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RIGIDITY OF HYPERSPACES

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ABSTRACT. Given a metric continuum X, we consider the following hyperspaces of X: 2^X, $C_n(X)$ and $F_n(X)$ ($n \in \mathbb{N}$). Let $F_1(X) = \{\{x\} : x \in X\}$. A hyperspace $K(X)$ of X is said to be rigid, provided that for every homeomorphism $h : K(X) \to K(X)$, we have $h(F_1(X)) = F_1(X)$. In this paper, we study conditions under which a continuum X has a rigid hyperspace $C_n(X)$. Among others, we consider families of continua, such as dendroids, Peano continua, hereditarily indecomposable continua and smooth fans.

1. Introduction. A *continuum* is a nondegenerate compact connected metric space. Given a continuum X, with metric d, we consider the following *hyperspaces* of X.

$$\begin{aligned} 2^X &= \{A \subset X : A \text{ is nonempty and closed in } X\}, \\ C_n(X) &= \{A \in 2^X : A \text{ has at most } n \text{ components}\}, \\ F_n(X) &= \{A \in 2^X : A \text{ has at most } n \text{ points}\}, \\ C(X) &= C_1(X). \end{aligned}$$

All hyperspaces are considered with the *Hausdorff metric* H [**19,** Remark 0.4] defined as

$$H(A, B) = \max\{\max\{d(a, B) : a \in A\}, \max\{d(b, A) : b \in B\}\},$$

where $d(a, B) = \min\{d(a, b) : b \in B\}$.

The hyperspace $F_n(X)$ is known as the *nth symmetric product* of X. The hyperspace $F_1(X)$ is an isometric copy of X embedded in each

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one of the hyperspaces. We extend the definition of $C_n(X)$ by defining $C_0(X) = \emptyset$.

A hyperspace $K(X) \in \{2^X, C_n(X), F_n(X)\}$ is said to be *rigid* provided that, for each homeomorphism $h : K(X) \to K(X)$, we have $h(F_1(X)) = F_1(X)$. The continuum X is said to have *unique hyperspace* $K(X)$ provided that the following implication holds: if Y is a continuum such that $K(X)$ is homeomorphic to $K(Y)$, then X is homeomorphic to Y.

Uniqueness of hyperspaces has been widely studied (see **[3, 8, 9, 10, 11, 15]** for recent references). The paper **[16]** provides a detailed survey of what is known about this topic. In the study of hyperspaces, a useful technique is to find a topological property that characterizes the elements of $F_1(X)$ in the hyperspace $K(X)$. When it is possible to find such a characterization, the hyperspace $K(X)$ is rigid. This technique has been used in studying uniqueness of hyperspaces, so both topics are closely related. Moreover, the topic of this paper leads us to new results on unique hyperspaces.

In this paper, we study rigidity of the hyperspaces $C_n(X)$. In **[10]**, rigidity of the symmetric products $F_n(X)$ is studied. In **[8]**, rigidity of hyperspaces $C_n(X)$ for indecomposable continua such that all their proper nondegenerate subcontinua are arcs is considered.

Among others, we consider families of continua, such as dendroids, Peano continua, hereditarily indecomposable continua and smooth fans.

2. Definitions and conventions. A *map* is a continuous function. Suppose that d is a metric for X. Given $\varepsilon > 0$, $p \in X$ and $A \in 2^X$, let $B(\varepsilon, p)$ be the open ε-ball around p in X, $N(\varepsilon, A) = \{p \in X$: there exists $a \in A$ such that $d(p, a) < \varepsilon\}$ and $B^H(\varepsilon, A) = \{E \in 2^X : H(A, E) < \varepsilon\}$ (we write $B_X(\varepsilon, p)$ and $N_X(\varepsilon, A)$ when the space X needs to be mentioned). A *simple n-od* is a finite graph G that is the union of n arcs emanating from a single point, v, and otherwise disjoint from one another. The point v is called the *vertex* of G. Simple 3-ods are called *simple triods*. Given subsets $A_1, \ldots, A_m$ of X, let $\langle A_1, \ldots, A_m \rangle = \{B \in 2^X : B \cap A_i \neq \emptyset$ for each $i \in \{1, \ldots, m\}$ and $B \subset A_1 \cup \ldots \cup A_m\}$.

We denote by S^1 the unit circle in the Euclidean plane. A *free arc* in the continuum X is an arc α with end points a and b such that

$\alpha - \{a, b\}$ is open in X. A *tail* in a continuum X is an arc α with end points a and b such that $\alpha - \{a\}$ is open in X. An *end point* in X is a point $p \in X$ such that p is an end point of every arc containing it.

Given a continuum X, let

$$\mathcal{G}(X) = \{p \in X : p \text{ has a neighborhood } M \text{ in } X \text{ such that } M \text{ is a finite graph}\},$$
$$\text{and } \mathcal{P}(X) = X - \mathcal{G}(X).$$

The continuum X is said to be *almost meshed* [**9**] provided that the set $\mathcal{G}(X)$ is dense in X.

Proceeding as in [**4,** Lemma 2.1] and using Lemma 1.48 of [**19**], the following lemma can be proved.

Lemma 2.1. *Let X be a continuum, and let $\mathcal{A}$ be a connected subset of 2^X such that $\mathcal{A} \cap C_n(X) \neq \emptyset$. Let $A_0 = \bigcup\{A : A \in \mathcal{A}\}$. Then*

(a) *A_0 has at most n components,*
(b) *if $\mathcal{A}$ is closed in 2^X, then $A_0 \in C_n(X)$,*
(c) *for each $A \in \mathcal{A}$, each component of A_0 intersects A.*

The continuum X is said to be *indecomposable*, provided that X cannot be put as the union of two of its proper subcontinua. And X is called *hereditarily indecomposable*, provided that each one of its subcontinua is indecomposable. The simplest indecomposable continuum is the so-called *Buckethandle continuum* which is described in [**20**]. The reader is referred to [**20**] where more examples of indecomposable and hereditarily indecomposable continua can be found.

A *wire* in a continuum X is a subset α of X such that α is homeomorphic to one of the spaces $(0, 1)$, $[0, 1)$, $[0, 1]$ or S^1 and α is a component of an open subset of X. By [**19,** Theorem 20.3], if a wire α in X is compact, then $\alpha = X$. So, if a wire is homeomorphic to $[0, 1]$ or S^1, then X is an arc or a simple closed curve. Given a continuum X, let

$$W(X) = \bigcup\{\alpha \subset X : \alpha \text{ is a wire in } X\}.$$

The continuum X is said to be *wired* provided that $W(X)$ is dense in X.

Notice that, if α is a free arc of a continuum X and p, q are the end points of α, then $\alpha - \{p, q\}$ is a wire in X. Thus, a continuum for which the union of its free arcs is dense is a wired continuum. Therefore, the class of wired continua includes finite graphs, dendrites with a closed set of end points, almost meshed continua **[9]**, compactifications of the ray $[0, \infty)$, compactifications of the real line and indecomposable continua whose nondegenerate proper subcontinua are arcs, which will be called indecomposable arc continua (see Lemma 2.2).

Lemma 2.2. *Let X be an indecomposable arc continuum. Then X is a wired continuum.*

Proof. Let $p \in X$. Let U be an open subset of X such that $cl_X(U) \neq X$. Let D be the component of U containing p. By **[19,** Theorem 20.3], $cl_X(D) \cap (X - U) \neq \emptyset$. Thus, D is not compact and $cl_X(D)$ is a proper nondegenerate subcontinuum of X. Hence, $cl_X(D)$ is an arc and D is a non compact connected subset of D. This implies that D is homeomorphic to $(0, 1)$ or $[0, 1)$. Hence, D is a wire. □

3. Wired continua. In this section, we present some technical results that will be used later for proving that some hyperspaces are rigid.

Given a point p in a continuum X, let $\dim_p[X]$ denote the inductive dimension of the continuum X at the point p **[20,** Definition 13.53]. An *m-od* in a continuum X is a subcontinuum B of X for which there exists $A \in C(B)$ such that $B - A$ has at least m components. By **[17,** Theorem 70.1], a continuum X contains an m-od if and only if $C(X)$ contains an m-cell. Given $A, B \in 2^X$ such that $A \subsetneq B$, an *order arc* from A to B is a continuous function $\alpha : [0, 1] \to C(X)$ such that $\alpha(0) = A$, $\alpha(1) = B$ and $\alpha(s) \subsetneq \alpha(t)$ if $0 \leq s < t \leq 1$. It is known **[19,** Theorem 1.25] that there exists an order arc from A to B if and only if $A \subsetneq B$ and each component of B intersects A.

Given a continuum X and $n \in \mathbb{N}$, let

$$\mathcal{W}_n(X) = \{A \in C_n(X) : \text{each component of } A \text{ is contained in a wire of } X\};$$

and

$$\mathcal{Z}_n(X) = \{A \in \mathcal{W}_n(X) : \text{there is a neighborhood } \mathcal{M} \text{ of } A \text{ in } C_n(X) \text{ such that the component } \mathcal{C} \text{ of } \mathcal{M} \text{ that contains } A \text{ is a } 2n\text{-cell}\}$$

A continuum X is said to be *n-wired preserving* provided that, for each homeomorphism $h : C_n(X) \to C_n(X)$, $h(\mathcal{W}_n(X)) = \mathcal{W}_n(X)$.

Lemma 3.1. *If X is a Peano continuum and $n \geq 1$, then*

$$\mathcal{W}_n(X) = \{A \in C_n(X) : \dim_A[C_n(X)] = 2n\}.$$

Proof. By [**18,** Theorem 2.4], if X is either an arc or a simple closed curve, then $\mathcal{W}_n(X) = C_n(X) = \{A \in C_n(X) : \dim_A[C_n(X)] = 2n\}$. Thus, we may assume that X is neither an arc nor a simple closed curve. Let $A \in \mathcal{W}_n(X)$ and let C be a component of A. Then there exists a wire W of X such that $C \subset W$. If W is a compact wire, by [**19,** Theorem 20.3], $W = X$ and X is an arc or a simple closed curve, which is contrary to our assumption. Thus, W is homeomorphic either to $(0,1)$ or $[0,1)$ and W is a component of an open set U of X. Since X is a Peano continuum, W is open in X. Let B be an arc such that $C \subset int_X(B) \subset B \subset W$. Thus, each point of C has a neighborhood in X that is an arc. Therefore, each point of A has a neighborhood in X that is an arc. By [**9,** Theorem 4], $\dim_A[C_n(X)]$ is finite, and there exists a finite graph D contained in X such that $A \subset \mathrm{int}_X(D)$. Thus, $C_n(D)$ is a neighborhood of A in $C_n(X)$. Since each point of A has a neighborhood in X that is an arc, A does not have ramification points of D, so by [**18,** Theorem 2.4] $2n = \dim_A[C_n(D)] = \dim_A[C_n(X)]$.

Now, take $A \in C_n(X)$ such that $\dim_A[C_n(X)] = 2n$. By [**9,** Theorem 4], there exists a finite graph D contained in X such that $A \subset \mathrm{int}_X(D)$. Let $R(D)$ be the set of ramification points of D. Notice that $\dim_A[C_n(D)] = \dim_A[C_n(X)] = 2n$. Thus, $A \cap R(D) = \emptyset$ by [**18,** Theorem 2.4]. Let C be a component of A. Then C is contained in an edge J of D. Let W be the component of $\mathrm{int}_X(D) \cap (X - R(D))$ that contains C. Then W is a connected subset of J. Since J is either an arc or a simple closed curve, W is a wire. Therefore, $A \in \mathcal{W}_n(X)$. $\square$

Lemma 3.2. *Let X be a continuum, and let*

$$\begin{aligned}\mathcal{V} = \{A \in C_n(X) :\ & \textit{there exists a neighborhood } \mathcal{M} \textit{ of } A \textit{ in } C_n(X) \\ & \textit{such that if } \mathcal{C} \textit{ is the component of } \mathcal{M} \textit{ containing } A, \\ & \textit{then } \dim[\mathcal{C}] = 2n\}.\end{aligned}$$

Then

(a) $\mathcal{W}_n(X) \subset \mathcal{V}$,
(b) *if X is a dendroid, then $\mathcal{W}_n(X) = \mathcal{V}$.*

Proof. By [**18,** Theorem 2.4], if X is either an arc or a simple closed curve, then $\mathcal{W}_n(X) = C_n(X) = \mathcal{V}$, and we are done. Thus, suppose that X is neither an arc nor a simple closed curve.

(a) Let $A \in \mathcal{W}_n(X)$. Let $A_1, \ldots, A_m$ be the components of A, where $m \leq n$. For each $i \in \{1, \ldots, m\}$, let U_i be an open subset of X such that the component W_i of U_i containing A_i is homeomorphic either to $(0, 1)$ or $[0, 1)$. Let $V_1, \ldots, V_m$ be open subsets of X such that $cl_X(V_1), \ldots, cl_X(V_m)$ are pairwise disjoint and, for each $i \in \{1, \ldots, m\}$, $A_i \subset V_i \subset U_i$. Let Z_i be the component of V_i containing A_i. Then Z_i is a nondegenerate connected subset of W_i. Thus, Z_i is homeomorphic either to $(0, 1)$ or $[0, 1)$. Let $\mathcal{M} = \langle V_1, \ldots, V_m \rangle \cap C_n(X)$, and let $\mathcal{D}$ be the component of $\mathcal{M}$ containing A. Then $\mathcal{M}$ is a neighborhood of A in $C_n(X)$. We claim that $\mathcal{D} = \langle Z_1, \ldots, Z_m \rangle \cap C_n(X)$. Let $\mathcal{C} = \langle Z_1, \ldots, Z_m \rangle \cap C_n(X)$. Note that $A \in \mathcal{C}$. For each $i \in \{1, \ldots, m\}$, fix a point $z_i \in Z_i$. Let $B \in \mathcal{C}$. For each $i \in \{1, \ldots, m\}$, $(B \cap Z_i) \cup \{z_i\} = (B \cap cl_X(V_i)) \cup \{z_i\}$ is a compact subset of Z_i, so there exists an arc $L_i \subset Z_i$ such that $(B \cap Z_i) \cup \{z_i\} \subset L_i$. Let α be an order arc from B to $L = L_1 \cup \cdots \cup L_m$, and let β be an order arc from $\{z_1, \ldots, z_m\}$ to L. Then $\operatorname{Im} \alpha \cup \operatorname{Im} \beta$ defines a path in $\mathcal{C}$ joining B and $\{z_1, \ldots, z_m\}$. This proves that $\mathcal{C}$ is a connected subset of $\mathcal{M}$. Hence, $\mathcal{C} \subset \mathcal{D}$. Let $D = \bigcup\{E : E \in \mathcal{D}\}$. Since $A \in \mathcal{D}$, by Lemma 2.1, D has at most m components, and each one of them intersects A. For each $i \in \{1, \ldots, m\}$, let D_i be the component of D containing A_i. Since $\mathcal{D} \subset \mathcal{M}$, $D \subset V_1 \cup \cdots \cup V_m$, so $A_i \subset D_i \subset V_i$. Thus, $D_i \subset Z_i$. This proves that $D \in \mathcal{C}$. Given $E \in \mathcal{D}$, $E \subset D$ and each component of D intersects E. Thus, $E \in \langle Z_1, \ldots, Z_m \rangle \cap C_n(X) = \mathcal{C}$. We have shown that $\mathcal{D} = \langle Z_1, \ldots, Z_m \rangle \cap C_n(X)$.

Let $J_1, \ldots, J_m$ be subintervals of $[0,1]$ such that $\mathrm{cl}_{[0,1]}(J_1), \ldots, \mathrm{cl}_{[0,1]}(J_m)$ are pairwise disjoint and, for each $i \in \{1, \ldots, m\}$, there exists a homeomorphism $f_i : Z_i \to J_i$. Let $f : \mathcal{D} \to \langle J_1, \ldots, J_m \rangle \cap C_n([0,1])$ be given by $f(B) = f_1(B \cap Z_1) \cup \cdots \cup f_m(B \cap Z_m)$. It is easy to show that f is a homeomorphism. Thus, $\dim_{\mathcal{D}}[C_n(X)] = \dim_{\langle J_1,\ldots,J_m\rangle \cap C_n([0,1])}[C_n([0,1])]$. By [**18,** Theorem 2.4], the dimension of this set is equal to $2n$. Hence, $\dim[\mathcal{D}] = 2n$. This ends the proof of (a).

(b) Suppose that X is a dendroid. Let $A \in C_n(X)$, and let $\mathcal{M}$ be a neighborhood of A in $C_n(X)$ such that, if $\mathcal{C}$ is the component of $\mathcal{M}$ containing A, then $\dim[\mathcal{C}] = 2n$. Let $A_1, \ldots, A_m$ be the components of A. Let $\varepsilon > 0$ be such that $B^H(2\varepsilon, A) \cap C_n(X) \subset \mathcal{M}$ and the sets $N(2\varepsilon, A_1), \ldots, N(2\varepsilon, A_m)$ are pairwise disjoint. For each $i \in \{1, \ldots, m\}$, let D_i be the component of $cl_X(N(\varepsilon, A_i))$ containing A_i. Let $D = D_1 \cup \cdots \cup D_m \in C_n(X)$. Notice that $H(A, D) < 2\varepsilon$. Let $\alpha : [0,1] \to C_n(X)$ be an order arc from A to D. Then, for each $t \in [0,1]$, $H(\alpha(t), A) \le H(D, A) < 2\varepsilon$. This implies that $D \in \mathcal{C}$.

We claim that D does not contain simple triods. Suppose, to the contrary, that there exists a simple triod T in D. Let v be the vertex of T. For each $i \in \{1, \ldots, m\}$, D_i is a dendroid. Then D_i is a limit of its subtrees. Thus, there exists a tree $S_i \subset D_i$ such that, if $S = S_1 \cup \ldots \cup S_m$, then $T \subset S$ and $H(D, S) < 2\varepsilon$. Using an order arc from S to D, we conclude that $S \in \mathcal{C}$. Since X is arcwise connected, we can join the different components of S by arcs and obtain a finite graph $G \subset X$ such that $S \subset G$. Since $C_n(G)$ is locally connected, there exists a compact connected neighborhood $\mathcal{N}$ of S in $C_n(G)$ such that $\mathcal{N} \subset \mathcal{M}$. Thus, $\mathcal{N} \subset \mathcal{C}$. By [**18,** Theorem 2.4], $2n < \dim_S[C_n(G)] = \dim_S[\mathcal{N}] \le \dim_S[\mathcal{C}] \le \dim[\mathcal{C}]$. This contradicts the choice of $\mathcal{M}$ and proves that D does not contain a simple triod.

Using that each arc in a dendroid is contained in a maximal one with respect to the inclusion (this follows from [**21,** Theorem 3.3]), it is possible to prove that a dendroid without simple triods is an arc. Thus, $D_1, \ldots, D_m$ are arcs. For each $i \in \{1, \ldots, m\}$, let W_i be the component of $N(\varepsilon, A_i)$ containing A_i. Then W_i is a nondegenerate connected subset of D_i. Thus, W_i is a wire. Hence, $A \in \mathcal{W}_n(X)$. This completes the proof of (b). $\square$

Lemmas 3.1 and 3.2 give topological characterizations of $\mathcal{W}_n(X)$

for the cases when X is a Peano continuum or a dendroid. As a consequence, we have the following theorem.

Theorem 3.3. *Continua that are in one of the following classes are n-wired preserving continua for every $n \in \mathbb{N}$.*

(a) *Peano continua,*
(b) *dendroids.*

The following example shows that arcwise connectedness is not enough to conclude that a continuum is 1-wired preserving.

Example 3.4. Let X be the Warzaw circle.

We will see that X is not a 1-wired preserving continuum. Let L be the limit arc of X. By [**17,** Theorem 7.4], $C(X)$ is homeomorphic to Cone$\,(X)$ (the topological cone over X). Let $h : C(X) \to$ Cone$\,(X)$ be a homeomorphism. Let M be a subcontinuum of X such that $L \subsetneq M$, and let N be an arc in X such that $L \cap N = \emptyset$. It is easy to see that L and M have neighborhoods $\mathcal{L}$ and $\mathcal{M}$, respectively, in $C(X)$ such that $\mathcal{L}$ and $\mathcal{M}$ are 2-cells and L (respectively, M) belongs to the manifold interior of $\mathcal{L}$ (respectively, $\mathcal{M}$). Clearly, for any two points p, q in Cone$\,(X)$ that have neighborhoods with these characteristics (2-cells having the point in its manifold interior) there exists a homeomorphism $h :$ Cone$\,(X) \to$ Cone$\,(X)$ such that $h(p) = q$. Hence, there exists a homeomorphism $g : C(X) \to C(X)$ such that $g(M) = N$. However, $M \notin \mathcal{W}_1(X)$ and $L \in \mathcal{W}_1(X)$. This proves that X is not a 1-wired preserving continuum. □

We finish this section by generalizing results that have been used in studying uniqueness of hyperspaces in other papers (see [**9**] for the more recent development). Here we consider components of neighborhoods instead of neighborhoods, in this way, we can use our results for more general continua.

Theorem 3.5. *Let X be a continuum. Then*

(a) $\mathcal{W}_n(X) \cap (C_n(X) - C_{n-1}(X)) \subset \mathcal{Z}_n(X)$,
(b) *if $n \geq 3$, then* $\mathcal{W}_n(X) \cap (C_n(X) - C_{n-1}(X)) = \mathcal{Z}_n(X)$.

Proof. (a) In the case that $n = 1$ and X is either an arc or a simple closed curve, $C(X)$ is a 2-cell, so $\mathcal{W}_n(X) = C(X) = \mathcal{Z}_n(X)$. Thus, we assume that $n > 1$ or X is neither an arc nor a simple closed curve. Take $A \in \mathcal{W}_n(X) \cap (C_n(X) - C_{n-1}(X))$. Let $A_1, \ldots, A_n$ be the components of A. For each $i \in \{1, \ldots, n\}$, let W_i be an open subset of X such that $A_i \subset W_i$, and the component C_i of W_i containing A_i is homeomorphic either to $(0, 1)$ or $[0, 1)$. For each $i \in \{1, \ldots, n\}$, let V_i be an open subset of X such that $A_i \subset V_i \subset cl_X(V_i) \subset W_i$ and the sets $cl_X(V_1), \ldots, cl_X(V_n)$ are pairwise disjoint. Let D_i be the component of $cl_X(V_i)$ containing A_i. Then D_i is a nondegenerate subcontinuum of C_i. Hence, D_i is an arc. Let $\mathcal{M} = \langle cl_X(V_1), \ldots, cl_X(V_n)\rangle \cap C_n(X)$ and $\mathcal{C} = \langle D_1, \ldots, D_n\rangle \cap C_n(X)$. Then $\mathcal{M}$ is a neighborhood of A in $C_n(X)$ and $\mathcal{C}$ is compact and connected. In fact, the map $\varphi : C(D_1) \times \ldots \times C(D_n) \to \mathcal{C}$ given by $\varphi(B_1, \ldots, B_n) = B_1 \cup \ldots \cup B_n$ is a homeomorphism. Hence, $\mathcal{C}$ is a $2n$-cell [**17,** Example 5.1]. We claim that $\mathcal{C}$ is a component of $\mathcal{M}$. Since $\mathcal{C}$ is connected, we can take the component $\mathcal{D}$ of $\mathcal{M}$ that contains A. Then $\mathcal{C} \subset \mathcal{D}$. Let $E = \bigcup\{F : F \in \mathcal{D}\}$. By Lemma 2.1, $E \in C_n(X)$. Notice that $A \subset E \subset cl_X(V_1) \cup \cdots \cup cl_X(V_n)$. This implies that E has exactly n components, in fact, the components of E are $E_1 = E \cap cl_X(V_1), \ldots, E_n = E \cap cl_X(V_n)$. Since $A_1 \subset E_1, \ldots, A_n \subset E_n$ and $D_1 \cup \cdots \cup D_n \in \mathcal{C} \subset \mathcal{D}$, we have $E_1 = D_1, \ldots, E_n = D_n$. Hence, for each $G \in \mathcal{D}$, $G \subset E$ and, by Lemma 2.1, each E_i intersects G, so $G \in \mathcal{M}$ and $G \cap cl_X(V_1) \subset D_1, \ldots, G \cap cl_X(V_n) \subset D_n$. This implies that $G \in \mathcal{C}$. We have shown that $\mathcal{C} = \mathcal{D}$. Therefore, $A \in \mathcal{Z}_n(X)$.

(b) Let $A \in \mathcal{Z}_n(X)$, and let $\mathcal{M}$ be a neighborhood of A in $C_n(X)$ such that the component $\mathcal{C}$ of $\mathcal{M}$ that contains A is a $2n$-cell. Let $A_1, \ldots, A_m$ be the components of A. For each $i \in \{1, \ldots, m\}$, let W_i be an open subset of X such that $A_i \subset W_i$, $\langle W_i, \ldots, W_m\rangle \subset \mathcal{M}$ and the component C_i of W_i containing A_i is homeomorphic either to $(0, 1)$ or $[0, 1)$ (or $[0, 1]$ or S^1 in the case that X is either an arc or a simple closed curve). Let $\varepsilon > 0$ be such that $B^H(3\varepsilon, A) \cap C_n(X) \subset \mathcal{M}$, the sets $N(3\varepsilon, A_1), \ldots, N(3\varepsilon, A_m)$ are pairwise disjoint and $N(3\varepsilon, A_i) \subset W_i$ for each $i \in \{1, \ldots, m\}$.

We need to show that $m = n$. Suppose to the contrary that $m < n$. In the case that $m < n - 1$, if X is neither an arc nor a simple closed curve, since $A_1 \neq C_1$, there exist subarcs $A_{m-1}, \ldots, A_{n-1}$ of C_1 such that $A_1, \ldots, A_{n-1}$ are pairwise disjoint. Let $B = A_1 \cup \cdots \cup A_{n-1}$. Then $B \in \mathcal{M}$ and $B \in \mathcal{C}$. If $A = X$ and X is either an arc or a simple closed curve, then it is easy to find pairwise disjoint subarcs $A_1, \dots, A_{n-1}$ such that the set $B = A_1 \cup \dots \cup A_{n-1} \in \mathcal{M}$ and $B \in \mathcal{C}$. In the case that $m = n-1$, define $B = A$. In any case, changing A for B if necessary, we can assume that A has exactly $n-1$ components (and then $A \neq X$).

For each $i \in \{1, \dots, n-1\}$, let D_i be the component of $cl_X(N(2\varepsilon, A_i))$ that contains A_i. Since $D_i \subset C_i$, D_i is an arc. Let $\mathcal{E} = \langle D_1, \dots, D_{n-1}\rangle \cap C_n(X)$. For each $i \in \{1, \dots, n-1\}$, let $\varphi_i : C(D_1) \times \ldots \times C(D_{i-1}) \times C_2(D_i) \times C(D_{i+1}) \times \ldots \times C(D_{n-1}) \to \mathcal{E}$ be given by $\varphi_i(E_1, \dots, E_{n-1}) = E_1 \cup \dots \cup E_{n-1}$. Since $C_2([0,1])$ is a 4-cell [**14,** Lemma 2.2], $C([0,1])$ is a 2-cell and φ_i is an embedding, $Im\varphi_i$ is a $2n$-cell. Notice that $\mathcal{E} = Im\varphi_1 \cup \ldots \cup Im\varphi_{n-1}$ and $Im\varphi_i \cap Im\varphi_j = \langle D_1, \dots, D_{n-1}\rangle \cap C_{n-1}(X)$, if $i \neq j$. Let $\mathcal{E}_0 = \langle D_1, \dots, D_{n-1}\rangle \cap C_{n-1}(X)$. Then $\mathcal{E}_0$ is a $2(n-1)$-cell.

Let $\mathcal{F}$ be the component of $cl_{C_n(X)}(B^H(\varepsilon, A) \cap C_n(X)) \cap \mathcal{C}$ such that $A \in \mathcal{F}$. Let $F = \bigcup\{G : G \in \mathcal{F}\}$. Since $A \in \mathcal{F}$, F has at most $n-1$ components (Lemma 1) and $F \cap N(2\varepsilon, A_i) \neq \emptyset$ for each $i \in \{1, \dots, n-1\}$. This implies that F has exactly $n-1$ components and they are $F_1 = F \cap N(2\varepsilon, A_1), \dots, F_{n-1} = F \cap N(2\varepsilon, A_{n-1})$. Thus, $F_1 \subset D_1, \dots, F_{n-1} \subset D_{n-1}$. Given $G \in \mathcal{F}$, $H(G, A) < 2\varepsilon$. This implies that $G \cap N(2\varepsilon, A_i) \neq \emptyset$ for each $i \in \{1, \dots, n-1\}$. Since $G \subset F \subset D_1 \cup \dots \cup D_{n-1}$ and by Lemma 2.1 (c), $G \cap D_i \neq \emptyset$ for each $i \in \{1, \dots, n-1\}$, so we conclude that $G \in \mathcal{E}$.

We have shown that $\mathcal{F} \subset \mathcal{E}$. Since $\mathcal{F}$ is a neighborhood of A in the $2n$-cell $\mathcal{C}$, there exists a $2n$-cell $\mathcal{K}$ such that $A \in \mathrm{int}_{\mathcal{C}}(\mathcal{K}) \subset \mathcal{K} \subset \mathcal{F}$. Let $\delta > 0$ be such that $\delta < \varepsilon$ and $B^H(\delta, A) \cap C_n(X) \cap \mathcal{C} \subset \mathcal{K}$. Since $A_1 \subset C_1$ and $A \neq X$, there exists an arc α_1 in C_1 such that diameter$(\alpha_1) < \delta$ and $\alpha_1 \cap A_1$ is a one-point set. Then $\mathcal{L} = \{A \cup \{x\} : x \in \alpha_1\}$ is a connected subset of $\mathcal{M}$ containing A. Thus, $\mathcal{L} \subset \mathcal{C}$ and $\mathcal{L} \subset B^H(\delta, A) \cap C_n(X)$. Hence, $\mathcal{L} \subset \mathcal{K}$. Fix a point $p_1 \in \alpha_1 - A$. Then $A \cup \{p_1\} \in Im\varphi_1 - \mathcal{E}_0$. This proves that $\mathcal{K} \cap Im\varphi_1 - \mathcal{E}_0$ is nonempty. Similarly, $\mathcal{K} \cap Im\varphi_2 - \mathcal{E}_0$ is nonempty (here, we are using that $n \geq 3$). Since $\mathcal{K} \subset \mathcal{E} = Im\varphi_1 \cup \dots \cup Im\varphi_{n-1}$ and $Im\varphi_i \cap Im\varphi_j = \mathcal{E}_0$, if $i \neq j$, we have that $\mathcal{K} \cap Im\varphi_1 - \mathcal{E}_0$ and $\mathcal{K} \cap Im\varphi_2 - \mathcal{E}_0$ are separated by $\mathcal{K} \cap \mathcal{E}_0$. This contradicts [**12,** Theorem IV 4] since $\mathcal{K}$ is a $2n$-cell and $\dim[\mathcal{K} \cap \mathcal{E}_0] \leq 2n-2$. Therefore, $m = n$. This completes the proof of the theorem. □

Theorem 3.6. *Let X be a continuum, and let $n \geq 3$. Then*

$$\mathcal{W}_1(X) = \{A \in \mathcal{W}_n(X) - \mathcal{Z}_n(X) : A \text{ has a basis } \mathcal{B} \text{ of neighborhoods of } A \text{ in } C_n(X) \text{ such that for each } \mathcal{U} \in \mathcal{B}, \text{ if } \mathcal{C} \text{ is the component of } \mathcal{U} \text{ that contains } A, \text{ then } \mathcal{C} \cap \mathcal{Z}_n(X) \text{ is connected}\}.$$

Proof. By Theorem 3.5, $\mathcal{Z}_n(X) = \mathcal{W}_n(X) \cap (C_n(X) - C_{n-1}(X))$.

($\subset$). Let $A \in \mathcal{W}_1(X)$. If $A = X$, then X is either an arc or a simple closed curve. Thus, $\mathcal{W}_n(X) = C_n(X)$ and $\mathcal{Z}_n(X) = C_n(X) - C_{n-1}(X)$. Hence, $A \in \mathcal{W}_n(X) - \mathcal{Z}_n(X)$ and since $A = X$ is homeomorphic to $[0,1]$ or S^1, it is easy to show that, for each $\varepsilon > 0$, $B^H(\varepsilon, A) \cap C_n(X)$ is a connected neighborhood of A with the property that $B^H(\varepsilon, A) \cap C_n(X) \cap \mathcal{Z}_n(X)$ is connected. So, we may assume that $A \neq X$. Then A is connected and there exists an open set W of X such that the component C of W containing A is homeomorphic to one of the spaces $(0,1)$ or $[0,1)$. By Theorem 3.5 (b), $A \in \mathcal{W}_n(X) - \mathcal{Z}_n(X)$. Let $\varepsilon > 0$ be such that $N(\varepsilon, A) \subset W$. Let $\mathcal{B} = \{B^H(\delta, A) \cap C_n(X) : 0 < \delta < \varepsilon\}$. Then $\mathcal{B}$ is a basis of neighborhoods of A in $C_n(X)$. Let $\delta > 0$ be such that $\delta < \varepsilon$, and let $\mathcal{U} = B^H(\delta, A) \cap C_n(X)$. Let $\mathcal{D}$ be the component of $\mathcal{U}$ that contains A. Let $T = \bigcup\{E : E \in \mathcal{D}\}$. By Lemma 1, T is a connected subset of X. Since $A \subset T \subset N(\varepsilon, A)$, $T \subset C$. Hence, T is a connected subset of $C \cap N(\delta, A)$. Hence, T is homeomorphic to an interval in the real line.

Given $K, L \in \mathcal{D} \cap \mathcal{Z}_n(X)$, we have $K \cup L \subset T$. Let J be an arc such that $K \cup L \subset J \subset T$. Then $J \subset C \cap N(\delta, A)$. Since $H(A, K) < \delta$, $H(A, L) < \delta$ and $J \subset N(\delta, A)$, every element $Q \in C_n(X)$ such that $K \subset Q \subset J$ (or $L \subset Q \subset J$) has the property that $H(Q, A) < \delta$. In particular, $H(J, A) < \delta$. Let $\eta > 0$ be such that $\eta < \delta - H(J, A)$. Then $B^H(\eta, J) \cap C_n(X) \subset B^H(\delta, A) \cap C_n(X)$. Let $\varphi : [0,1] \to J$ be a homeomorphism. Then there exists $\xi > 0$ such that if $W \in C_n([0,1]) - C_{n-1}([0,1])$ and $\lambda([0,1] - W) < \xi$, then $\varphi(W) \in B^H(\eta, J) \cap C_n(X)$, where λ is the Lebesgue measure in $[0,1]$. It is easy to show that $\mathcal{R} = \{R \in C_n([0,1]) - C_{n-1}([0,1]) : \lambda([0,1] - R) \leq \xi\}$ is pathwise connected. Now, we can describe a path joining K and L in $\mathcal{D} \cap \mathcal{Z}_n(X)$. Recall that $\mathcal{Z}_n(X) = \mathcal{W}_n(X) \cap (C_n(X) - C_{n-1}(X))$. Since K and L have exactly n components, $\varphi^{-1}(K)$ and $\varphi^{-1}(L)$ have exactly n components. Then we can give paths $\alpha_1, \alpha_2 : [0,1] \to$

$C_n([0,1]) - C_{n-1}([0,1])$ such that $\alpha_1(0) = K$, $\alpha_2(0) = L$, for each $i \in \{1,2\}$, $\lambda[0,1] - \alpha_i(1) \leq \xi$; and if $0 \leq s \leq t \leq 1$, then $\alpha_i(s) \subset \alpha_i(t)$. Let $\alpha_3 : [0,1] \to C_n([0,1]) - C_{n-1}([0,1])$ be a path in $\mathcal{R}$ joining $\alpha_1(1)$ and $\alpha_2(1)$. Then, combining the paths $\varphi \circ \alpha_1$, $\varphi \circ \alpha_2$ and $\varphi \circ \alpha_3$, we obtain a path joining K and L in $\mathcal{D} \cap \mathcal{Z}_n(X)$.

($\supset$). Now, take $A \in \mathcal{W}_n(X) - \mathcal{Z}_n(X)$ such that there exists a basis $\mathcal{B}$ of neighborhoods of A in $C_n(X)$ such that, for each $\mathcal{U} \in \mathcal{B}$, if $\mathcal{C}$ is the component of $\mathcal{U}$ that contains A, then $\mathcal{C} \cap \mathcal{Z}_n(X)$ is connected. In order to prove that $A \in \mathcal{W}_1(X)$, we only need to show that A is connected. Suppose to the contrary that A has at least two components. Let $A = A_1 \cup \cdots \cup A_m$, where the sets $A_1, \ldots, A_m$ are the different components of A. Since $A \notin \mathcal{Z}_n(X)$, by Theorem 7 (b), $1 < m < n$. Since $A \in \mathcal{W}_n(X)$, for each $i \in \{1, \ldots, m\}$ there exists an open subset U_i of X such that $A_i \subset U_i$ and the component C_i of U_i containing A_i is homeomorphic to $(0,1)$ or $[0,1)$. Since A is not connected, $A \neq X$. So we may assume that $U_i \neq X$ and then U_i is homeomorphic either to $(0,1)$ or $[0,1)$. Let $\delta > 0$ be such that, for each $i \in \{1, \ldots, m\}$, $N(\delta, A_i) \subset U_i$ and the sets $cl_X(N(\delta, A_1)), \ldots, cl_X(N(\delta, A_m))$ are pairwise disjoint. Let $\mathcal{U} \in \mathcal{B}$ be such that $\mathcal{U} \subset B^H(\delta, A) \cap C_n(X)$, and let $\mathcal{C}$ be the component of $\mathcal{U}$ that contains A. Then $\mathcal{C} \cap \mathcal{Z}_n(X)$ is connected.

Let $\varepsilon > 0$ be such that $B^H(\varepsilon, A) \cap C_n(X) \subset \mathcal{U}$ and $\varepsilon < \delta$. Since $A_1 \subset C_1$, A_1 is an arc or a one-point set. Then there exists a path $\alpha : [0,1] \to C_n(X)$ such that $\alpha(0) = A_1$, $\alpha(t)$ has exactly $n - m + 1$ components, $\alpha(t) \subset C_1$ and $H(A_1, \alpha(t)) < \varepsilon$ for each $t > 0$. Thus, the function $\beta : [0,1] \to C_n(X)$ given by $\beta(t) = \alpha(t) \cup A_2 \cup \cdots \cup A_m$ is continuous and $H(A, \beta(t)) < \varepsilon$ for each $t \in [0,1]$. Hence, $\beta(1) \in \mathcal{C} - C_{n-1}(X)$. By Theorem 3.5 (b), $\beta(1) \in \mathcal{C} \cap \mathcal{Z}_n(X)$. Let $B = \beta(1)$. In a similar way, it is possible to construct an element $D \in \mathcal{C} \cap \mathcal{Z}_n(X)$ of the form $D = A_1 \cup D_1 \cup A_3 \cup \ldots \cup A_m$, where $D_1 \subset C_2$ and D_1 has exactly $n - m + 1$ components. Let $\mathcal{K} = \{E \in \mathcal{C} \cap \mathcal{Z}_n(X) : E \cap cl_X(N(\delta, A_i))$ is connected for each $i \in \{2, \ldots, m\}\}$ and $\mathcal{L} = \{E \in \mathcal{C} \cap \mathcal{Z}_n(X) : E \cap cl_X(N(\delta, A_1))$ has at most $n - m$ components$\}$.

It is easy to show that $\mathcal{C} \cap \mathcal{Z}_n(X) = \mathcal{K} \cup \mathcal{L}$, $\mathcal{K}$ and $\mathcal{L}$ are closed in $\mathcal{C} \cap \mathcal{Z}_n(X)$, $B \in \mathcal{K}$ and $D \in \mathcal{L}$. By the connectedness of $\mathcal{C} \cap \mathcal{Z}_n(X)$, there exists an element $E \in \mathcal{K} \cap \mathcal{L}$. Thus, E has most $n-1$ components. This contradicts the fact that the elements of $\mathcal{Z}_n(X)$ have exactly n components (Theorem 3.5 (b)). This contradictions proves that A is connected and completes the proof of the theorem. □

4. Peano continua. Let X be a continuum with metric d. A closed subset A of X is said to be a *Z-set* in X provided that, for each $\varepsilon > 0$, there is a map $f : X \to X - A$ such that $d(x, f(x)) < \varepsilon$ for each $x \in X$.

Theorem 4.1. *Let X be a Peano continuum that is not almost meshed and $n \in \mathbb{N}$. Then $C_n(X)$ is not rigid.*

Proof. Let P denote the set of points $x \in X$ such that no neighborhood of x in X is a finite graph, and let $C_n(X, P) = \{A \in C_n(X) : A \cap P \neq \emptyset\}$. Note that P is closed in X. By [**9,** Theorem 16], $C_n(X, P)$ is homeomorphic to the Hilbert cube. By hypothesis, there exist an open set $U \subset P$ and a point $p \in U$. Choose a nondegenerate subcontinuum T of X such that $p \in T \subset U$.

Let $\mathcal{A} = bd_{C_n(X)}(C_n(X, P))$ and $\mathcal{B} = \{\{p\}, T\}$. Notice that $\mathcal{A}$ and $\mathcal{B}$ are disjoint closed subsets of $C_n(X, P)$. By [**9,** Claim 1, Theorem 20] $\mathcal{A}$ is a Z-set of $C_n(X, P)$ and, since $\mathcal{B}$ is a finite subset of the Hilbert cube $C_n(X, P)$, it is a Z-set of $C_n(X, P)$ as well. Then $\mathcal{A} \cup \mathcal{B}$ is a Z-set of $C_n(X, P)$ [**17,** Exercise 9.4]. Consider the homeomorphism $h : \mathcal{A} \cup \mathcal{B} \to \mathcal{A} \cup \mathcal{B}$ that is the identity restricted to $\mathcal{A}$ and satisfies $h(\{p\}) = T$.

By Anderson's homogeneity theorem (see [**17,** 11.9.1]), there exists a homeomorphism $g : C_n(X, P) \to C_n(X, P)$ that extends h. Thus, we can extend g to a homeomorphism $f : C_n(X) \to C_n(X)$ by defining f as the identity on $C_n(X) - C_n(X, P)$. Since $f(\{p\}) = T$, $C_n(X)$ is not rigid. □

Lemma 4.2. *Let X be a Peano continuum with no tails. Then*

(a) *if $p \in \alpha - \{a, b\}$ for some free arc α with end points a and b, $\alpha \subset X$ and $h : C(X) \to C(X)$ is a homeomorphism, then $h(\{p\}) \in F_1(X)$,*
(b) *if X is almost meshed, then $C(X)$ is rigid.*

Proof. (a) If X is a simple closed curve, then $C(X)$ is a 2-cell such that its manifold boundary is $F_1(X)$. This implies that $h(F_1(X)) = F_1(X)$. Hence, we may assume that X is not a simple closed curve. Since X does not contain tails, we have that X is not an arc. Let

$A = h(\{p\})$. Let $\mathcal{U} = \langle \alpha - \{a, b\} \rangle \cap C(X)$. Then $\mathcal{U}$ is an open subset of $C(X)$ containing $\{p\}$. By **[17,** Example 5.1], there is a homeomorphism from $\mathcal{U}$ to the space $C = [0, 1) \times [0, 1)$ that takes $\{p\}$ to the point $(0, 0)$. Hence, A has a neighborhood $\mathcal{M}$ in $C(X)$ such that $\mathcal{M}$ is a 2-cell and A belongs to the manifold boundary of $\mathcal{M}$. In particular, $\dim_A[C(X)] = 2$.

By Lemma 3.1, $A \in \mathcal{W}_1(X)$. Thus, there exists an open subset U of X such that the component B of U containing A is homeomorphic either to $(0, 1)$ or $[0, 1)$. By local connectedness of X, B is open in X. If there exists a homeomorphism $f : [0, 1) \to B$, then $f([0, \frac{1}{2}])$ is an arc such that $f([0, \frac{1}{2})) = B - f([\frac{1}{2}, 1])$ is open in X. Thus $f([0, \frac{1}{2}])$ is a tail in X, contrary to the hypothesis. This proves that B is homeomorphic to $(0, 1)$. Since $A \subset B$, A is either an arc or a one-point set. If A is an arc, by **[17,** Example 5.1], A is in the manifold interior of a 2-cell $\mathcal{D} \subset C(X)$. We may assume that $\mathcal{D} \subset \mathcal{M}$. This contradicts the invariance of domain theorem and proves that A is not an arc. Hence, $A \in F_1(X)$ and (a) is proved.

(b) is immediate from (a) and the fact that in meshed continua the set of points p satisfying the conditions described in (a) is dense in X. □

Corollary 4.3. *Let X be an almost meshed Peano continuum. Then $C(X)$ is rigid if and only if X contains no tails.*

Proof. Let α, a and b be as in the definition of a tail. We may assume that $\alpha \neq X$. Then a is a cut point of X, so $E = X - (\alpha - \{a\})$ is a subcontinuum of X. Let Δ be the solid triangle in the Euclidean plane $\mathbb{R}^2$ with vertices $(0, 0)$, $(0, 1)$ and $(1, 1)$. Given two different points $p, q \in \mathbb{R}^2$, let pq denote the convex segment joining them. By **[17,** Example 5.1], there exists a homeomorphism $f : C(\alpha) \to \Delta$ such that $f(\{A \in C(\alpha) : a \in A\}) = (0, 0)(0, 1)$ and $f(\{\{x\} : x \in \alpha\}) = (0, 0)(1, 1)$. Let $k : \Delta \to \Delta$ be a homeomorphism such that $k|(0, 0)(0, 1)$ is the identity map and $k(\frac{1}{2}, \frac{1}{2}) = (\frac{1}{2}, 1)$. Let $h : C(X) \to C(X)$ be given by

$$h(A) = \begin{cases} A, & \text{if } A \cap E \neq \emptyset, \\ f^{-1}(k(f(A))), & \text{if } A \in C(\alpha). \end{cases}$$

Clearly, h is a homeomorphism and $h(F_1(X)) \nsubseteq F_1(X)$. □

Theorem 4.4. *If X is an almost meshed Peano continuum with no tails, then $C_n(X)$ is rigid for every $n \neq 2$.*

Proof. By Lemma 4.2 (b), we consider only the case $n \geq 3$. Let $h : C_n(X) \to C_n(X)$ be a homeomorphism. By Theorem 3.3 (a), $h(\mathcal{W}_n(X)) = \mathcal{W}_n(X)$. This implies that $h(\mathcal{Z}_n(X)) = \mathcal{Z}_n(X)$. By Theorem 3.6, $h(\mathcal{W}_1(X)) = \mathcal{W}_1(X)$. Proceeding as in the proof of Lemma 4.2 (a), it is possible to prove that if $p \in X$ is such that $p \in \alpha - \{a, b\}$ for some free arc $\alpha \subset X$ with endpoints a and b, then $h(\{p\}) \in F_1(X)$. Since X is almost meshed, we conclude that $h(F_1(X)) = F_1(X)$ and $C_n(X)$ is rigid. □

Question 4.5. *Is $C_3([0,1])$ rigid?*

Question 4.6. *Let X be a finite graph and $n \geq 3$. Is $C_n(X)$ rigid?*

Question 4.7. *Let X be an almost meshed Peano continua such that X contains a tail, and let $n \geq 3$. Can $C_n(X)$ be rigid?*

Lemma 4.8. *If $n \geq 3$ and $h : C_n([0,1]) \to C_n([0,1])$ is a homeomorphism, then $h(C_{n-1}([0,1])) = C_{n-1}([0,1])$.*

Proof. By definition, $\mathcal{W}_n([0,1]) = C_n([0,1])$. Then $h(\mathcal{Z}_n([0,1])) = \mathcal{Z}_n([0,1])$. By Theorem 3.5, $h(C_n([0,1]) - C_{n-1}([0,1])) = C_n([0,1]) - C_{n-1}([0,1])$. Hence, $h(C_{n-1}([0,1])) = C_{n-1}([0,1])$. □

As a consequence of Lemma 4.8, we have the following.

Theorem 4.9. *$C_3([0,1])$ is rigid if and only if $C_n([0,1])$ is rigid for each $n \geq 3$.*

Theorem 4.10. *If X contains a free arc, then $C_2(X)$ is not rigid.*

Proof. Let α be a free arc in X with end points a and b. Fix a point $p \in \alpha - \{a, b\}$ and let $\varepsilon > 0$ be such that $B(\varepsilon, p) \subset \alpha - \{a, b\}$. Let $\mathcal{M} = C_2(\alpha)$. By [**14,** Lemma 2.2], $\mathcal{M}$ is a 4-cell. Let $\mathcal{U} = B^H(\varepsilon, \{p\}) \cap C_2(X)$. Then $\mathcal{U}$ is an open subset of $C_2(X)$ such that $\mathcal{U} \subset \mathcal{M}$. From the construction in [**14,** Lemma 2.2], it can be seen that $F_1(\alpha)$ is an arc in the manifold boundary $\partial\mathcal{M}$ of $\mathcal{M}$. Since $int_{\partial\mathcal{M}}(F_1(\alpha)) = \emptyset$, there exists a homeomorphism $g : \mathcal{M} \to \mathcal{M}$ such that $g(\{p\}) \notin F_1(\alpha)$, and g is the identity in $\mathcal{M} - \mathcal{U}$. Let $h : C_2(X) \to C_2(X)$ be given by

$$h(A) = \begin{cases} g(A), & \text{if } A \in \mathcal{M}, \\ A & \text{if } A \in C_2(X) - \mathcal{U}. \end{cases}$$

Clearly, g is a homeomorphism and $h(\{p\}) \notin F_1(X)$. Therefore, $C_2(X)$ is not rigid. □

Next, we summarize the results of this section.

Remark 4.11. Let X be a Peano continuum. Then

(a) if X is not almost meshed, then $C_n(X)$ is not rigid for all $n \in \mathbb{N}$,
(b) $C_2(X)$ is not rigid,
(c) if X is almost meshed and contains no tails, then $C_n(X)$ is rigid for each $n \neq 2$,
(d) if X is almost meshed, then X contains no tails if and only if $C(X)$ is rigid,
(e) $C_3([0,1])$ is rigid if and only if $C_n([0,1])$ is rigid for each $n \geq 3$,
(f) it is not known if $C_3([0,1])$ is rigid.

5. Smooth fans. Given a dendroid X and points $p, q \in X$, we denote the unique arc joining p and q in X by pq if $p \neq q$ and $pq = \{p\}$ if $p = q$. A *fan* is a dendroid X with exactly one ramification point v, called the *vertex* of X. The fan X, with vertex v, is said to be *smooth*, provided that for each sequence $\{x_i\}_{i=1}^{\infty}$ converging to a point $x \in X$, we have $\lim vx_i = vx$. It is known that the class of smooth fans coincides with the class of subcontinua of the Cantor fan that are not arcs [**6,** Corollary 4]. Given a smooth fan X, let E be the set of end points of X. The *Lelek fan* (see [**5**]) is the unique smooth fan such that E is dense in X. Eberhart and Nadler, in [**7,** Corollary 3.3], proved that a smooth fan X has unique hyperspace $C(X)$ in the class of smooth fans. That is, if X and Y are smooth fans and $C(X)$ is homeomorphic to $C(Y)$, then X is homeomorphic to Y. This result was extended by Acosta who proved that a smooth fan X has unique hyperspace $C(X)$ in the class of fans [**1,** Theorem 5.4]. The second-named author showed Acosta's result cannot be generalized to fans by constructing a fan X such that X does not have unique hyperspace $C(X)$ in the class of fans [**13**]. As a consequence of Corollary 4.3, we have that if Y is a simple n-od, then the hyperspace $C(X)$ is not rigid (although it has unique hyperspace $C_n(X)$ for each $n \in \mathbb{N}$).

This subsection is devoted to characterizing smooth fans X for which $C(X)$ is rigid.

The following discussion is partially contained in [**7,** Sections 2, 3].

Throughout this section, X denotes a smooth fan with vertex v, set of end points E and $E_0 = cl(E) - E$.

Notice that E_0 can be the empty set, as in the case when X is homeomorphic to the cone over the Cantor set, and it is also possible that E_0 is dense X, as in the case of the Lelek fan. We assume that X is contained in the cone over the Cantor set, where the Cantor set is the usual middle third set constructed in $[0, 1] \times \{0\}$ in the plane and its vertex is the point $v = (0, 1)$. Consider the projection on the second coordinate $\pi : X \to [0, 1]$. Then $\pi(v) = 1$. Let

$$X_0 = \{v\} \cup cl_X\Big(\bigcup\{vx : x \in E_0\}\Big).$$

Notice that X_0 is a subcontinuum of X.

Define $\psi : E \to [0, 1]$ as $\psi(e) = \min \pi(X_0 \cap ve)$. Notice that ψ is not necessarily continuous.

Let $\mathcal{V} = \{A \in C(X) : v \in A\}$. For each $e \in E$, let $\mathcal{L}(e) = \{A \in C(X) : A \subset ve - \{v\}\}$. Notice that

$$C(X) = \mathcal{V} \cup \Big(\bigcup\{\mathcal{L}(e) : e \in E\}\Big).$$

Lemma 5.1. *Let $h : C(X) \to C(X)$ be a homeomorphism. Then there exists a bijection $h_0 : E \to E$ (not necessarily continuous) such that*

(a) *$h(\mathcal{V}) = \mathcal{V}$,*
(b) *for each $e \in E$, $h(\mathcal{L}(e)) = \mathcal{L}(h_0(e))$,*
(c) *$h(\{v\}) = \{v\}$,*
(d) *for each $e \in E$, $h(F_1(ve)) \subset F_1(vh_0(e)) \cup \{xh_0(e) : x \in vh_0(e)\}$ and $h(ve) = vh_0(e)$.*

Proof. (a) If E is finite with m elements, then $\mathcal{V} = \{A \in C(X) : \dim_A C(X) = m\}$ and, if E is infinite, $\mathcal{V} = \{A \in C(X) : \dim_A C(X) = \infty\}$. This implies that $h(\mathcal{V}) = \mathcal{V}$.

(b) The existence of h_0 and (b) follow from the fact that the components of $C(X) - \mathcal{V}$ are the elements of the set $\{\mathcal{L}(e) : e \in E\}$.

(c) Notice that $\mathcal{V} \cap (\bigcap\{cl_{C(X)}(\mathcal{L}(e)) : e \in E\}) = \{\{v\}\}$. This implies that $h(\{v\}) = \{v\}$.

(d) By **[17,** Example 5.1], for each $e \in E$, $\mathcal{L}(e)$ is homeomorphic to $(0,1) \times [0,1)$, and its manifold boundary is $\partial\mathcal{L}(e) = F_1(ve - \{v\}) \cup \{xe : x \in ve - \{v\}\}$. Moreover, $cl_{C(X)}(\partial\mathcal{L}(e))$ is an arc with end points $\{v\}$ and ve and $cl_{C(X)}(\partial\mathcal{L}(e)) - \{\{v\}, ve\} = \partial\mathcal{L}(e)$. Thus, we conclude that $h(F_1(ve)) \subset F_1(vh_0(e)) \cup \{xh_0(e) : x \in vh_0(e)\}$, and since $h(\{v\}) = \{v\}$, $h(ve) = vh_0(e)$. $\square$

Lemma 20 limits the options for the image $h(\{x\})$, for $x \in ve$, when $h : C(X) \to C(X)$ is a homeomorphism. Since we are trying to determine when $C(X)$ is rigid, we are interested in determining what conditions X must satisfy in order that singletons are mapped to singletons. The following results will help us to do this.

Lemma 5.2. *Let $h : C(X) \to C(X)$ be a homeomorphism. If $x \in E_0$, then*:

(a) $h(\{x\}) \in F_1(X)$,
(b) $h(F_1(vx)) \subset F_1(X)$,
(c) $h(F_1(X_0)) \subset F_1(X)$.

Proof. Let $e \in E$ be such that $x \in ve$. Suppose contrary to (a) that $h(\{x\}) \notin F_1(X)$. By Lemma 5.1 (d), $h(\{x\}) = zh_0(e)$, where h_0 is as in Lemma 5.1 and $z \in vh_0(e) - \{h_0(e)\}$. Let $\{e_m\}_{m=1}^{\infty}$ be a sequence in E such that $\lim e_m = x$. We may assume that $\lim h_0(e_m) = u$ for some $u \in X$. By the smoothness of X, $\lim vh_0(e_m) = vu$. By continuity of h, $\lim h(\{e_m\}) = h(\{x\}) = zh_0(e)$. By Lemma 5.1 (b), for each $m \in \mathbb{N}$, $h(\{e_m\}) \subset vh_0(e_m)$, so $zh_0(e) \subset vu$. Since $h_0(e) \in E$, we conclude that $u = h_0(e)$. Using smoothness of X again, we have $\lim ve_m = vx$, $\lim h(ve_m) = h(vx)$ and (Lemma 5.1 (d)) $\lim h(ve_m) = \lim vh_0(e_m) = vu = vh_0(e) = h(ve)$. Thus, $vx = ve$. Hence, $x = e \in E$, contrary to the hypothesis. This completes the proof of (a).

(b) By Lemma 5.1 (d), $h(F_1(vx))$ is an arc contained in the set $\mathcal{A} = F_1(vh_0(e)) \cup \{xh_0(e) : x \in vh_0(e)\}$. Notice that $\mathcal{A}$ is an arc with end points $\{v\}$ and $vh_0(e)$. Since $\{v\}$ and $\{x\}$ are the end points of the arc $F_1(vx)$, we have $h(\{v\})$ and $h(\{x\})$ are the end points of the arc $h(F_1(vx))$. By (a) and Lemma 5.1 (c), $h(\{v\})$ and $h(\{x\})$ are in the subarc $F_1(vh_0(e))$ of $\mathcal{A}$. Hence, $h(F_1(vx)) \subset F_1(vh_0(e)) \subset F_1(X)$.

(c) Is immediate from (b). □

Lemma 5.3. *Let $e_0 \in E$ be such that E is locally compact at e_0. Then there exist an open subset W of X and $t_1 \in (0,1)$ such that*

(a) $e_0 \in W$,
(b) $W \cap E$ *is compact,*
(c) $W \cap X_0 = \emptyset$, *and*
(d) *if $e \in W \cap E$, then $\pi(e) < t_1 < \psi(e)$.*

Proof. Let d be the Euclidean metric for $\mathbb{R}^2$. Let U be an open set of X such that $e_0 \in U$ and $cl_X(U) \cap E$ is compact. We claim that $e_0 \notin X_0$. Suppose to the contrary that there exist a sequence $\{x_m\}_{m=1}^{\infty}$ in X and a sequence $\{z_m\}_{m=1}^{\infty}$ in E_0 such that $\lim x_m = e_0$ and $x_m \in vz_m$ for each $m \in \mathbb{N}$. By the compactness of X, we may suppose that $\lim z_m = z$ for some $z \in X$. Given $m \in \mathbb{N}$, there exists $e_m \in E$ such that $d(e_m, z_m) < \frac{1}{m}$. Then $\lim e_m = z$. By the smoothness of x, $\lim ve_m = \lim vz_m = vz$. Since $x_m \in vz_m$ for each $m \in \mathbb{N}$, we have $e_0 \in vz$. Since $e_0 \in E$, we obtain that $e_0 = z$. Thus, there exists $m \in \mathbb{N}$ such that $z_m \in U$. Since $z_m \in cl_X(E) - E$, $z_m = \lim w_r$ for some sequence $\{w_r\}_{r=1}^{\infty}$ in $E \cap U$. Hence, $\{w_r\}_{r=1}^{\infty}$ is a sequence in $cl_X(U) \cap E$ whose limit is not in $cl_X(U) \cap E$. This shows that $cl_X(U) \cap E$ is not compact, a contradiction. Hence, $e_0 \notin X_0$. Therefore, there exists an open set V of X such that $e_0 \subset V$ and $cl_X(V) \subset U - X_0$. Thus, $cl_X(V) \cap E$ is compact.

Since $v \notin E$, it follows that E is totally disconnected. Thus, $cl_X(V) \cap E$ is compact and totally disconnected and then $cl_X(V) \cap E$ is 0-dimensional. By the way we are considering the cone over the Cantor set in $\mathbb{R}^2$, we can assume that there exists an open set $Y \subset [0,1]$ and numbers $s_0, s_1 \in (0,1)$ such that $V = X \cap (\{(0,1)(y,0) : y \in Y\}) \cap ([0,1] \times (s_0, s_1))$, where $(0,1)(y,0)$ is the convex segment joining the points $(0,1)$ and $(y,0)$. Since $e_0 \in V$, $s_0 < \pi(e_0) < s_1$. Given $e \in V \cap E$, there exists $y \in Y$ such that $e \in (0,1)(y,0)$. Let p be the point in $(0,1)(y,0)$ such that $\pi(p) = s_1$. Then $(ep - \{p\}) \subset V$. This implies that $(ep - \{p\}) \cap X_0 = \emptyset$. Notice that if $w \in X_0$, then $vw \subset X_0$. Hence, $\psi(e) \geq s_1$.

Since $cl_X(V) \cap E$ is 0-dimensional, there exists an open and closed subset R of $cl_X(V) \cap E$ such that $e_0 \in R$ and $R \subset V \cap E$. Let W be an open subset of X such that $R = W \cap cl_X(V) \cap E$ and $W \subset V$. Notice that $e_0 \in W$, $W \cap E = R$ is compact, $W \cap X_0 = \emptyset$. Let $t_1 \in (\max\{\pi(e) : e \in W \cap E\}, s_1)$. Clearly, t_1 satisfies (d). □

Theorem 5.4. *The following are equivalent*:

(a) $C(X)$ *is rigid,*
(b) $X = X_0$,
(c) E *has no points of local compactness.*

Proof. (b) ⇒ (a) is immediate from Lemma 5.2 (c) applied to h and h^{-1}.

(c) ⇒ (b). Suppose that $X \neq X_0$. Take $x \in X - X_0$, and let $e \in E$ be such that $x \in ve$. If $e \in X_0$, since $v \in X_0$ and X_0 is a subcontinuum of X, then $x \in ev \subset X_0$, a contradiction. Thus, $e \notin X_0$. Let U be an open subset of X such that $e \in U$ and $cl_X(U) \cap X_0 = \emptyset$. We now prove that $cl_X(U) \cap E$ is a compact neighborhood of e in E. Since X is compact, it is enough to prove that, for any sequence $\{e_m\}_{m=1}^{\infty}$ in $cl_X(U) \cap E$ that converges to a point $z \in cl_X(U)$, it follows that $z \in E$. Since $E_0 \subset X_0$, $z \notin E_0$. Since $z \in cl_X(E)$ and $z \notin E_0$, we have that $z \in E$. Thus, E is locally compact at e.

(a) ⇒ (c). Suppose that E has a point e_0 of local compactness. Let W be an open subset of X and $t_1 \in (0, 1)$ satisfying properties (a)–(d) of Lemma 5.3. For each $e \in E \cap W$, let $x_e \in ve$ be such that $\pi(x_e) = t_1$. Notice that the map $e \to x_e$ is continuous. Let $Z = \bigcup\{ex_e \subset X : e \in W \cap E\}$. Let $g : (W \cap E) \times [0, 1] \to Z$ be given by $g(e, s) = s \cdot e + (1 - s) \cdot x_e$. Clearly, g is a continuous one-to-one onto map. Since $W \cap E$ is compact, g is a homeomorphism. In particular, we have that Z and $Z_0 = \{x_e \in X : e \in W \cap E\}$ are compact subsets of X.

We claim that Z_0 is the boundary of Z (in X). Since, for each $e \in W \cap E$, $\pi(e) < t_1 < 1$, $Z_0 \subset \mathrm{bd}_X(Z)$. Now, let $x \in Z \cap cl_X(X - Z)$, and let $\{x_m\}_{m=1}^{\infty}$ be a sequence of $X - Z$ such that $\lim x_m = x$. For each $m \in \mathbb{N}$, let $e_m \in E$ be such that $x_m \in ve_m$, and let $e \in W \cap E$ be such that $x \in ex_e$. We may suppose that $\lim e_m = x_0$ for some $x_0 \in X$. Since $x \in Z$, $\pi(x) \leq t_1$, so $x \neq v$. By the smoothness of X, $\lim ve_m = vx_0$, so $x \in vx_0 \cap (ve - \{v\})$. This implies that $vx_0 \subset ve$. If $x_0 \neq e$, then $x_0 \in E_0$ and $x_0 \in X_0$. By Lemma 5.3 (d), $t_1 < \psi(e) = \min \pi(X_0 \cap ve)$. Thus, $t_1 < \pi(x_0)$. Since $x \in vx_0$, $\pi(x_0) \leq \pi(x)$. This contradicts the fact that $x \in Z$ and proves that $x_0 = e$. Hence, $x_0 \in W$. Since W is open, there exists $M \in \mathbb{N}$ such that, for each $m \geq M$, $e_m \in W$. Given $m \geq M$, since $e_m \in W \cap E$ and $x_m \in (X - Z) \cap ve_m$, we have that $\pi(x_m) > t_1$. Hence, $\pi(x) \geq t_1$. Since $x \in Z$, we conclude that $\pi(x) = t_1$. Thus, $x = x_e \in Z_0$. This completes the proof that $Z_0 = \mathrm{bd}_X(Z)$.

Define
$$\mathcal{L} = \bigcup \{C(ex_e) \subset C(X) : e \in W \cap E\}.$$

Since $\mathcal{L} = \{A \in C(X) : A \subset Z\}$, $\mathcal{L}$ is closed in $C(X)$.

Using the geometric model for $C([0,1])$ **[17,** Example 5.1], it is possible to define a homeomorphism $f : C([0,1]) \to C([0,1])$ such that: $f(A) = A$ for each $A \in C([0,1])$, $0 \in A$ and $f(\{1\}) = [\frac{1}{2}, 1]$.

Let $h_C : \mathcal{L} \to \mathcal{L}$ be given by $h_C(A) = (g \circ (\mathrm{id}_{W \cap E} \times f) \circ g^{-1})(A)$.

It is easy to show that h_C is a homeomorphism with the following properties:

(1) if $A \in \mathcal{L}$ and $x_e \in A$ for some $e \in E \cap W$, then $h_C(A) = A$,
(2) if $e \in E \cap W$, then $h_C(\{e\}) \notin F_1(X)$.

We extend h_C to a homeomorphism $h : C(X) \to C(X)$ given by

$$h(A) = \begin{cases} h_C(A), & \text{if } A \in \mathcal{L}, \\ A, & \text{if } A \cap \mathrm{cl}_X(X - Z) \neq \emptyset. \end{cases}$$

By (1) and the equality $Z_0 = \mathrm{bd}_X(Z)$, h is a well-defined continuous function. It is easy to check that h is one-to-one and surjective. By (2), we conclude that $C(X)$ is not rigid. $\square$

Theorem 5.5. *If $C(X)$ is rigid, then $C_n(X)$ is rigid for each $n \neq 2$.*

Proof. Let $n \geq 3$, and let $h : C_n(X) \to C_n(X)$ be a homeomorphism. By Theorem 3.3, $h(\mathcal{W}_n(X)) = \mathcal{W}_n(X)$. Then $h(\mathcal{Z}_n(X)) = \mathcal{Z}_n(X)$, and by Theorem 3.6, $h(\mathcal{W}_1(X)) = \mathcal{W}_1(X)$. Notice that $\mathcal{W}_1(X) = \{A \in$

$C(X) : v \notin A\}$. Let v, $\mathcal{V}$, E and for each $e \in E$, $\mathcal{L}(e)$ be defined as at the beginning of this section. By [**7,** Theorem 3.1 (3)] $\mathcal{V}$ is homeomorphic to the Hilbert cube. Let $\mathcal{A} = cl_{C_n(X)}(\mathcal{W}_1(X))$. Then $h(\mathcal{A}) = \mathcal{A}$ and $h(\mathcal{A} - \mathcal{W}_1(X)) = \mathcal{A} - \mathcal{W}_1(X)$. Clearly, $\mathcal{A} = \{A \in C(X)$: there exists $e \in E$ such that $A \subset \mathcal{L}(e)\}$ and $\mathcal{A} - \mathcal{W}_1(X) = \mathcal{V} \cap \mathcal{A}$. Hence, $h(\mathcal{V} \cap \mathcal{A}) = \mathcal{V} \cap \mathcal{A}$. By [**7,** Theorem 3.1 (2)], $\mathcal{V} \cap \mathcal{A}$ is a Z-set of $\mathcal{V}$. By [**17,** 11.9.1], the homeomorphism $h|\mathcal{V} \cap \mathcal{A} : \mathcal{V} \cap \mathcal{A} \to \mathcal{V} \cap \mathcal{A}$ can be extended to a homeomorphism $g : \mathcal{V} \to \mathcal{V}$. Let $f : \mathcal{V} \cup \mathcal{A} \to \mathcal{V} \cup \mathcal{A}$ be given by

$$f(A) = \begin{cases} g(A), & \text{if } A \in \mathcal{V}, \\ h(A), & \text{if } A \in \mathcal{A}. \end{cases}$$

Then f is a homeomorphism. Notice that $\mathcal{V} \cup \mathcal{A} = C(X)$. Since $C(X)$ is rigid, $f(F_1(X)) = F_1(X)$. Given $p \in X - \{v\}$, $\{p\} \in \mathcal{W}_1(X)$. Thus, $h(\{p\}) = f(\{p\}) \in F_1(X)$. Since $X - \{v\}$ is dense in X, we conclude that $h(F_1(X)) \subset F_1(X)$. Similarly, $h^{-1}(F_1(X)) \subset F_1(X)$. Hence, $h(F_1(X)) = F_1(X)$. Therefore, $C_n(X)$ is rigid. □

Corollary 5.6. *The following are equivalent*:

(a) $C(X)$ *is rigid,*
(b) $X = X_0$,
(c) E *has no points of local compactness,*
(d) $C_n(X)$ *is rigid for every* $n \neq 2$.

Related to Question 4.5 and Corollary 5.6, we can pose the following problem.

Problem 5.7. Letting Z be a continuum, is it true that if $C(Z)$ is rigid, then $C_3(Z)$ is rigid?

Example 5.8. Another smooth fan Y for which $C(Y)$ is rigid.

Recall that the Lelek fan is characterized as the unique smooth fan X such that the set of end points of X is dense in X. Given $p \in X - (E(X) \cup \{v\})$, $p \in E_0 \subset X_0$. Thus, $X = cl_X(X - (E(X) \cup \{v\})) \subset X_0$. By Theorem 23, $C(X)$ is rigid. Another smooth fan Y with this property can be constructed as follows. Let Z be the cone over the Cantor set C. Suppose $X \subset Z$ and v is the vertex of X and Z. Let

$g : C \times [0,1] \to Z$ be the natural quotient map such that $g(c,1) = v$ for each $c \in C$. Let $Q = C \times [0,2]$. Let $P = (C \times [1,2]) \cup g^{-1}(X)$, and let Y be the quotient space that results from P by identifying the set $C \times \{2\}$ to a point. It is easy to check that Y is a smooth fan, Y is not homeomorphic to the Lelek fan and $C(Y)$ is rigid. $\square$

Question 5.9. *Supposing that $C_2(X)$ is rigid, does it follow that $C(X)$ is rigid?*

6. Hereditarily indecomposable continua.

Theorem 6.1. *If X is hereditarily indecomposable, then 2^X and $C_n(X)$ are rigid for each $n \in \mathbb{N}$.*

Proof. This theorem was proved in Claim 4 of Theorem 4.4 of **[15]**. $\square$

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