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POSITIVE, NEGATIVE AND MIXED-TYPE SOLUTIONS FOR PERIODIC VECTOR DIFFERENTIAL EQUATIONS

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ABSTRACT. This paper is devoted to the study of periodic solutions of the first-order vector differential equations $x'(t) + f(t, x(t)) = 0$. We first introduce the concepts of positive, negative and mixed-type solutions. Then, by using a fixed point theorem in cones, we obtain some existence and multiplicity results of such solutions. Furthermore, we also present some examples to illustrate our main results.

1. Introduction. Consider the periodic boundary value problem (PBVP)

$$x'(t) + f(t, x(t)) = 0, \quad \text{almost everywhere } t \in [0, T], \tag{1.1}$$

$$x(0) = x(T), \tag{1.2}$$

where $T > 0$ and $f : [0, T] \times \mathbf{R}^n \to \mathbf{R}^n$ is a Carathéodory function, namely, $f(t, x)$ satisfies the following Carathéodory conditions:

(C1) for all $x \in \mathbf{R}^n$, $f(\cdot, x)$ is Lebesgue measurable;
(C2) for almost every $t \in [0, T]$, $f(t, \cdot)$ is continuous.

By a solution of (1.1) we mean an absolutely continuous function $x(t) = (x^1(t), x^2(t), \ldots, x^n(t))^\top$ satisfying (1.1) almost everywhere. Furthermore, we say that a nonzero solution $x(t) = (x^1(t), x^2(t), \ldots, x^n(t))^\top$ of (1.1) is a positive solution if $x^i(t) \geq 0$ for

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$i = 1, 2, \ldots, n$; a negative solution if $x^i(t) \le 0$ for $i = 1, 2, \ldots, n$; and a mixed-type solution if there exist I^+ and I^- satisfying $I^+ \cup I^- = \{1, 2, \ldots, n\}$ such that $x^i(t) \ge 0$ for $i \in I^+$, $x^i(t) \le 0$ for $i \in I^-$.

In recent years, the PBVPs for differential equations have been extensively studied: refer to **[8, 17, 18, 19, 20, 21]** for the method of upper and lower solutions coupled with monotone iterative techniques; to **[1, 3, 7, 22]** for the method of the fixed point theorem; to **[2, 4, 16]** for the method of calculus of variations and optimization; to **[2, 5, 10, 11, 12, 13, 14, 15]** for related results on partial differential equations. Different from the above works, in this paper, we consider the first-order vector differential equations (1.1) with the periodic boundary value conditions (1.2). We first introduce the concepts of positive, negative and mixed-type solutions. Then, by using a fixed point theorem in cones, we obtain some existence and multiplicity results of such solutions. The result can be viewed as an extension of our previous work **[9]**, where the existence and multiplicity results on positive solutions are obtained.

The outline of this paper is as follows. In Section 2, we introduce some notation and preliminary results. Section 3 presents some existence results of positive solutions, negative solutions and mixed-type solutions. The multiplicity results are given in Section 4. Finally, we give some concluding remarks in Section 5.

2. Notation and preliminary results. Throughout this paper, x^i denotes the ith component of x, $|x| = \max\{|x^i|, i = 1, 2, \ldots, n\}$ denotes the norm of x, C_T denotes the Banach space of all continuous functions $x : [0, T] \to \mathbf{R}^n$ endowed with the norm $\|x\| = \sup\{|x(t)|, t \in [0, T]\}$, $L^1[0, T]$ denotes the set of all integrable functions from $[0, T]$ to $\mathbf{R}$, $C(\mathbf{R}^n, \mathbf{R})$ denotes the set of all continuous functions from $\mathbf{R}^n$ to $\mathbf{R}$.

The following hypotheses are assumed in this paper:

(H1) there exist $\varphi \in L^1[0, T]$ and $\psi \in C(\mathbf{R}^n, \mathbf{R})$ such that

$$|f(t, x)| \le \varphi(t)\psi(x), \quad \text{for almost every } t \in [0, T],\ x \in \mathbf{R}^n;$$

(H2) there exists $M(t) = \operatorname{diag}(m_i(t))$ such that

$$Ef(t, x) \le EM(t)x, \quad \text{for almost every } t \in [0, T],\ Ex \ge 0,$$

where $m_i \in L^1[0,T]$ satisfies $m_i(t) \geq 0$ for almost every $t \in [0,T]$ and $\int_0^T m_i(\tau)\,\mathrm{d}\tau \neq 0$, $E = \mathrm{diag}\,(\sigma_i)$, $\sigma_i = \{-1,1\}$, $i = 1,2,\ldots,n$.

We consider the first order linear PBVP

$$x'(t) + M(t)x(t) = h(t), \quad \text{a.e. } t \in [0,T], \tag{2.1}$$

$$x(0) = x(T), \tag{2.2}$$

where $h \in L^1[0,T]$ and $M(t)$ is given in (H2).

Let

$$U(t,s) = \mathrm{diag}\,(u_i(t,s)), \quad (t,s) \in [0,T]\times[0,T], \tag{2.3}$$

where $u_i(t,s) = (\mathrm{e}^{-\int_s^t m_i(\tau)\,\mathrm{d}\tau})/(1-\mathrm{e}^{-\int_0^T m_i(\tau)\,\mathrm{d}\tau})$, $i = 1,2,\ldots,n$. It is obvious that $u_i(t,s)$ is an absolutely continuous nonnegative function on $[0,T]\times[0,T]$ and satisfies

$$\begin{cases} \frac{1}{1-\mathrm{e}^{-\int_0^T m_i(\tau)\,\mathrm{d}\tau}} \leq u_i(t,s) \leq \frac{\mathrm{e}^{\int_0^T m_i(\tau)\,\mathrm{d}\tau}}{1-\mathrm{e}^{-\int_0^T m_i(\tau)\,\mathrm{d}\tau}}, & t<s; \\ u_i(t,s) = \frac{1}{1-\mathrm{e}^{-\int_0^T m_i(\tau)\,\mathrm{d}\tau}}, & t=s; \\ \frac{\mathrm{e}^{-\int_0^T m_i(\tau)\,\mathrm{d}\tau}}{1-\mathrm{e}^{-\int_0^T m_i(\tau)\,\mathrm{d}\tau}} \leq u_i(t,s) \leq \frac{1}{1-\mathrm{e}^{-\int_0^T m_i(\tau)\,\mathrm{d}\tau}}, & t>s. \end{cases} \tag{2.4}$$

We can easily claim that the following lemma holds.

Lemma 2.1. *Assume that $m_i \in L^1[0,T]$ satisfies $m_i(t) \geq 0$ for almost every $t \in [0,T]$ and $\int_0^T m_i(\tau)\,\mathrm{d}\tau \neq 0$, $i = 1,2,\ldots,n$. Let $M(t) = \mathrm{diag}\,(m_i(t))$. Then, for any $h \in L^1[0,T]$,* PBVP (2.1), (2.2) *has a unique solution*

$$x(t) = \int_0^t U(t,s)h(s)\,\mathrm{d}s + \Delta\int_t^T U(t,s)h(s)\,\mathrm{d}s,$$

where $\Delta = \mathrm{diag}\,(\delta_i), \delta_i = \mathrm{e}^{-\int_0^T m_i(\tau)\,\mathrm{d}\tau}$.

For the study of PBVP (1.1), (1.2) by utilizing the fixed point theorem in cones, we consider the following auxiliary PBVP

$$x'(t) + M(t)x(t) = M(t)z(t) - f(t,z(t)), \quad \text{a.e. } t \in [0,T], \tag{2.5}$$

$$x(0) = x(T), \tag{2.6}$$

where $z \in C_T$.

By Lemma 2.1, it is known that, for any $z \in C_T$, PBVP (2.5), (2.6) has a unique solution

$$x_z(t) = \int_0^t U(t,s)h_z(s)\,\mathrm{d}s + \Delta \int_t^T U(t,s)h_z(s)\,\mathrm{d}s,$$

where $h_z(s) = M(s)z(s) - f(s, z(s))$.

Let the operator $T : C_T \to C_T$ be defined by

$$T(z)(t) = \int_0^t U(t,s)h_z(s)\,\mathrm{d}s + \Delta \int_t^T U(t,s)h_z(s)\,\mathrm{d}s, \quad t \in [0,T].$$

Clearly, if T has a fixed point $z \in C_T$, then $z(t)$ is a solution of PBVP (1.1), (1.2).

Let $I^+ = \{i : \sigma_i = 1\}$, $I^- = \{i : \sigma_i = -1\}$. Then $I^+ \cup I^- = \{1, 2, \cdots, n\}$. Define a cone in C_T by

$$K_\Delta = \{x \in C_T : x^i(t) \geq \delta_i \|x^i\|, \quad i \in I^+; \\ x^i(t) \leq -\delta_i \|x^i\|, \quad i \in I^-;\ t \in [0,T]\},$$

where $\|x^i\| = \sup\{|x^i(t)|, t \in [0,T]\}$, $i = 1, 2, \ldots, n$.

Lemma 2.2. $T(K_\Delta) \subset K_\Delta$.

Proof. For any $t \in [0,T]$ and $z \in K_\Delta$, let $h_z^i(t) = m_i(t)z^i(t) - f^i(t, z(t))$. Then, when $i \in I^+$, by (H2) and (2.4), we have

$$\begin{aligned}
(T(z))^i(t) &= \int_0^t u_i(t,s)h_z^i(s)\,\mathrm{d}s + \delta_i \int_t^T u_i(t,s)h_z^i(s)\,\mathrm{d}s \\
&\leq \frac{1}{1 - \mathrm{e}^{-\int_0^T m_i(\tau)\,\mathrm{d}\tau}} \int_0^t h_z^i(s)\,\mathrm{d}s \\
&\quad + \delta_i \frac{\mathrm{e}^{\int_0^T m_i(\tau)\,\mathrm{d}\tau}}{1 - \mathrm{e}^{-\int_0^T m_i(\tau)\,\mathrm{d}\tau}} \int_t^T h_z^i(s)\,\mathrm{d}s \\
&= \frac{1}{1 - \mathrm{e}^{-\int_0^T m_i(\tau)\,\mathrm{d}\tau}} \int_0^T h_z^i(s)\,\mathrm{d}s,
\end{aligned}$$

which implies

$$\|(T(z))^i\| \leq \frac{1}{1 - \mathrm{e}^{-\int_0^T m_i(\tau)\,\mathrm{d}\tau}} \int_0^T h_z^i(s)\,\mathrm{d}s.$$

Thus,

$$
\begin{aligned}
(T(z))^i(t) &= \int_0^t u_i(t,s)h_z^i(s)\,\mathrm{d}s + \delta_i \int_t^T u_i(t,s)h_z^i(s)\,\mathrm{d}s \\
&\geq \frac{\mathrm{e}^{-\int_0^T m_i(\tau)\,\mathrm{d}\tau}}{1-\mathrm{e}^{-\int_0^T m_i(\tau)\,\mathrm{d}\tau}} \int_0^t h_z^i(s)\,\mathrm{d}s \\
&\quad + \delta_i \frac{1}{1-\mathrm{e}^{-\int_0^T m_i(\tau)\,\mathrm{d}\tau}} \int_t^T h_z^i(s)\,\mathrm{d}s \\
&= \frac{\mathrm{e}^{-\int_0^T m_i(\tau)\,\mathrm{d}\tau}}{1-\mathrm{e}^{-\int_0^T m_i(\tau)\,\mathrm{d}\tau}} \int_0^T h_z^i(s)\,\mathrm{d}s \\
&\geq \delta_i \|(T(z))^i\|. \qquad (2.7)
\end{aligned}
$$

On the other hand, when $i \in I^-$, by (H2) and (2.4), we have

$$
\begin{aligned}
(T(z))^i(t) &= \int_0^t u_i(t,s)h_z^i(s)\,\mathrm{d}s + \delta_i \int_t^T u_i(t,s)h_z^i(s)\,\mathrm{d}s \\
&\geq \frac{1}{1-\mathrm{e}^{-\int_0^T m_i(\tau)\,\mathrm{d}\tau}} \int_0^t h_z^i(s)\,\mathrm{d}s \\
&\quad + \delta_i \frac{\mathrm{e}^{\int_0^T m_i(\tau)\,\mathrm{d}\tau}}{1-\mathrm{e}^{-\int_0^T m_i(\tau)\,\mathrm{d}\tau}} \int_t^T h_z^i(s)\,\mathrm{d}s \\
&= \frac{1}{1-\mathrm{e}^{-\int_0^T m_i(\tau)\,\mathrm{d}\tau}} \int_0^T h_z^i(s)\,\mathrm{d}s,
\end{aligned}
$$

which implies

$$
-\|(T(z))^i\| \geq \frac{1}{1-\mathrm{e}^{-\int_0^T m_i(\tau)\,\mathrm{d}\tau}} \int_0^T h_z^i(s)\,\mathrm{d}s.
$$

Thus,

$$
\begin{aligned}
(T(z))^i(t) &= \int_0^t u_i(t,s)h_z^i(s)\,\mathrm{d}s + \delta_i \int_t^T u_i(t,s)h_z^i(s)\,\mathrm{d}s \\
&\leq \frac{\mathrm{e}^{-\int_0^T m_i(\tau)\,\mathrm{d}\tau}}{1-\mathrm{e}^{-\int_0^T m_i(\tau)\,\mathrm{d}\tau}} \int_0^T h_z^i(s)\,\mathrm{d}s \\
&\leq -\delta_i \|(T(z))^i\|. \qquad (2.8)
\end{aligned}
$$

Therefore, $T(K_\Delta) \subset K_\Delta$ follows from (2.7) and (2.8). $\square$

Lemma 2.3. *Let $\eta > 0$, $\Omega = \{x \in C_T : \|x\| < \eta\}$. Then $T : K_\Delta \cap \bar{\Omega} \to K_\Delta$ is completely continuous.*

Note that f is a Carathéodory function and satisfies (H1). The proof is easily obtained by using the Arzelà-Ascoli theorem and the Lebesgue dominated convergence theorem. We omit it here.

In the following lemma, we recall the fixed point theorem in cones (see [**6**]).

Lemma 2.4. *Let K be a cone in a Banach space X and Ω_1, Ω_2 two bounded open sets in X such that $0 \in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_2$. Let $T : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \to K$ be a completely continuous operator such that either*:

(i) *there exists $z_0 \in K \setminus \{0\}$ such that $z - Tz \neq \lambda z_0$, $z \in K \cap \partial\Omega_2$, $\lambda \geq 0$; $Tz \neq \mu z$, $z \in K \cap \partial\Omega_1$, $\mu \geq 1$, or*

(ii) *there exists $z_0 \in K \setminus \{0\}$ such that $z - Tz \neq \lambda z_0$, $z \in K \cap \partial\Omega_1$, $\lambda \geq 0$; $Tz \neq \mu z$, $z \in K \cap \partial\Omega_2$, $\mu \geq 1$.*

Then T has at least one fixed point in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

3. Existence of solutions.

Let

$$\frac{f(t,x)}{x} = \left(\frac{f^1(t,x)}{x^1}, \frac{f^2(t,x)}{x^2}, \ldots, \frac{f^n(t,x)}{x^n}\right)^\top.$$

For convenience, we introduce the following notations:

$$\overline{f}_0 = \limsup_{|x|\to 0} \ \operatorname*{ess\,sup}_{t\in[0,T]} \frac{f(t,x)}{x},$$

$$\underline{f}_0 = \liminf_{|x|\to 0} \ \operatorname*{ess\,inf}_{t\in[0,T]} \frac{f(t,x)}{x},$$

$$\overline{f}_\infty = \limsup_{|x|\to \infty} \ \operatorname*{ess\,sup}_{t\in[0,T]} \frac{f(t,x)}{x},$$

$$\underline{f}_\infty = \liminf_{|x|\to \infty} \ \operatorname*{ess\,inf}_{t\in[0,T]} \frac{f(t,x)}{x}.$$

From now on, we denote $\delta = \min\{\delta_i, i = 1, 2, \ldots, n\}$, and $x > 0$ means $x^i > 0$ for $i = 1, 2, \ldots, n$.

Theorem 3.1. *Let f be a Carathéodory function satisfying* (H1), (H2). *If*

(H3) $\underline{f}_0 > 0$, $\overline{f}_\infty < 0$; *or*
(H4) $\underline{f}_\infty > 0$, $\overline{f}_0 < 0$.

Then PBVP (1.1), (1.2) *has at least one mixed-type solution $x(t)$. Moreover, $x^i(t) \geq 0$ for $i \in I^+$ and $x^i(t) \leq 0$ for $i \in I^-$.*

Proof. At first, assume that (H3) holds. Then there exist $\varepsilon > 0$, $r_1 > 0$, $R_0 > 0$ (ε and r_1 are small enough, R_0 is large enough) such that

$$\begin{aligned} &\frac{f^i(t,x)}{x^i} \geq \varepsilon, \quad \text{almost everywhere } t \in [0,T], \\ &0 < |x| \leq r_1, \quad i = 1,2,\dots,n, \end{aligned} \tag{3.1}$$

$$\begin{aligned} &\frac{f^i(t,x)}{x^i} \leq -\varepsilon, \quad \text{almost everywhere } t \in [0,T], \\ &|x| \geq R_0, \quad i = 1,2,\dots,n. \end{aligned} \tag{3.2}$$

Let $\Omega_1 = \{x \in C_T : \|x\| < r_1\}$. We now prove that

$$T(z) \neq \mu z, \quad \text{for all } z \in K_\Delta \cap \partial\Omega_1, \quad \mu \geq 1. \tag{3.3}$$

To this aim, we note that, for any $z \in K_\Delta \cap \partial\Omega_1$, there exists $i_0 \in \{1,2,\dots,n\}$ such that $\|z^{i_0}\| = r_1$. In addition, it is not difficult to check that $0 < \delta r_1 \leq |z(t)| \leq r_1$ for all $t \in [0,T]$. By (3.1), we have

$$\frac{f^{i_0}(t,z(t))}{z^{i_0}(t)} \geq \varepsilon, \quad \text{almost everywhere } t \in [0,T]. \tag{3.4}$$

If $i_0 \in I^+$, (3.4) implies $f^{i_0}(t,z(t)) \geq \varepsilon z^{i_0}(t)$ for almost every $t \in [0,T]$. Thus,

$$\begin{aligned} (T(z))^{i_0}(t) &= \int_0^t u_{i_0}(t,s) h_z^{i_0}(s)\,\mathrm{d}s + \delta_{i_0} \int_t^T u_{i_0}(t,s) h_z^{i_0}(s)\,\mathrm{d}s \\ &\leq \frac{1}{1 - \mathrm{e}^{-\int_0^T m_{i_0}(\tau)\,\mathrm{d}\tau}} \\ &\quad \times \int_0^t \mathrm{e}^{-\int_s^t m_{i_0}(\tau)\,\mathrm{d}\tau} (m_{i_0}(s) z^{i_0}(s) - \varepsilon z^{i_0}(s))\,\mathrm{d}s \end{aligned}$$

$$
\begin{aligned}
&+\frac{\mathrm{e}^{-\int_0^T m_{i_0}(\tau)\,\mathrm{d}\tau}}{1-\mathrm{e}^{-\int_0^T m_{i_0}(\tau)\,\mathrm{d}\tau}}\\
&\times\int_t^T \mathrm{e}^{-\int_s^t m_{i_0}(\tau)\,\mathrm{d}\tau}(m_{i_0}(s)z^{i_0}(s)-\varepsilon z^{i_0}(s))\,\mathrm{d}s\\
\le\ &\frac{\|z^{i_0}\|}{1-\mathrm{e}^{-\int_0^T m_{i_0}(\tau)\,\mathrm{d}\tau}}\mathrm{e}^{-\int_0^t m_{i_0}(\tau)\,\mathrm{d}\tau}\\
&\bigg\{\int_0^t \mathrm{e}^{\int_0^s m_{i_0}(\tau)\,\mathrm{d}\tau}(m_{i_0}(s)-\varepsilon)\,\mathrm{d}s\\
&\quad+\int_t^T \mathrm{e}^{-\int_s^T m_{i_0}(\tau)\,\mathrm{d}\tau}(m_{i_0}(s)-\varepsilon)\,\mathrm{d}s\bigg\}.
\end{aligned}\tag{3.5}
$$

Integration by parts yields

$$
\begin{aligned}
&\int_0^t \mathrm{e}^{\int_0^s m_{i_0}(\tau)\,\mathrm{d}\tau} m_{i_0}(s)\,\mathrm{d}s\\
&\quad+\int_t^T \mathrm{e}^{-\int_s^T m_{i_0}(\tau)\,\mathrm{d}\tau} m_{i_0}(s)\,\mathrm{d}s\\
&\qquad=\mathrm{e}^{\int_0^t m_{i_0}(\tau)\,\mathrm{d}\tau}-\mathrm{e}^{-\int_t^T m_{i_0}(\tau)\,\mathrm{d}\tau}.
\end{aligned}\tag{3.6}
$$

In addition,

$$
\begin{aligned}
&\mathrm{e}^{-\int_0^t m_{i_0}(\tau)\,\mathrm{d}\tau}\int_0^t \mathrm{e}^{\int_0^s m_{i_0}(\tau)\,\mathrm{d}\tau}\,\mathrm{d}s\\
&\quad+\mathrm{e}^{-\int_0^t m_{i_0}(\tau)\,\mathrm{d}\tau}\int_t^T \mathrm{e}^{-\int_s^T m_{i_0}(\tau)\,\mathrm{d}\tau}\,\mathrm{d}s\\
&=\int_0^t \mathrm{e}^{-\int_s^t m_{i_0}(\tau)\,\mathrm{d}\tau}\,\mathrm{d}s+\mathrm{e}^{-\int_0^T m_{i_0}(\tau)\,\mathrm{d}\tau}\int_t^T \mathrm{e}^{-\int_s^t m_{i_0}(\tau)\,\mathrm{d}\tau}\,\mathrm{d}s\\
&\qquad\ge T\mathrm{e}^{-\int_0^T m_{i_0}(\tau)\,\mathrm{d}\tau}.
\end{aligned}\tag{3.7}
$$

Combining (3.5), (3.6) and (3.7), we derive

$$
(T(z))^{i_0}(t)\le\left(1-\frac{\varepsilon T}{\mathrm{e}^{\int_0^T m_{i_0}(\tau)\,\mathrm{d}\tau}-1}\right)\|z^{i_0}\|<\|z^{i_0}\|.
$$

If $i_0\in I^-$, (3.4) implies $f^{i_0}(t,z(t))\le\varepsilon z^{i_0}(t)$ for almost every $t\in[0,T]$.

Thus,

$$
\begin{aligned}
(T(z))^{i_0}(t) &= \int_0^t u_{i_0}(t,s)h_z^{i_0}(s)\,\mathrm{d}s + \delta_{i_0}\int_t^T u_{i_0}(t,s)h_z^{i_0}(s)\,\mathrm{d}s \\
&\geq \frac{1}{1-\mathrm{e}^{-\int_0^T m_{i_0}(\tau)\,\mathrm{d}\tau}} \\
&\quad\times \int_0^t \mathrm{e}^{-\int_s^t m_{i_0}(\tau)\,\mathrm{d}\tau}(m_{i_0}(s)z^{i_0}(s)-\varepsilon z^{i_0}(s))\,\mathrm{d}s \\
&\quad+ \frac{\mathrm{e}^{-\int_0^T m_{i_0}(\tau)\,\mathrm{d}\tau}}{1-\mathrm{e}^{-\int_0^T m_{i_0}(\tau)\,\mathrm{d}\tau}} \\
&\quad\times \int_t^T \mathrm{e}^{-\int_s^t m_{i_0}(\tau)\,\mathrm{d}\tau}(m_{i_0}(s)z^{i_0}(s)-\varepsilon z^{i_0}(s))\,\mathrm{d}s \\
&\geq \frac{-\|z^{i_0}\|}{1-\mathrm{e}^{-\int_0^T m_{i_0}(\tau)\,\mathrm{d}\tau}}\mathrm{e}^{-\int_0^t m_{i_0}(\tau)\,\mathrm{d}\tau} \\
&\quad\times \left\{\int_0^t \mathrm{e}^{\int_0^s m_{i_0}(\tau)\,\mathrm{d}\tau}(m_{i_0}(s)-\varepsilon)\,\mathrm{d}s\right. \\
&\quad\quad \left.+\int_t^T \mathrm{e}^{-\int_s^T m_{i_0}(\tau)\,\mathrm{d}\tau}(m_{i_0}(s)-\varepsilon)\,\mathrm{d}s\right\} \\
&\geq -\left(1-\frac{\varepsilon T}{\mathrm{e}^{\int_0^T m_{i_0}(\tau)\,\mathrm{d}\tau}-1}\right)\|z^{i_0}\| > -\|z^{i_0}\|.
\end{aligned}
$$

Therefore, (3.3) is proved.

On the other hand, let $r_2 = R_0/\delta$, $\Omega_2 = \{x \in C_T : \|x\| < r_2\}$ and $z_0 = (z_0^1, z_0^2, \ldots, z_0^n)^\top$ with $z_0^i = \sigma_i$, $i = 1, 2, \ldots, n$. It is easy to see that $z_0 \in K_\Delta \setminus \{0\}$. We now prove that

$$
z - T(z) \neq \lambda z_0, \quad \text{for all } z \in K_\Delta \cap \partial\Omega_2,\ \lambda \geq 0. \tag{3.8}
$$

Suppose on the contrary that there exists $\overline{z} \in K_\Delta \cap \partial\Omega_2$ such that $\overline{z} - T(\overline{z}) = \lambda_0 z_0$ for some $\lambda_0 \geq 0$. By $\overline{z} \in K_\Delta \cap \partial\Omega_2$, we know that there exists $i_0 \in \{1, 2, \ldots, n\}$ such that $\|\overline{z}^{i_0}\| = r_2$. In addition, it is easy to check that $|\overline{z}(t)| \geq \delta r_2 = R_0$ for all $t \in [0,T]$. By (3.2), we have

$$
\frac{f^{i_0}(t,\overline{z}(t))}{\overline{z}^{i_0}(t)} \leq -\varepsilon, \quad \text{for almost every } t \in [0,T]. \tag{3.9}
$$

If $i_0 \in I^+$, (3.9) implies $f^{i_0}(t,\overline{z}(t)) \leq -\varepsilon\overline{z}^{i_0}(t)$ for almost every

$t \in [0,T]$. Taking into account that $\overline{z} \in K_\Delta$, it follows that $\overline{z}^{i_0}(t) \geq \delta_{i_0}\|\overline{z}^{i_0}\| \geq \delta r_2 = R_0 > 0$. Let $\xi = \min_{t\in[0,T]} \overline{z}^{i_0}(t)$. Then $\xi \geq R_0$. Thus,

$$
\begin{aligned}
\overline{z}^{i_0}(t) &= \int_0^t u_{i_0}(t,s)h_{\overline{z}}^{i_0}(s)\,\mathrm{d}s \\
&\quad + \delta_{i_0}\int_t^T u_{i_0}(t,s)h_{\overline{z}}^{i_0}(s)\,\mathrm{d}s + \lambda_0 \\
&= \frac{1}{1-\mathrm{e}^{-\int_0^T m_{i_0}(\tau)\,\mathrm{d}\tau}} \\
&\quad \times \int_0^t \mathrm{e}^{-\int_s^t m_{i_0}(\tau)\,\mathrm{d}\tau}(m_{i_0}(s)\overline{z}^{i_0}(s) - f^{i_0}(s,\overline{z}(s)))\,\mathrm{d}s \\
&\quad + \frac{\mathrm{e}^{-\int_0^T m_{i_0}(\tau)\,\mathrm{d}\tau}}{1-\mathrm{e}^{-\int_0^T m_{i_0}(\tau)\,\mathrm{d}\tau}} \\
&\quad \times \int_t^T \mathrm{e}^{-\int_s^t m_{i_0}(\tau)\,\mathrm{d}\tau}(m_{i_0}(s)\overline{z}^{i_0}(s) - f^{i_0}(s,\overline{z}(s)))\,\mathrm{d}s + \lambda_0 \\
&\geq \frac{1}{1-\mathrm{e}^{-\int_0^T m_{i_0}(\tau)\,\mathrm{d}\tau}} \\
&\quad \times \int_0^t \mathrm{e}^{-\int_s^t m_{i_0}(\tau)\,\mathrm{d}\tau}(m_{i_0}(s)+\varepsilon)\overline{z}^{i_0}(s)\,\mathrm{d}s \\
&\quad + \frac{\mathrm{e}^{-\int_0^T m_{i_0}(\tau)\,\mathrm{d}\tau}}{1-\mathrm{e}^{-\int_0^T m_{i_0}(\tau)\,\mathrm{d}\tau}} \\
&\quad \times \int_t^T \mathrm{e}^{-\int_s^t m_{i_0}(\tau)\,\mathrm{d}\tau}(m_{i_0}(s)+\varepsilon)\overline{z}^{i_0}(s)\,\mathrm{d}s + \lambda_0 \\
&\geq \frac{\xi \mathrm{e}^{-\int_0^t m_{i_0}(\tau)\,\mathrm{d}\tau}}{1-\mathrm{e}^{-\int_0^T m_{i_0}(\tau)\,\mathrm{d}\tau}} \\
&\quad \times \left(\int_0^t \mathrm{e}^{\int_0^s m_{i_0}(\tau)\,\mathrm{d}\tau} m_{i_0}(s)\,\mathrm{d}s + \int_t^T \mathrm{e}^{-\int_s^T m_{i_0}(\tau)\,\mathrm{d}\tau} m_{i_0}(s)\,\mathrm{d}s\right) \\
&\quad + \frac{\varepsilon\xi}{1-\mathrm{e}^{-\int_0^T m_{i_0}(\tau)\,\mathrm{d}\tau}} \\
&\quad \times \left(\int_0^t \mathrm{e}^{-\int_s^t m_{i_0}(\tau)\,\mathrm{d}\tau}\,\mathrm{d}s + \mathrm{e}^{-\int_0^T m_{i_0}(\tau)\,\mathrm{d}\tau}\int_t^T \mathrm{e}^{-\int_s^t m_{i_0}(\tau)\,\mathrm{d}\tau}\,\mathrm{d}s\right) \\
&\quad + \lambda_0.
\end{aligned}
$$

Then, by (3.6) and (3.7), we derive

$$\begin{aligned}\overline{z}^{i_0}(t) &\geq \xi + \frac{\varepsilon\xi}{1-\mathrm{e}^{-\int_0^T m_{i_0}(\tau)\,\mathrm{d}\tau}} T\mathrm{e}^{-\int_0^T m_{i_0}(\tau)\,\mathrm{d}\tau} + \lambda_0 \\ &= \left(1 + \frac{\varepsilon T}{\mathrm{e}^{\int_0^T m_{i_0}(\tau)\,\mathrm{d}\tau} - 1}\right)\xi + \lambda_0.\end{aligned}$$

Hence, $\xi \geq (1 + (\varepsilon T)/(\mathrm{e}^{\int_0^T m_{i_0}(\tau)\,\mathrm{d}\tau} - 1))\xi + \lambda_0 > \xi$, which is a contradiction.

If $i_0 \in I^-$, (3.9) yields $f^{i_0}(t, \overline{z}(t)) \geq -\varepsilon\overline{z}^{i_0}(t)$ for almost every $t \in [0, T]$. In addition, $\overline{z} \in K_\Delta$ implies $\overline{z}^{i_0}(t) \leq -\delta_{i_0}\|\overline{z}^{i_0}\| \leq -\delta r_2 = -R_0 < 0$. Let $\xi = \max_{t\in[0,T]} \overline{z}^{i_0}(t)$. Then $\xi \leq -R_0$. Thus,

$$\begin{aligned}\overline{z}^{i_0}(t) &= \int_0^t u_{i_0}(t,s)h_{\overline{z}}^{i_0}(s)\,\mathrm{d}s + \delta_{i_0}\int_t^T u_{i_0}(t,s)h_{\overline{z}}^{i_0}(s)\,\mathrm{d}s - \lambda_0 \\ &= \frac{1}{1-\mathrm{e}^{-\int_0^T m_{i_0}(\tau)\,\mathrm{d}\tau}} \\ &\quad\times \int_0^t \mathrm{e}^{-\int_s^t m_{i_0}(\tau)\,\mathrm{d}\tau}(m_{i_0}(s)\overline{z}^{i_0}(s) - f^{i_0}(s,\overline{z}(s)))\,\mathrm{d}s \\ &\quad+ \frac{\mathrm{e}^{-\int_0^T m_{i_0}(\tau)\,\mathrm{d}\tau}}{1-\mathrm{e}^{-\int_0^T m_{i_0}(\tau)\,\mathrm{d}\tau}} \\ &\quad\times \int_t^T \mathrm{e}^{-\int_s^t m_{i_0}(\tau)\,\mathrm{d}\tau}(m_{i_0}(s)\overline{z}^{i_0}(s) - f^{i_0}(s,\overline{z}(s)))\,\mathrm{d}s - \lambda_0 \\ &\leq \frac{1}{1-\mathrm{e}^{-\int_0^T m_{i_0}(\tau)\,\mathrm{d}\tau}} \\ &\quad\times \int_0^t \mathrm{e}^{-\int_s^t m_{i_0}(\tau)\,\mathrm{d}\tau}(m_{i_0}(s)+\varepsilon)\overline{z}^{i_0}(s)\,\mathrm{d}s \\ &\quad+ \frac{\mathrm{e}^{-\int_0^T m_{i_0}(\tau)\,\mathrm{d}\tau}}{1-\mathrm{e}^{-\int_0^T m_{i_0}(\tau)\,\mathrm{d}\tau}} \\ &\quad\times \int_t^T \mathrm{e}^{-\int_s^t m_{i_0}(\tau)\,\mathrm{d}\tau}(m_{i_0}(s)+\varepsilon)\overline{z}^{i_0}(s)\,\mathrm{d}s - \lambda_0 \\ &\leq \frac{\xi\mathrm{e}^{-\int_0^t m_{i_0}(\tau)\,\mathrm{d}\tau}}{1-\mathrm{e}^{-\int_0^T m_{i_0}(\tau)\,\mathrm{d}\tau}} \\ &\quad\times \left(\int_0^t \mathrm{e}^{\int_0^s m_{i_0}(\tau)\,\mathrm{d}\tau} m_{i_0}(s)\,\mathrm{d}s + \int_t^T \mathrm{e}^{-\int_s^T m_{i_0}(\tau)\,\mathrm{d}\tau} m_{i_0}(s)\,\mathrm{d}s\right)\end{aligned}$$

$$
\begin{aligned}
&+ \frac{\varepsilon\xi}{1-\mathrm{e}^{-\int_0^T m_{i_0}(\tau)\,\mathrm{d}\tau}} \\
&\times \left(\int_0^t \mathrm{e}^{-\int_s^t m_{i_0}(\tau)\,\mathrm{d}\tau}\,\mathrm{d}s + \mathrm{e}^{-\int_0^T m_{i_0}(\tau)\,\mathrm{d}\tau} \int_t^T \mathrm{e}^{-\int_s^t m_{i_0}(\tau)\,\mathrm{d}\tau}\,\mathrm{d}s \right) \\
&- \lambda_0 \\
\le{}& \xi + \frac{\varepsilon\xi}{1-\mathrm{e}^{-\int_0^T m_{i_0}(\tau)\,\mathrm{d}\tau}} T\mathrm{e}^{-\int_0^T m_i(\tau)\,\mathrm{d}\tau} - \lambda_0 \\
={}& \left(1 + \frac{\varepsilon T}{\mathrm{e}^{\int_0^T m_{i_0}(\tau)\,\mathrm{d}\tau} - 1}\right)\xi - \lambda_0.
\end{aligned}
$$

Hence, $\xi \le (1 + (\varepsilon T)/(\mathrm{e}^{\int_0^T m_{i_0}(\tau)\,\mathrm{d}\tau} - 1))\xi - \lambda_0 < \xi$, which is a contradiction. Therefore, (3.8) is proved.

By Lemma 2.3, $T : K_\Delta \cap (\overline{\Omega}_2 \setminus \Omega_1) \to K_\Delta$ is completely continuous. According to Lemma 2.4, there exists $x \in K_\Delta \cap (\bar{\Omega}_2 \setminus \Omega_1)$ such that $T(x)(t) = x(t)$, $\|x\| \ge r_1 > 0$, $x^i(t) \ge \delta_i \|x^i\| \ge 0$, $i \in I^+$; $x^i(t) \le -\delta_i \|x^i\| \le 0$, $i \in I^-$. Therefore, $x(t)$ is a mixed-type solution of PBVP (1.1), (1.2).

Next, assume that (H4) holds. Then there exist $\varepsilon_1 > 0$, $r_3 > 0$, $R_1 > 0$ (ε_1 and r_3 are small enough, R_1 is large enough) such that

$$
\begin{gathered}
\frac{f^i(t,x)}{x^i} \le -\varepsilon_1, \quad \text{almost everywhere } t \in [0,T], \\
0 < |x| \le r_3, \quad i = 1, 2, \cdots, n,
\end{gathered} \tag{3.10}
$$

$$
\begin{gathered}
\frac{f^i(t,x)}{x^i} \ge \varepsilon_1, \quad \text{almost everywhere } t \in [0,T], \\
|x| \ge R_1, \quad i = 1, 2, \ldots, n.
\end{gathered} \tag{3.11}
$$

Let $\Omega_3 = \{x \in C_T : \|x\| < r_3\}$. Then, for any $z \in K_\Delta \cap \partial\Omega_3$, there exists $i_0 \in \{1, 2, \ldots, n\}$ such that $\|z^{i_0}\| = r_3$. By (3.10), similar to the proof of (3.8), we have

$$
z - T(z) \ne \lambda z_0, \quad \text{for all } z \in K_\Delta \cap \partial\Omega_3,\ \lambda \ge 0. \tag{3.12}
$$

Let $r_4 = R_1/\delta$ and $\Omega_4 = \{x \in C_T : \|x\| < r_4\}$. Then, for any $z \in K_\Delta \cap \partial\Omega_4$, there exists $i_0 \in \{1, 2, \cdots, n\}$ such that $\|z^{i_0}\| = r_4$. Thus, if $i_0 \in I^+$, $z^{i_0}(t) \ge \delta_{i_0}\|z^{i_0}\| \ge R_1$ and if $i_0 \in I^-$, $z^{i_0}(t) \le -\delta_{i_0}\|z^{i_0}\| \le -R_1$. Hence, $|z(t)| \ge R_1$. In view of (3.11), similar to the proof of (3.3), we can obtain

$$T(z) \neq \mu z, \quad \text{for all } u \in K_\Delta \cap \partial\Omega_4,\ \mu \geq 1.$$

By Lemma 2.4, T has a fixed point $x \in K_\Delta \cap (\overline{\Omega}_4 \backslash \Omega_3)$, $\|x\| \geq r_3 > 0$, $x^i(t) \geq \delta_i \|x^i\| \geq 0$, $i \in I^+$; $x^i(t) \leq -\delta_i \|x^i\| \leq 0$, $i \in I^-$. Therefore, $x(t)$ is a mixed-type solution of PBVP (1.1), (1.2). □

Remark 3.1.

(i) When $E = \operatorname{diag}(1, 1, \dots, 1)$, Theorem 3.1 gives the existence result of positive solutions for PBVP (1.1), (1.2).
(ii) When $E = \operatorname{diag}(-1, -1, \dots, -1)$, Theorem 3.1 gives the existence result of negative solutions for PBVP (1.1), (1.2).

Example 3.1. Consider the two-dimensional PBVP of the following form

$$x' + (t+1)\sin x + xy - x(x^2 + y^2) = 0, \quad 0 \leq t \leq 1, \tag{3.13}$$

$$y' + \arctan y - (t^2+1)y^3 - x^2 y = 0, \quad 0 \leq t \leq 1, \tag{3.14}$$

$$x(0) = x(1), \quad y(0) = y(1). \tag{3.15}$$

In this example, $T = 1$, and

$$f(t, x, y) = \begin{pmatrix} (t+1)\sin x + xy - x(x^2 + y^2) \\ \arctan y - (t^2+1)y^3 - x^2 y \end{pmatrix}.$$

By choosing

$$M(t) = \begin{pmatrix} t+1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad E = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

we can easily verify that (H1) and (H2) hold. In addition, by a direct calculation, we get $\underline{f}_0 = (1, 1)^\top > 0$, $\overline{f}_\infty \to (-\infty, -\infty)^\top$. Theorem 3.1 shows the PBVP (3.13), (3.14), (3.15) has at least one mixed-type solution $(x(t), y(t))^\top$. Moreover, $x(t) \geq 0$, $y(t) \leq 0$.

4. Multiplicity of solutions.

Theorem 4.1. *Let f be a Carathéodory function which satisfies* (H1), (H2). *If*

(H5) $\overline{f}_0 < 0, \overline{f}_\infty < 0$; *and*
(H6) *there exists* $\rho > 0$ *such that*

$$\inf_{\delta\rho \le |x| \le \rho} \operatorname*{ess\,inf}_{t\in[0,T]} \frac{f(t,x)}{x} > 0.$$

Then PBVP (1.1), (1.2) *has at least two mixed-type solutions* $x_1(t)$ *and* $x_2(t)$. *Moreover,* $x_1^i(t)$ *and* $x_2^i(t) \ge 0$ *for* $i \in I^+$ *and* $x_1^i(t)$ *and* $x_2^i(t) \le 0$ *for* $i \in I^-$.

Proof. By (H5) and the proof of Theorem 3.1, we know that there exist $r_5 > 0$, $r_6 > 0$ (r_5 is small enough, and r_6 is large enough, $r_5 < \rho < r_6$) such that

$$z - T(z) \ne \lambda z_0, \quad \text{for all } z \in K_\Delta \cap \partial\Omega_5,\ \lambda \ge 0,$$
$$z - T(z) \ne \lambda z_0, \quad \text{for all } z \in K_\Delta \cap \partial\Omega_6,\ \lambda \ge 0,$$

where $\Omega_5 = \{x \in C_T : \|x\| < r_5\}$ and $\Omega_6 = \{x \in C_T : \|x\| < r_6\}$.

By (H6), there exists $\varepsilon > 0$ (ε is small enough) such that

$$\frac{f^i(t,x)}{x^i} \ge \varepsilon, \quad \text{almost everywhere } t \in [0,T],$$
$$\delta\rho \le |x| \le \rho, \quad i = 1, 2, \ldots, n. \tag{4.1}$$

Let $\Omega_7 = \{x \in C_T : \|x\| < \rho\}$. Then, for any $z \in K_\Delta \cap \partial\Omega_7$, there exists $i_0 \in \{1, 2, \ldots, n\}$ such that $\|z^{i_0}\| = \rho$. Thus, if $i_0 \in I^+$, $z^{i_0}(t) \ge \delta_{i_0}\|z^{i_0}\| \ge \delta\rho$ and if $i_0 \in I^-$, $z^{i_0}(t) \le -\delta_{i_0}\|z^{i_0}\| \le -\delta\rho$. Hence, $\delta\rho \le |z(t)| \le \rho$. In view of (4.1), by using a similar method as used in the proof of Theorem 3.1, we have

$$T(z) \ne \mu z, \quad \text{for all } z \in K_\Delta \cap \partial\Omega_7,\ \mu \ge 1.$$

It is obvious that $\Omega_5 \subset \Omega_7 \subset \Omega_6$, and by Lemma 2.4, we conclude that T has at least two fixed points $x_1 \in K_\Delta \cap (\overline{\Omega}_7 \backslash \Omega_5)$ and $x_2 \in K_\Delta \cap (\overline{\Omega}_6 \backslash \Omega_7)$. Moreover, $r_5 \le \|x_1\| < \rho$, $\rho < \|x_2\| \le r_6$, $x_1^i(t)$, $x_2^i(t) \ge 0$, $i \in I^+$; $x_1^i(t)$, $x_2^i(t) \le 0$, $i \in I^-$. Therefore, $x_1(t)$ and $x_2(t)$ are mixed-type solutions of PBVP (1.1), (1.2). □

Similar to the proof of Theorem 4.1, we have the following theorem.

Theorem 4.2. *Let* f *be a Carathéodory function which satisfies* (H1), (H2). *If*

(H7) $\underline{f}_0 > 0, \underline{f}_\infty > 0$; *and*
(H8) *there exists* $\rho > 0$ *such that*

$$\sup_{\delta\rho\le|x|\le\rho} \operatorname*{ess\,sup}_{t\in[0,T]} \frac{f(t,x)}{x} < 0.$$

Then PBVP (1.1), (1.2) *has at least two mixed-type solutions* $x_1(t)$ *and* $x_2(t)$. *Moreover,* $x_1^i(t)$, $x_2^i(t) \ge 0$ *for* $i \in I^+$; $x_1^i(t)$, $x_2^i(t) \le 0$ *for* $i \in I^-$.

Remark 4.1.

(i) When $E = \operatorname{diag}(1, 1, \ldots, 1)$, Theorems 4.1 and 4.2 give the multiplicity results of positive solutions for PBVP (1.1) and (1.2).
(ii) When $E = \operatorname{diag}(-1, -1, \ldots, -1)$, Theorems 4.1 and 4.2 give the multiplicity results of negative solutions for PBVP (1.1) and (1.2).

Example 4.1. Consider the following two-dimensional PBVP

$$x' + \frac{1}{4}(t^2+1)\,x - 2\sqrt{x^2+y^2}\,\mathrm{e}^{-\sqrt{x^2+y^2}}\,x = 0, \tag{4.2}$$

$$y' + \frac{1}{2\pi}\,y \arctan(1+x^2+y^2) - (t+1)\sqrt{x^2+y^2}\,\mathrm{e}^{-\sqrt{x^2+y^2}}\,y = 0, \tag{4.3}$$

$$x(0) = x(1), \quad y(0) = y(1), \tag{4.4}$$

where $0 \le t \le 1$. In this example, $T = 1$, and

$$f(t,x,y) = \begin{pmatrix} \frac{1}{4}(t^2+1)\,x - 2\sqrt{x^2+y^2}\,\mathrm{e}^{-\sqrt{x^2+y^2}}\,x \\ \frac{1}{2\pi}\,y \arctan(1+x^2+y^2) - (t+1)\sqrt{x^2+y^2}\,\mathrm{e}^{-\sqrt{x^2+y^2}}\,y \end{pmatrix}.$$

By choosing

$$M(t) = \begin{pmatrix} \frac{1}{4}(t^2+1) & 0 \\ 0 & \frac{1}{8} \end{pmatrix}, \qquad E = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

we can easily verify that (H1) and (H2) hold. A direct calculation yields $\underline{f}_0 = (1/4, 1/8)^\top > 0$, $\underline{f}_\infty = (1/4, 1/4)^\top > 0$. Let $\delta = \mathrm{e}^{-1/3}$, $\rho = 1$. By means of the Matlab, we can compute that $\max\{(f(t,x,y))/(x,y) \mid t \in [0,T], \delta\rho \le |(x,y)^\top| \le \rho\} \le (-0.1876, -0.0938)^\top < 0$. Theorem 4.2

shows that PBVP (4.2), (4.3), (4.4) has at least two mixed-type solutions, $(x_1(t), y_1(t))^\top$ and $(x_2(t), y_2(t))^\top$. Moreover, $x_1(t)$, $x_2(t) \geq 0$; $y_1(t)$, $y_2(t) \leq 0$.

5. Conclusions. This paper has presented some existence and multiplicity results of positive, negative and mixed-type solutions for PBVP (1.1), (1.2) in vector form under some proper conditions. In special case $n = 1$, the existence and multiplicity conditions can be weakened as follows.

As regards the existence and multiplicity results of positive solutions, we only require

$$\overline{f}_0 = \limsup_{x \to 0+} \ \operatorname*{ess\ sup}_{t \in [0,T]} \frac{f(t,x)}{x},$$

$$\underline{f}_0 = \liminf_{x \to 0+} \ \operatorname*{ess\ inf}_{t \in [0,T]} \frac{f(t,x)}{x},$$

$$\overline{f}_\infty = \limsup_{x \to +\infty} \ \operatorname*{ess\ sup}_{t \in [0,T]} \frac{f(t,x)}{x},$$

$$\underline{f}_\infty = \liminf_{x \to +\infty} \ \operatorname*{ess\ inf}_{t \in [0,T]} \frac{f(t,x)}{x},$$

and the multiplicity conditions (H6) and (H8), respectively, can be replaced by

(H6)$'$ there exists $\rho > 0$ such that

$$\inf_{\delta\rho \leq x \leq \rho} \ \operatorname*{ess\ inf}_{t \in [0,T]} f(t,x) > 0;$$

(H8)$'$ there exists $\rho > 0$ such that

$$\sup_{\delta\rho \leq x \leq \rho} \ \operatorname*{ess\ sup}_{t \in [0,T]} f(t,x) < 0.$$

As regards the existence and multiplicity results of negative solutions, we only require

$$\overline{f}_0 = \limsup_{x \to 0-} \ \operatorname*{ess\ sup}_{t \in [0,T]} \frac{f(t,x)}{x},$$

$$\underline{f}_0 = \liminf_{x \to 0-} \ \operatorname*{ess\ inf}_{t \in [0,T]} \frac{f(t,x)}{x},$$

$$\overline{f}_\infty = \limsup_{x \to -\infty} \operatorname*{ess\,sup}_{t \in [0,T]} \frac{f(t,x)}{x},$$

$$\underline{f}_\infty = \liminf_{x \to -\infty} \operatorname*{ess\,inf}_{t \in [0,T]} \frac{f(t,x)}{x},$$

and the multiplicity conditions (H6) and (H8), respectively, can be replaced by

(H6)″ there exists $\rho > 0$ such that

$$\sup_{-\rho \le x \le -\delta\rho} \operatorname*{ess\,sup}_{t \in [0,T]} f(t,x) < 0.$$

(H8)″ there exists $\rho > 0$ such that

$$\inf_{-\rho \le x \le -\delta\rho} \operatorname*{ess\,inf}_{t \in [0,T]} f(t,x) > 0.$$

In addition, we can also use the methods in this paper to deal with the PBVP of the following form

$$-x'(t) + f(t, x(t)) = 0, \quad \text{almost everywhere } t \in [0,T], \tag{5.1}$$

$$x(0) = x(T), \tag{5.2}$$

where $f : [0,T] \times \mathbf{R}^n \to \mathbf{R}^n$ is a Carathéodory function satisfying (H1) and (H2). For PBVP (5.1), (5.2), the function $u_i(t,s)$ in (2.3) is replaced by

$$u_i(t,s) = \frac{\mathrm{e}^{\int_s^t m_i(\tau)\,\mathrm{d}\tau}}{\mathrm{e}^{\int_0^T m_i(\tau)\,\mathrm{d}\tau} - 1}, \quad i = 1, 2, \ldots, n.$$

Letting $\delta_i = \mathrm{e}^{-\int_0^T m_i(\tau)\,\mathrm{d}\tau}$, we can also prove that Theorems 3.1, 4.1 and 4.2 are valid for PBVP (5.1), (5.2).

Finally, we also remark that, if the function $f(t,x)$ is periodic with respect to t, the existence and multiplicity of positive, negative and mixed-type solutions can also be discussed by using our method.

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