ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 45, Number 1, 2015

CLASSIFICATION OF TOTALLY UMBILICAL $\xi^{\perp}$ CR-SUBMANIFOLDS OF COSYMPLECTIC MANIFOLDS

SIRAJ UDDIN, VIQAR AZAM KHAN AND CENAP OZEL

ABSTRACT. In **[6]**, Cabras, Ianus and Pitis proved that in a cosymplectic manifold there does not exist any extrinsic sphere tangent to the structure vector field ξ. We consider the structure vector field ξ normal to the submanifold in the sense of Papaghiuc **[12]** and derive that a totally umbilical CR-submanifold of a cosymplectic manifold is either (i) totally geodesic, (ii) anti-invariant or (iii) an extrinsic sphere.

1. Introduction. A submanifold M tangent to the structure vector field ξ is called a *contact* CR-*submanifold* if it admits a pair of differentiable distributions $\mathcal{D}$ and $\mathcal{D}^{\perp}$ such that $\mathcal{D}$ is invariant and its orthogonal complementary distribution $\mathcal{D}^{\perp}$ is anti-invariant, i.e., $TM = \mathcal{D} \oplus \mathcal{D}^{\perp} \oplus \langle \xi \rangle$ with $\phi(\mathcal{D}_x) \subseteq \mathcal{D}_x$ and $\phi(\mathcal{D}_x^{\perp}) \subset T_x^{\perp}M$, for every $x \in M$. Thus, a CR-submanifold M tangent to ξ is *invariant* if $\mathcal{D}^{\perp}$ is identically zero and an *anti-invariant* if $\mathcal{D}$ is identically zero, respectively. If neither $\mathcal{D} = \{0\}$ nor $\mathcal{D}^{\perp} = \{0\}$, then M is a *proper* CR-*submanifold*.

A submanifold M of a Riemannian manifold $\widetilde{M}$ is said to be *totally umbilical* if $h(X,Y) = g(X,Y)H$. If $h(X,Y) = 0$, for any X and Y tangent to M, then M is said to be a *totally geodesic submanifold*. If $H = 0$, then it is called a *minimal submanifold*.

A submanifold M of $\dim M \geq 2$ is said to be an *extrinsic sphere* **[7]** if it is totally umbilical and has a non-zero parallel mean curvature vector H.

2010 AMS *Mathematics subject classification.* Primary 53B25, 53C40, 53C42.
Keywords and phrases. Totally umbilical, CR-submanifold, extrinsic sphere, cosymplectic manifold.
Received by the editors on September 10, 2012, and in revised form on December 1, 2012.
DOI:10.1216/RMJ-2015-45-1-361

In **[1]**, Bejancu studied the CR-submanifolds of a Kaehler manifold. Later on, many research articles have been published on CR-submanifolds for different structures **[4]**. These submanifolds are the natural generalization of both holomorphic and totally real submanifolds of a Kaehler manifold. Totally umbilical CR-submanifolds of a Kaehler manifold have been studied by Bejancu **[2]**, Chen (see **[7, 8]**), and Deshmukh and Husain **[9]**.

An odd-dimensional counterpart of a Kaehler manifold is given by a cosymplectic manifold, which is locally a product of a Kaehler manifold with a circle or a line **[5]**. Indeed, a cosymplectic structure on a $(2n+1)-$dimensional manifold $\tilde{M}$ is a normal almost contact metric structure (ϕ, ξ, η, g) on $\tilde{M}$ such that the $1-$form η and the fundamental $2-$form Φ are closed (see **[3, 10]**). A trivial example of a cosymplectic manifold is given by the product of a $2n$-dimensional Kaehler manifold with a 1-dimensional manifold.

The submanifolds of a cosymplectic manifold have been studied by Ludden **[10]**. Later on, Cabras, Ianus and Pitis **[6]** proved that in a cosymplectic manifold there does not exist any extrinsic sphere tangent to the structure vector field ξ. Thus, to study extrinsic spheres in a cosymplectic manifold, we consider the structure vector field ξ normal to the submanifold in the sense of Papaghiuc, the submanifold in this case is called a $\xi^{\perp}-$submanifold **[12]**.

For a totally umbilical contact CR-submanifold tangent to the structure vector filed ξ of a cosymplectic manifold we proved the following theorem.

Theorem 1.1 ([**13**]). *Let M be a totally umbilical CR-submanifold of a cosymplectic manifold $\widetilde{M}$. Then at least one of the following statements is true.*

(i) *M is totally geodesic,*
(ii) *the anti-invariant distribution $\mathcal{D}^{\perp}$ is one-dimensional, i.e.,*

$$\dim \mathcal{D}^{\perp} = 1,$$

(iii) *the mean curvature vector $H \in \Gamma(\mu)$.*

In this paper we study a totally umbilical contact CR-submanifold of a cosymplectic manifold when the structure vector field ξ is normal to the submanifold. We discuss all possible cases on the classification of a totally umbilical $\xi^\perp$ CR-submanifold of cosymplectic manifolds.

2. Preliminaries. Let $\widetilde{M}$ be a $(2n+1)$-dimensional smooth manifold with an almost contact structure (ϕ, ξ, η), that is, ϕ is a $(1,1)$ tensor field, ξ is a vector field and η is a 1-form, satisfying the following properties

$$\phi^2 = -I + \eta \otimes \xi, \qquad \phi\xi = 0, \qquad \eta \circ \phi = 0, \qquad \eta(\xi) = 1. \tag{2.1}$$

In this case we call $(\widetilde{M}, \phi, \xi, \eta)$ an almost contact manifold. From **[3]**, there exists a Riemannian metric g on an almost contact manifold $\widetilde{M}$ satisfying the following compatibility condition

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y). \tag{2.2}$$

for any X, Y tangent to $\widetilde{M}$.

From **[3]**, we have the following definition. An almost contact structure (ϕ, ξ, η) is said to be *normal* if $[\phi, \phi] + 2d\eta \otimes \xi$ vanishes identically on $\widetilde{M}$, where

$$[\phi, \phi](X, Y) = \phi^2[X, Y] + [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y]$$

is the Nijenhuis tensor of ϕ for any vector fields X, Y tangent to $\widetilde{M}$.

The fundamental 2-form Φ on $\widetilde{M}$ is defined as $\Phi(X, Y) = g(X, \phi Y)$, for any vector fields X, Y tangent to $\widetilde{M}$. If $\Phi = d\eta$, the almost contact structure is called a *contact structure*. A normal almost contact structure with Φ and η is called a *cosymplectic structure*. It is well known that the cosymplectic structure is characterized by

$$\widetilde{\nabla}_X \phi = 0 \quad \text{and} \quad \widetilde{\nabla}_X \eta = 0, \tag{2.3}$$

where $\widetilde{\nabla}$ is the Levi-Civita connection of g on $\widetilde{M}$. From (2.3), it follows that $\widetilde{\nabla}_X \xi = 0$.

If we denote the curvature tensor of a cosymplectic manifold $\widetilde{M}$ by $\widetilde{R}$, then we have the following equalities

$$\widetilde{R}(\phi X, \phi Y)Z = \widetilde{R}(X, Y)Z \tag{2.4}$$

and

$$\widetilde{R}(X,Y)\phi Z = \phi\widetilde{R}(X,Y)Z.$$

Let M be a submanifold of an almost contact metric manifold $\widetilde{M}$ with the induced metric g, and let ∇ and $\nabla^{\perp}$ be the induced connections on the tangent bundle TM and the normal bundle $T^{\perp}M$ of M, respectively. We denote by $\mathcal{F}(M)$ the algebra of smooth functions on M and by $\Gamma(TM)$ the $\mathcal{F}(M)$-module of all smooth sections of a vector bundle TM over M. Then, the Gauss and Weingarten formulae are given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X,Y) \tag{2.5}$$

$$\widetilde{\nabla}_X N = -A_N X + \nabla^{\perp}_X N, \tag{2.6}$$

for each $X,\ Y \in \Gamma(TM)$ and $N \in \Gamma(T^{\perp}M)$, where h is the second fundamental form and A_N is the shape operator for the immersion of M into $\widetilde{M}$. They are related as

$$g(h(X,Y),N) = g(A_N X,Y), \tag{2.7}$$

where g denotes the Riemannian metric on $\widetilde{M}$ as well as induced on M. The mean curvature vector H on M is given by

$$H = \frac{1}{m}\sum_{i=1}^{m} h(e_i,e_i), \tag{2.8}$$

where m is the dimension of M and $\{e_1,\ e_2,\dots,e_m\}$ is a local orthonormal frame of the vector fields on M.

The covariant derivative of the second fundamental form h is defined as

$$(\nabla_X h)(Y,Z) = \nabla^{\perp}_X(h(Y,Z)) - h(\nabla_X Y,Z) - h(Y,\nabla_X Z). \tag{2.9}$$

The equations of Gauss and Codazzi are given, respectively, by

$$\begin{aligned} R(X,Y;Z,W) = {}& \widetilde{R}(X,Y;Z,W) + g(h(X,W),h(Y,Z)) \\ & - g(h(X,Z),h(Y,W)), \end{aligned} \tag{2.10}$$

$$\widetilde{R}(X,Y;Z,N) = g((\nabla_X h)(Y,Z),N) - g((\nabla_Y h)(X,Z),N), \tag{2.11}$$

where R denotes the curvature tensor of M,

$$R(X,Y;Z,W)=g(R(X,Y)Z,W),$$

for any X,Y,Z,W vector fields tangent to M and N normal to M.

By the definition of totally umbilical submanifold, the equations (2.5), (2.6), (2.9), (2.10) and (2.11) reduce to the following five equations, respectively:

$$\widetilde{\nabla}_X Y=\nabla_X Y+g(X,Y)H, \tag{2.12}$$

$$\widetilde{\nabla}_X N=-g(H,N)X+\nabla^{\perp}_X N, \tag{2.13}$$

$$(\nabla_X h)(Y,Z)=g(Y,Z)\nabla^{\perp}_X H, \tag{2.14}$$

$$\begin{aligned} R(X,Y;Z,W)=\widetilde{R}&(X,Y;Z,W)\\ &+\alpha^2\{g(X,W)g(Y,Z)-g(X,Z)g(Y,W)\} \end{aligned} \tag{2.15}$$

$$\widetilde{R}(X,Y;Z,N)=g(Y,Z)g(\nabla^{\perp}_X H,N)-g(X,Z)g(\nabla^{\perp}_Y H,N) \tag{2.16}$$

where $\alpha=\|H\|^2$.

3. Totally umbilical $\xi^{\perp}$ CR-submanifolds. Throughout this section, the structure vector field ξ is normal to the submanifold M, and we say that M is a $\xi^{\perp}$ submanifold. Thus, in this case we define CR-submanifolds as follows. A $\xi^{\perp}$ submanifold M of an almost contact metric manifold $\widetilde{M}$ is called a $\xi^{\perp}$ *CR-submanifold* if there exists a pair of differentiable distributions $\mathcal{D}$ and $\mathcal{D}^{\perp}$ on M such that $\mathcal{D}$ is invariant and its orthogonal complementary distribution $\mathcal{D}^{\perp}$ is anti-invariant, i.e., $TM=\mathcal{D}\oplus\mathcal{D}^{\perp}$ with $\phi(\mathcal{D}_x)\subseteq\mathcal{D}_x$ and $\phi(\mathcal{D}^{\perp}_x)\subset T^{\perp}_x M$, for every $x\in M$. Thus, a $\xi^{\perp}$ CR-submanifold M is *invariant* if $\mathcal{D}^{\perp}=\{0\}$ and *anti-invariant* if $\mathcal{D}=\{0\}$, respectively. If neither $\mathcal{D}=\{0\}$ nor $\mathcal{D}^{\perp}=\{0\}$, then M is a proper $\xi^{\perp}$ CR-submanifold. In the case of a $\xi^{\perp}$ CR-submanifold of an almost contact metric manifold, the normal bundle $T^{\perp}M$ is decomposed as

$$T^{\perp}M=\mu\oplus\langle\xi\rangle\oplus\phi D^{\perp}.$$

Now we give the following main result for a totally umbilical $\xi^{\perp}$ CR-submanifold.

Theorem 3.1. *Let M be a totally umbilical $\xi^\perp$ CR-submanifold of a cosymplectic manifold $\widetilde{M}$. Then M is one of the following.*

(i) *It is totally geodesic.*
(ii) *It is an anti-invariant submanifold.*
(iii) *It is an extrinsic sphere.*

Here we note that case (iii) *occurs when* $\dim M$ *is odd.*

Proof. Here we consider the structure vector field ξ normal to M. Then, by direct calculations as in Theorem 1.1 [13], we get the following equality

$$g(H, \phi Z)\left\{1 - \frac{g(Z, W)^2}{\|Z\|^2\|W\|^2}\right\} = 0, \tag{3.1}$$

for any $Z, W \in \Gamma(\mathcal{D}^\perp)$. From (3.1), we obtain that either $H = 0$, which is case (i) or $H \neq 0$ and $H \in \Gamma(\mu \oplus \langle\xi\rangle)$ or $H \neq 0$ and $H \notin \Gamma(\mu \oplus \langle\xi\rangle)$.

Now, if $H \neq 0$ and $H \in \Gamma(\mu \oplus \langle\xi\rangle)$, then for any $X \in \Gamma(\mathcal{D})$ and $N \in \Gamma(\phi\mathcal{D}^\perp)$, we have

$$\widetilde{\nabla}_X \phi N = \phi \widetilde{\nabla}_X N.$$

Using (2.12) and (2.13), we obtain

$$\nabla_X \phi N + g(X, \phi N)H = -g(H, N)\phi X + \phi \nabla^\perp_X N. \tag{3.2}$$

The second part of the left hand side and the first part of the right hand side are zero by the orthogonality of two distributions; hence, we get $\nabla_X \phi N = \phi \nabla^\perp_X N$. This means that $\nabla^\perp_X N \in \Gamma(\phi\mathcal{D}^\perp)$. Since $H \in \Gamma(\mu \oplus \langle\xi\rangle)$, then

$$0 = g(\nabla^\perp_X N, H) = -g(N, \nabla^\perp_X H). \tag{3.3}$$

Thus, it follows from (3.3) that $\nabla^\perp_X H \in \Gamma(\mu \oplus \langle\xi\rangle)$, for any $X \in \Gamma(\mathcal{D})$. Also, for a cosymplectic manifold, we have

$$\widetilde{\nabla}_X \phi H = \phi \widetilde{\nabla}_X H.$$

Then, from (2.13), we derive

$$-g(\phi H, H)X + \nabla^\perp_X \phi H = -g(H, H)\phi X + \phi \nabla^\perp_X H.$$

By orthogonality of H and ϕH, the above equation takes the form

$$\nabla^\perp_X \phi H = -g(H,H)\phi X + \phi\nabla^\perp_X H. \tag{3.4}$$

Hence, equation (3.4) gives $\phi X = 0$, for all $X \in \Gamma(\mathcal{D})$, i.e., $\mathcal{D} = \{0\}$. This proves case (ii) of the theorem.

Now, suppose $H \neq 0$ and $H \notin \Gamma(\mu \oplus \langle\xi\rangle)$. Then, again by (3.1), we obtain $\dim \mathcal{D}^\perp = 1$. Also, if we consider $\dim M \geq 5$, then there are at most two unit orthonormal vectors $X, Y \in \Gamma(\mathcal{D})$ such that $g(X,Y) = g(\phi X, \phi Y) = g(\phi X, Y) = -g(X, \phi Y) = 0$. Then, from (2.16), we have

$$\widetilde{R}(\phi Y, \phi X; X, N) = 0,$$

for any non zero vector field $N \in \Gamma(T^\perp M)$. Using (2.4), we obtain $\widetilde{R}(Y, X; X, N) = 0$. Again, using (2.16), we deduce that $g(\nabla^\perp_Y H, N) = 0$, for all $Y \in \Gamma(\mathcal{D})$ and $N \in \Gamma(T^\perp M)$, this means that

$$\nabla^\perp_Y H = 0, \quad \text{for all } Y \in \Gamma(\mathcal{D}). \tag{3.5}$$

Now, for any $X \in \Gamma(\mathcal{D})$ and $Z \in \Gamma(\mathcal{D}^\perp)$, from (2.16), we can derive $\widetilde{R}(Z, X; \phi X, N) = 0$, for any $N \in \Gamma(\mu \oplus \langle\xi\rangle)$. Using (2.4), we obtain $\widetilde{R}(Z, X; X, N) = 0$. Then, from (2.16), we obtain $g(\nabla^\perp_Z H, N) = 0$, for any $N \in \Gamma(\mu \oplus \langle\xi\rangle)$, which implies that

$$\nabla^\perp_Z H \in \Gamma(\phi\mathcal{D}^\perp), \quad \text{for all } Z \in \Gamma(\mathcal{D}^\perp). \tag{3.6}$$

Also, from (2.16), we get $\mathcal{R}(Z, X; \phi X, \phi Z) = 0$, for any $X \in \Gamma(\mathcal{D})$ and $Z \in \Gamma(\mathcal{D}^\perp)$. Hence, by (2.4), we derive $\widetilde{R}(Z, X; X, \phi Z) = 0$. Thus, from (2.16), we obtain that $g(\nabla^\perp_Z H, \phi Z) = 0$, for any $Z \in \Gamma(\mathcal{D}^\perp)$, that is

$$\nabla^\perp_Z H \in \Gamma(\mu \oplus \langle\xi\rangle), \quad \text{for all } \ Z \in \Gamma(\mathcal{D}^\perp). \tag{3.7}$$

Then from (3.6) and (3.7), we conclude that $\nabla^\perp_Z H \in \Gamma(\mu \oplus \langle\xi\rangle) \cap \Gamma(\phi\mathcal{D}^\perp)$, i.e.,

$$\nabla^\perp_Z H = 0, \quad \text{for all } Z \in \Gamma(\mathcal{D}^\perp). \tag{3.8}$$

Equations (3.5) and (3.8) imply that $\nabla^\perp_X H = 0$, for all $X \in \Gamma(TM)$. Hence, by definition, M is an extrinsic sphere in $\widetilde{M}$. This completes the proof of the theorem. $\square$

Given a Riemannian manifold $\widetilde{M}$, for any two linearly independent vectors $X,\ Y \in \Gamma(T\widetilde{M})$, the sectional curvature denoted by $\widetilde{K}(X,Y)$ is defined as

$$\widetilde{K}(X,Y) = \frac{\widetilde{R}(X,Y;Y,X)}{\|X\|^2\|Y\|^2 - g(X,Y)^2} \tag{3.9}$$

where $\widetilde{R}$ is the Riemannian curvature tensor. If X and Y are orthonormal vector fields on $\widetilde{M}$, then their sectional curvature is

$$\widetilde{K}(X,Y) = \widetilde{R}(X,Y;Y,X). \tag{3.10}$$

For a CR-submanifold M normal to the structure vector field ξ, the plane section $X \wedge Z$ with $X \in \Gamma(\mathcal{D})$ and $Z \in \Gamma(\mathcal{D}^\perp)$ is called a CR-*section*. The sectional curvature $\widetilde{K}(X \wedge Z)$ of a CR-section $X \wedge Z$ is called a CR-*sectional curvature*. Now we are ready to give the following result.

Theorem 3.2. *Let M be a totally umbilical $\xi^\perp$ CR-submanifold of a cosymplectic manifold $\widetilde{M}$. Then all CR-sectional curvatures of $\widetilde{M}$ vanish.*

Proof. For a totally umbilical submanifold we have

$$\widetilde{R}(X,Y;Z,N) = g(Y,Z)g(\nabla^\perp_X H, N) - g(X,Z)g(\nabla^\perp_Y H, N),$$

for any $X, Y, Z \in \Gamma(TM)$ and $N \in \Gamma(T^\perp M)$. In particular, if for any unit vectors $X \in \Gamma(\mathcal{D})$ and $Z \in \Gamma(\mathcal{D}^\perp)$, then the above equation takes the form

$$\widetilde{R}(X,Z;\phi X,\phi Z) = 0.$$

Using the property of Riemannian curvature tensor, we obtain

$$\widetilde{R}(\phi X,\phi Z;X,Z) = 0.$$

Then, from (2.4) and the property of Riemannian curvature tensor, we get

$$\widetilde{R}(X,Z;X,Z) = -\widetilde{R}(X,Z;Z,X) = 0. \tag{3.11}$$

Hence, by equations (3.10) and (3.11), we obtain $\widetilde{K}(X \wedge Z) = 0$ which is the desired result. $\square$

REFERENCES

1. A. Bejancu, *CR-submanifolds of a Kaehler manifold*, Proc. Amer. Math. Soc. **69** (1978), 135–142.

2. ______, *Umbilical CR-submanifolds of a Kaehler manifold*, Rend Math. J. **13** (1980), 431–466.

3. D.E. Blair, *Contact manifolds in Riemannian geometry*, Lect. Notes Math. **509**, Springer-Verlag, Berlin, 1976.

4. D.E. Blair and B.Y. Chen, *On CR-submanifolds of Hermitian manifold*, Israel J. Math. **34** (1980), 353–363.

5. D.E. Blair and S.I. Goldberg, *Topology of almost contact manifolds*, J. Differ. Geom. **1** (1967), 347–354.

6. A. Cabras, S. Ianus and GH. Pitis, *Extrinsic spheres and parallel submanifolds in cosymplectic manifolds*, Math. J. Tayama **17** (1994), 31–53.

7. B.Y. Chen, *Extrinsic spheres in Kaehler manifolds*, Mich. Math. J. **23** (1976), 327–330.

8. ______, *Totally umbilical submanifolds of Kaehler manifolds*, Arch. Math. J. **36** (1981), 83–91.

9. S. Deshmukh and S.I. Husain, *Totally umbilical CR-submanifolds of a Kaehler manifold*, Kodai Math. J. **9** (1986), 425–429.

10. G.D. Ludden, *Submanifolds of cosymplectic manifolds*, J. Differ. Geom. **4** (1970), 237–244.

11. K. Nomizu and K. Yanu, *On circles and spheres in Riemannian geometry*, Math. Ann. **210** (1974), 163–170.

12. N. Papaghiuc, *Semi-invariant submanifolds in a Kenmotsu manifold*, Rend. Mat. **3** (1983), 607–622.

13. Siraj Uddin and Cenap Ozel, *A classification theorem on totally umbilical submanifolds in a cosymplectic manifold*, Hacettepe J. Math. Stat. **43** (2014), 635–640.

Department of Mathematics, Faculty of Science, King Abdulaziz University, 21589 Jeddah, Saudi Arabia
Email address: siraj.ch@gmail.com

Department of Mathematics, Aligarh Muslim University, 202002 Aligarh, India
Email address: viqarster@gmail.com

Department of Mathematics, Dokuz Eylul University, 35160 Izmir, Turkey
Email address: cenap.ozel@gmail.com