma(t)) \le r + \Lambda^*, \quad \text{for all } t.$$

Furthermore, x_{n_k} satisfies the integral equation

$$x_{n_k}(t) = \int_0^T G(t,s) f(s, x_{n_k}(s) + \gamma(s))\,ds + \frac{\hbar}{n_k}.$$

Letting $k \to \infty$, we arrive at

$$x(t) = \int_0^T G(t,s) f(s, x(s) + \gamma(s))\, ds.$$

Therefore, x is a nontrivial T-periodic solution of (3.1). $\square$

Remark 3.2. From (3.9) in the above proof, we can weaken the condition $\Gamma_* \geq 0$ as

$$m \int_0^T \phi_{r+\Lambda^*}(s)\, ds + \Gamma_* > 0.$$

Therefore, we can deal with some cases that $(v, \gamma(t))$ changes sign.

The condition $\Gamma_* \geq 0$ is equivalent to the condition

$$(v, \gamma(t)) \geq 0 \tag{3.10}$$

for all t. Therefore, we do not require that the mean of each component of e is non-negative. Furthermore, we have improved those results in **[35]** since it required the condition

$$(v, e(t)) \geq 0. \tag{3.11}$$

It is easy to see that (3.10) is weaker than (3.11).

Corollary 3.3. *Assume that $f \in \mathbb{C}((\mathbb{R}/T\mathbb{Z}) \times \mathbb{R}^N \setminus \{0\}, \mathbb{R}^N)$, and there exists a vector $v \in \mathbb{R}_+^N$ and continuous positive functions b, c, d and $\alpha, \beta > 0$ such that*

$$\text{(F)} \quad \frac{d(t)}{|x|_v^\alpha} \leq (v, f(t,x)) \leq \frac{b(t)}{|x|_v^\alpha} + \mu c(t)|x|_v^\beta, \quad \textit{for all } t \textit{ and } x \in R_+^N.$$

Then, for each $e \in \mathbb{C}(\mathbb{R}/T\mathbb{Z}, \mathbb{R}^N)$ with $\Gamma_* \geq 0$,

(i) *if $\beta < 1$, then (1.1) has at least one nontrivial T-periodic solution for each $\mu > 0$,*
(ii) *if $\beta \geq 1$, then (1.1) has at least one nontrivial T-periodic solution for each $0 < \mu < \widetilde{\mu}$, where $\widetilde{\mu}$ is some positive constant.*

Proof. We will apply Theorem 3.1. To this end, for $s \in \mathbb{R}$, $s > 0$, we take

$$g(s) = b^* s^{-\alpha}, \qquad h(s) = \mu c^* s^\beta,$$

where

$$b^* = \max_t b(t), \qquad c^* = \max_t c(t).$$

Then it is easy to see that (H2) is satisfied, and the existence condition (H3) becomes

$$\mu < \frac{r(\sigma r + \Gamma_*)^\alpha - b^* MT}{c^* MT(r + \Lambda^*)^{\alpha+\beta}}$$

for some $r > 0$. So (H_3) is satisfied if

$$0 < \mu < \widetilde{\mu} := \sup_{r>0} \frac{r(\sigma r + \Gamma_*)^\alpha - b^* MT}{c^* MT(r + \Lambda^*)^{\alpha+\beta}}.$$

Note that $\widetilde{\mu} = \infty$ if $\beta < 1$ and $\widetilde{\mu} < \infty$ if $\beta \geq 1$. Finally (H_1) is satisfied if we take

$$\phi_{r+\Lambda^*}(t) = \frac{d(t)}{(r + \Lambda^*)^\alpha}.$$

Now we have the desired results (i) and (ii). □

Let us assume that the two-dimensional linear system

$$\begin{cases} x'' + h(t)x' + a(t)x = 0, \\ y'' + h(t)y' + a(t)y = 0, \end{cases}$$

satisfies condition (A). Then Corollary 3.3 can be applied to the following two 2-dimensional nonlinear systems

$$\begin{cases} x'' + h(t)x' + a(t)x = \dfrac{c(t)}{|x|^\alpha} + \mu b(t)|x|^\beta + e_1(t), \\ y'' + h(t)y' + a(t)y = \dfrac{c(t)}{|x|^\alpha} + \mu b(t)|x|^\beta + e_2(t), \end{cases} \tag{3.12}$$

and

$$\begin{cases} x'' + h(t)x' + a(t)x = \dfrac{c(t)}{|x|^\alpha} + e_1(t), \\ y'' + h(t)y' + a(t)y = \mu b(t)|x|^\beta + e_2(t), \end{cases} \tag{3.13}$$

with $b, c, e_1, e_2 \in \mathbb{C}[0, T]$, $\alpha, \beta > 0$ and $\mu \in \mathbb{R}$ is a given parameter. Note that system (3.13) is not covered by the results containing in the references mentioned above. Although system (3.12) has been studied in **[10, 17]**, we have improved those results contained.

Remark 3.4. In Theorem 3.1, we do not formulate the condition (1.2) because our conditions are more general. In fact, Theorem 3.1 can deal with the singular case as well as the non-singular case. For the singular case, we can deal with examples like (3.12) and (3.13), in which condition (1.2) is satisfied. For the non-singular case, we can deal with some examples like $f(t,x) = b(t)|x|^\beta$ with $b \geq 0$ and $\beta > 0$. Moreover, we can deal with more general nonlinearities which do not need to have a constant sign behavior.

Finally, in this section, we apply Theorem 3.1 to the following damped forced systems

$$x'' + h(t)x' + \nabla V(t,x) = e(t), \tag{3.14}$$

where $h \in \mathbb{C}(\mathbb{R}\backslash T\mathbb{Z}, R)$, $e \in \mathbb{C}(\mathbb{R}\backslash T\mathbb{Z}, \mathbb{R}^N)$ and $V \in \mathbb{C}^1(\mathbb{R}\times\mathbb{R}^N\backslash\{0\}, \mathbb{R})$ has a singularity of repulsive type at the origin, that is, there exists a fixed $v \in \mathbb{R}^N_+$ such that

$$\lim_{\substack{x\to 0\\ x\in\mathbb{R}^N_+}} (v, \nabla V(t,x)) = -\infty, \quad \text{uniformly in } t.$$

The following result is direct from Theorem 3.1.

Corollary 3.5. *Assume that there exists positive constant k satisfying* (2.2) *and r such that* (H_1)–(H_3) *are satisfied with $f(t,x)$ replaced by $k^2x - \nabla V(t,x)$. Then* (3.14) *has at least one nontrivial T-periodic solution if $(v,\gamma(t)) \geq 0$.*

Proof. Writing (3.14) as the equivalent system

$$x'' + h(t)x' + k^2x = e(t) + k^2x - \nabla V(t,x).$$

Now the proof is finished by a direct application of Theorem 3.1. □

An important application of Corollary 3.5 is the system (3.14) with the potential

$$V(x) = \frac{1}{|x|^\alpha} + k^2\frac{|x|^2}{2}. \tag{3.15}$$

Corollary 3.6. *Assume that there exists* $v \in \mathbb{R}_+^N$ *such that* $(v, \gamma(t)) \geq 0$ *for all* t *and* $\alpha > 0$. *Then* (3.14) *with* V *given as* (3.15) *has at least a nontrivial* T*-periodic solution for all* $k > 0$ *satisfying* (2.2).

Remark 3.7. In [**35**], Torres studied the system (1.3) and obtained existence results in the case $\alpha > 0$ and $(v, e(t)) \geq 0$ for some $v \in \mathbb{R}_+^N$. We have improved those results since we have the damping term, and we only require that $(v, \gamma(t)) \geq 0$ in Corollary 3.6.

Remark 3.8. In [**18, 35**], two existence results for the system (1.3) were established by Franco, Webb and Torres, respectively. Their proofs are based on fixed point theorems in cones for completely continuous operators, and the proofs are simpler and more clear than the proof presented in our paper. However, for us it seems difficult to obtain the same results in our paper using the fixed point theorems in cones. The main reason is that we have the term involved in (3.1). On the other hand, if we do not transform (1.1) into (3.1), we cannot get Theorem 3.1 under the condition $\Gamma_* \geq 0$.

Acknowledgments. We would like to show our thanks to the anonymous referee for his/her valuable comments.

REFERENCES

1. A. Ambrosetti and V. Coti Zelati, *Critical points with lack of compactness and singular dynamical systems*, Ann. Mat. Pura Appl. **149** (1987), 237–259.

2. ———, *Periodic solutions of singular Lagrangian systems*, Birkhauser, Boston, MA, 1993.

3. P. Amster and M. Clapp, *Periodic solutions of resonant systems with rapidly rotating nonlinearities*, Discr. Cont. Dynam. Syst. **31** (2011), 373–383.

4. A. Bahri and P.H. Rabinowitz, *A minimax method for a class of Hamiltonian systems with singular potentials*, J. Funct. Anal. **82** (1989), 412–428.

5. V. Bevc, J.L. Palmer and C. Süsskind, *On the design of the transition region of axi-symmetric magnetically focusing beam valves*, J. Brit. Inst. Radio Eng. **18** (1958), 696–708.

6. D. Bonheure and C. De Coster, *Forced singular oscillators and the method of lower and upper solutions*, Top. Meth. Nonlin. Anal. **22** (2003), 297–317.

7. J.L. Bravo and P.J. Torres, *Periodic solutions of a singular equation with indefinite weight*, Adv. Nonlin. Stud. **10** (2010), 927–938.

8. A. Cabada and J.A. Cid, *On the sign of the Green's function associated to Hill's equation with an indefinite potential*, Appl. Math. Comp. **205** (2008), 303–308.

9. J. Chu and P.J. Torres, *Applications of Schauder's fixed point theorem to singular differential equations*, Bull. Lond. Math. Soc. **39** (2007), 653–660.

10. J. Chu, P.J. Torres and M. Zhang, *Periodic solutions of second order non-autonomous singular dynamical systems*, J. Diff. Equat. **239** (2007), 196–212.

11. J. Chu and M. Zhang, *Rotation numbers and Lyapunov stability of elliptic periodic solutions*, Discr. Cont. Dynam. Syst. **21** (2008), 1071–1094.

12. J. Chu and Z. Zhang, *Periodic solutions of singular differential equations with sign-changing potential*, Bull. Austr. Math. Soc. **82** (2010), 437–445.

13. J. Chu, N. Fan and P.J. Torres, *Periodic solutions for second order singular damped differential equations*, J. Math. Anal. Appl. **388** (2012), 665–675.

14. M.A. del Pino and R.F. Manásevich, *Infinitely many T-periodic solutions for a problem arising in nonlinear elasticity*, J. Diff. Equat. **103** (1993), 260–277.

15. A. Fonda and R. Toader, *Periodic orbits of radially symmetric Keplerian-like systems*: *A topological degree approach*, J. Diff. Equat. **244** (2008), 3235–3264.

16. A. Fonda and A.J. Ureña, *Periodic, subharmonic, and quasi-periodic oscillations under the action of a central force*, Discr. Cont. Dynam. Syst. **29** (2011), 169–192.

17. D. Franco and P.J. Torres, *Periodic solutions of singular systems without the strong force condition*, Proc. Amer. Math. Soc. **136** (2008), 1229–1236.

18. D. Franco and J.R.L. Webb, *Collisionless orbits of singular and nonsingular dynamical systems*, Discr. Cont. Dynam. Syst. **15** (2006), 747–757.

19. W.B. Gordon, *Conservative dynamical systems involving strong forces*, Trans. Amer. Math. Soc. **204** (1975), 113–135.

20. A. Granas and J. Dugundji, *Fixed point theory*, Springer Mono. Math., Springer-Verlag, New York, 2003.

21. P. Habets and L. Sanchez, *Periodic solution of some Liénard equations with singularities*, Proc. Amer. Math. Soc. **109** (1990), 1135–1144.

22. R. Hakl and P.J. Torres, *On periodic solutions of second-order differential equations with attractive-repulsive singularities*, J. Diff. Equat. **248** (2010), 111–126.

23. ——, *Maximum and antimaximum principles for a second order differential operator with variable coefficients of indefinite sign*, Appl. Math. Comp. **217** (2011), 7599–7611.

24. R. Hakl, P.J. Torres and M. Zamora, *Periodic solutions of singular second order differential equations*: *Upper and lower functions*, Nonlin. Anal. **74** (2011), 7078–7093.

25. D. Jiang, J. Chu and M. Zhang, *Multiplicity of positive periodic solutions to superlinear repulsive singular equations*, J. Diff. Equat. **211** (2005), 282–302.

26. A.C. Lazer and S. Solimini, *On periodic solutions of nonlinear differential equations with singularities*, Proc. Amer. Math. Soc. **99** (1987), 109–114.

27. X. Li and Z. Zhang, *Periodic solutions for damped differential equations with a weak repulsive singularity*, Nonlin. Anal. **70** (2009), 2395–2399.

28. M. Meehan and D. O'Regan, *Existence theory for nonlinear Volterra integrodifferential and integral equations*, Nonlin. Anal. **31** (1998), 317–341.

29. I. Rachunková, M. Tvrdý and I. Vrkoč, *Existence of nonnegative and nonpositive solutions for second order periodic boundary value problems*, J. Diff. Equat. **176** (2001), 445–469.

30. M. Ramos and S. Terracini, *Noncollision periodic solutions to some singular dynamical systems with very weak forces*, J. Diff. Equat. **118** (1995), 121–152.

31. J. Ren, Z. Cheng and S. Siegmund, *Positive periodic solution for Brillouin electron beam focusing system*, Discr. Cont. Dynam. Syst. **16** (2011), 385–392.

32. S. Solimini, *On forced dynamical systems with a singularity of repulsive type*, Nonlin. Anal. **14** (1990), 489–500.

33. S. Terracini, *Remarks on periodic orbits of dynamical systems with repulsive singularities*, J. Funct. Anal. **111** (1993), 213–238.

34. P.J. Torres, *Existence of one-signed periodic solutions of some second-order differential equations via a Krasnoselskii fixed point theorem*, J. Diff. Equat. **190** (2003), 643–662.

35. ______, *Non-collision periodic solutions of forced dynamical systems with weak singularities*, Discr. Cont. Dynam. Syst. **11** (2004), 693–698.

36. ______, *Weak singularities may help periodic solutions to exist*, J. Diff. Equat. **232** (2007), 277–284.

37. P. Yan and M. Zhang, *Higher order nonresonance for differential equations with singularities*, Math. Meth. Appl. Sci. **26** (2003), 1067–1074.

38. M. Zhang, *Periodic solutions of damped differential systems with repulsive singular forces*, Proc. Amer. Math. Soc. **127** (1999), 401–407.

39. ______, *Periodic solutions of equations of Ermakov-Pinney type*, Adv. Nonlin. Stud. **6** (2006), 57–67.

40. ______, *Optimal conditions for maximum and antimaximum principles of the periodic solution problem*, Bound. Val. Prob. **2010**, Art. ID 410986, 26 pages.

Department of Mathematics, College of Science, Hohai University, Nanjing 210098, China
Email address: **jifengchu@126.com**

Department of Mathematics, College of Science, Hohai University, Nanjing 210098, China
Email address: **shjli_hn@126.com**

School of Statistics and Applied Mathematics, Anhui University of Finance and Economics, Bengbu 233030, China
Email address: **mmyddk@163.com**