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COMPOSITION OPERATORS ON WEIGHTED HARDY SPACES

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ABSTRACT. Suppose φ is an analytic self-map of open unit disk $\mathbf{D}$ and ψ is an analytic function on $\mathbf{D}$. Then a weighted composition operator induced by φ with weight ψ is given by $(W_{\psi,\varphi}f)(z) = \psi(z)f(\varphi(z))$, for $z \in \mathbf{D}$ and f analytic on $\mathbf{D}$. Necessary and sufficient conditions are given for the boundedness and compactness of the weighted composition operators $W_{\psi,\varphi}$. In terms of fixed points in the closed unit disk $\overline{\mathbf{D}}$, conditions under which $W_{\psi,\varphi}$ is compact are given. Necessary conditions for the compactness of C_φ are given in terms of the angular derivative $\varphi'(\zeta)$ where ζ is on the boundary of the unit disk. Moreover, we present sufficient conditions for the membership of composition operators in the Schatten p-class $S_p(H^s(\beta_1), H^q(\beta_2))$, where the inducing map has supremum norm strictly smaller than 1.

1. Introduction. A Hilbert space $\mathcal{H}$ whose vectors are functions analytic on the unit disk $\mathbf{D}$ is called a weighted Hardy space if the monomials $\{1, z, z^2, \ldots\}$ form a nonzero orthogonal set of vectors and the polynomials are dense in $\mathcal{H}$. The properties of a weighted Hardy space depend on the weight sequence $\{\beta(n)\}_{n=0}^\infty$, which is defined by $\beta(n) = \|z^n\|_{\mathcal{H}}$.

Let $\{\beta(n)\}_{n=0}^\infty$ be a sequence of positive numbers such that $\beta(0) = 1$. Define the set $H^p(\beta)$, $1 < p < \infty$, to be the set of all formal power series $f(z) = \sum_{n=0}^\infty a_n z^n$ with $z \in \mathbf{D}$ such that

$$\|f\|_{H^p(\beta)}^p = \sum_{n=0}^\infty |a_n|^p (\beta(n))^p < \infty.$$

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If $\lim_{n\to\infty}(\beta(n+1))/(\beta(n)) = 1$ or $\liminf_{n\to\infty}\beta(n)^{1/n} = 1$, then the weighted Hardy space $H^p(\beta)$ consists of functions analytic on the unit disk $\mathbf{D}$. In this paper, we consider weighted Hardy spaces $H^p(\beta)$ where $\{\beta(n)\}_{n=0}^{\infty}$ is a sequence of positive numbers with $\beta(0) = 1$ and $\liminf_{n\to\infty}\beta(n)^{1/n} = 1$.

In the case $p = 2$, the space $H^2(\beta)$ is a Hilbert space of functions analytic on the unit disk $\mathbf{D}$ with the inner product given by

$$\langle f, g\rangle = \sum_{n=0}^{\infty} a_n \overline{b_n}(\beta(n))^2,$$

where $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ are in $H^2(\beta)$. Some well-known special cases of this type of Hilbert space are the classical Hardy, the Bergman and the Dirichlet spaces with weights $\beta(n) \equiv 1$ for all n, $\beta(n) = (n+1)^{-1/2}$, and $\beta(n) = (n+1)^{1/2}$, respectively. Thus the terminology *weighted Hardy space* comes from the observation that, if $\beta(n) \equiv 1$, then $H^2(\beta)$ is the classical Hardy Hilbert space $H^2(\mathbf{D})$. Note that, if $\{\beta_1(n)\}$ and $\{\beta_2(n)\}$ are weight sequences with

$$c\beta_2(n) \le \beta_1(n) \le \frac{1}{c}\beta_2(n),$$

for some positive constant c, then $H^2(\beta_1) = H^2(\beta_2)$ with equivalent norms.

If ψ is an analytic map on the unit disk $\mathbf{D}$ and φ is an analytic self-map of $\mathbf{D}$, the weighted composition operator $W_{\psi,\varphi}$ is the operator on $H^p(\beta)$ given by

$$(W_{\psi,\varphi} f)(z) = \psi(z) f(\varphi(z)),$$

for $f \in H^p(\beta)$ and $z \in \mathbf{D}$. If $W_{\psi,\varphi}$ is an operator on $H^p(\beta)$, then it will map the constant function $f \equiv 1$ to ψ, so ψ must belong to $H^p(\beta)$. Moreover, the function $f(z) = z$ belongs to $H^p(\beta)$, so if $\psi \equiv 1$ and $W_{\psi,\varphi}$ is bounded, then $W_{\psi,\varphi}$ maps f to φ; hence, φ belongs to $H^p(\beta)$.

The Banach spaces $H^p(\beta)$, $1 < p < \infty$, are reflexive with norm $\|.\|_{H^p(\beta)}$ and the dual space of $H^p(\beta)$ is $H^q(\beta^{p/q})$ where $1/p + 1/q = 1$. So, if $g(z) = \sum_{n=0}^{\infty} b_n z^n$ in $H^q(\beta^{p/q})$, then

$$\|g\|^q_{H^q(\beta^{p/q})} = \sum_{n=0}^{\infty} |b_n|^q (\beta^{p/q}(n))^q = \sum_{n=0}^{\infty} |b_n|^q \beta^p(n).$$

Hence, if $f \in H^p(\beta)$ and $g \in (H^p(\beta))^*$, then

$$\langle f, g\rangle_\beta = \sum_{n=0}^{\infty} a_n \overline{b_n} \beta^p(n),$$

where $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$.

Now, we define a class of functions that play an important role in studying weighted composition operators on $H^p(\beta)$, $1 < p < \infty$. For a complex number w, define

$$K_w(z) = \sum_{n=0}^{\infty} \frac{\overline{w}^n}{\beta^p(n)} z^n.$$

For $1/p + 1/q = 1$ and $w \in \mathbf{D}$, we observe that

$$\begin{aligned} \|K_w\|^q_{(H^p(\beta))^*} &= \|K_w\|^q_{H^q(\beta^{p/q})} \\ &= \sum_{n=0}^{\infty} \frac{|w|^{nq}}{\beta^{pq}(n)} (\beta^{p/q}(n))^q \\ &= \sum_{n=0}^{\infty} \frac{|w|^{nq}}{\beta^q(n)} < \infty, \end{aligned} \tag{1}$$

the convergence of the series in (1) follows from the assumption $\liminf_{n\to\infty} \beta(n)^{1/n} = 1$ and root test. Moreover, if $f(z) = \sum_{n=0}^{\infty} a_n z^n$, then

$$\langle f, K_w\rangle_\beta = \sum_{n=0}^{\infty} \frac{a_n w^n}{\beta^p(n)} \beta^p(n) = f(w).$$

Therefore, $K_w \in (H^p(\beta))^*$ is a point evaluation kernel of $H^p(\beta)$ whenever $w \in \mathbf{D}$. The next result follows from (1).

Lemma 1.1. *For $w \in \mathbf{D}$, the norm $\|K_w\|_{(H^p(\beta))^*}$ is an increasing function of $|w|$.*

The following lemma is crucial when working with weighted composition operators.

Lemma 1.2. *Let $W_{\psi,\varphi}$ be bounded on $H^p(\beta)$, and let K_w be a point evaluation kernel. Then*

$$W^*_{\psi,\varphi}K_w = \overline{\psi(w)}K_{\varphi(w)}.$$

Proof. For any $f \in H^p(\beta)$, we have

$$\begin{aligned}\langle f, W^*_{\psi,\varphi}K_w\rangle_\beta &= \langle W_{\psi,\varphi}f, K_w\rangle_\beta \\ &= \langle \psi f\circ\varphi, K_w\rangle_\beta \\ &= \langle f, \overline{\psi(w)}K_{\varphi(w)}\rangle_\beta.\end{aligned}$$

Since this holds for all $f \in H^p(\beta)$, we get the desired result. □

The proof of the following lemma can be obtained by adapting the proof of ([**1,** Theorem 2.17]).

Lemma 1.3. *Let $1 < p < \infty$, and let $\beta(n)$ be a weight sequence such that $\sum_{n=0}^\infty \beta^{-q}(n) = \infty$, where $1/p + 1/q = 1$. Then the normalized reproducing kernel $K_w/\|K_w\|_{(H^p(\beta))^*}$ tends to zero weakly as $|w| \to 1$.*

This result will be helpful since a compact operator T in a Banach space $\mathcal{H}$ takes weakly convergent sequences to norm convergent sequences. In particular, if $\{x_n\} \to 0$ weakly and T is compact, then $\|Tx_n\|_{\mathcal{H}} \to 0$.

2. Boundedness of weighted composition operators. In this section we are interested in studying the boundedness of weighted composition operators acting on small weighted Hardy spaces. The weighted Hardy space $H^p(\beta)$ is called a small weighted Hardy space if the condition

$$\sum_{n=0}^\infty \frac{1}{\beta^r(n)} < \infty,$$

holds for $1/p+1/r = 1$. The functions in this space are analytic on the open unit disk $\mathbf{D}$ and continuous on $\overline{\mathbf{D}}$.

Theorem 2.1. *Let $1 < q \le p < \infty$, and let φ be an analytic self-map of $\mathbf{D}$ and $\psi \in H^\infty$. Let $\beta_1(n)$ and $\beta_2(n)$ be any two weight sequences such that $\beta_1(n) \le \beta_2(n)$ for all n, $\beta_2(n)$ is bounded, and $\sum_{n=0}^\infty \beta_1^{-r}(n) < \infty$*

where $1/p+1/r=1$. *If, for all* $f \in H^p(\beta_1)$, $f\circ\varphi \in H^\infty$, *then* $W_{\psi,\varphi}$ *is bounded from* $H^p(\beta_1)$ *into* $H^q(\beta_2)$.

Before proceeding with the proof of Theorem 2.1, we need the following lemma which is a slight modification of ([**8,** Proposition 2]).

Lemma 2.2. *Let* $1<p<\infty$. *Suppose* $\beta(n)$ *is a bounded sequence such that* $\beta(n)\le M$ *for all* n, *and* $f\in H^\infty$. *Then* $f\in H^p(\beta)$ *and* $\|f\|^p_{H^p(\beta)} \le M^p\|f\|_\infty$.

Proof. Suppose $\beta(n)\le M$ for all n. Let $f(z)=\sum_{n=0}^\infty a_n z^n$, then

$$\begin{aligned}\|f\|^p_{H^p(\beta)} &= \sum_{n=0}^\infty |a_n|^p\beta^p(n) \le M^p\sum_{n=0}^\infty |a_n|^p\\ &= M^p\|f\|^p_{H^p} \le M^p\|f\|_\infty.\end{aligned}$$

Moreover, we get $H^\infty \subseteq H^p \subseteq H^p(\beta)$. □

Proof of Theorem 2.1. Let $f\in H^p(\beta_1)$ and $f\circ\varphi\in H^\infty$. Since $\beta_2(n)$ is bounded, by Lemma 2.2, we get $f\circ\varphi\in H^q(\beta_2)$. From Lemma 2.2, $\|f\circ\varphi\|^q_{H^q(\beta_2)} \le c\|f\circ\varphi\|_\infty$. Since $\varphi(\mathbf{D})\subseteq\mathbf{D}$, we get

$$\|f\circ\varphi\|^q_{H^q(\beta_2)} \le c\|f\circ\varphi\|_\infty \le c\|f\|_\infty. \tag{2}$$

Now, let $f(z)=\sum_{n=0}^\infty a_n z^n$ be in $H^p(\beta_1)$ and for $1/p+1/r=1$, $\sum_{n=0}^\infty 1/(\beta_1^r(n))<\infty$, by Hölder's inequality we get for all $z\in\mathbf{D}$

$$\begin{aligned}|f(z)| &= \left|\sum_{n=0}^\infty a_n z^n\right| = \left|\sum_{n=0}^\infty a_n\beta_1(n)\frac{z^n}{\beta_1(n)}\right|\\ &\le \left(\sum |a_n|^p\beta_1^p(n)\right)^{1/p}\left(\sum_{n=0}^\infty \frac{|z|^{nr}}{\beta_1^r(n)}\right)^{1/r}\\ &\le \|f\|_{H^p(\beta_1)}\left(\sum_{n=0}^\infty \frac{1}{\beta_1^r(n)}\right)^{1/r}.\end{aligned}$$

Hence, for the constant $c_1=(\sum_{n=0}^\infty 1/(\beta_1^r(n)))^{1/r}$, we get

$$|f\|_\infty \le c_1\|f\|_{H^p(\beta_1)}. \tag{3}$$

Therefore, by using (2) and (3), we have

$$\begin{aligned}\|W_{\psi,\varphi}f\|_{H^q(\beta_2)} &= \|\psi(f\circ\varphi)\|_{H^q(\beta_2)} \le \|\psi\|_\infty\|f\circ\varphi\|_{H^q(\beta_2)}\\ &\le c^{1/q}\|\psi\|_\infty\|f\|_\infty^{1/q} \le c^{1/q}c_1^{1/q}\|\psi\|_\infty\|f\|_{H^p(\beta_1)}^{1/q}.\end{aligned}$$

This proves $W_{\psi,\varphi}$ is bounded. □

From the proof of Theorem 2.1, we get the following corollary if we remove the boundedness condition of the weight sequence $\{\beta_2(n)\}_{n=0}^{\infty}$.

Corollary 2.3. *Let $1 < q \le p < \infty$, and let φ be an analytic self-map of $\mathbf{D}$ and $\psi \in H^\infty$. Let $\beta_1(n)$ and $\beta_2(n)$ be any two weight sequences such that $\beta_1(n) \le \beta_2(n)$ for all n, and $\sum_{n=0}^{\infty}\beta_1^{-r}(n) < \infty$ where $1/p+1/r=1$. If $\|f\circ\varphi\|_{H^q(\beta_2)} \le c\|f\circ\varphi\|_\infty$ for all $f\in H^p(\beta_1)$, then $W_{\psi,\varphi}$ is bounded from $H^p(\beta_1)$ into $H^q(\beta_2)$.*

Now, we find necessary conditions for boundedness of the weighted composition in $H^p(\beta)$. Obviously, one of these necessary conditions is that $\psi \in H^p(\beta)$.

Theorem 2.4. *Let $1 < p < \infty$. Let $\beta(n)$ be a weight sequence such that $\sum_{n=0}^{\infty}\beta^{-q}(n) < \infty$ where $1/p + 1/q = 1$. Let φ be an analytic self-map of $\mathbf{D}$, and let ψ be any analytic function of $\mathbf{D}$. If $W_{\psi,\varphi}$ is bounded in $H^p(\beta)$, then*

$$\sup_{z\in\mathbf{D}}|\psi(z)|^q\sum_{n=0}^{\infty}\frac{|\varphi(z)|^{nq}}{\beta^q(n)} < \infty. \tag{4}$$

Proof. Suppose $W_{\psi,\varphi}$ is bounded, then there is a constant $M > 0$ such that $\|W_{\psi,\varphi}^*(K_z)\|_{(H^p(\beta))^*} \le M\|K_z\|_{(H^p(\beta))^*}$, for all $z\in\mathbf{D}$. Since $W_{\psi,\varphi}^*(K_z) = \overline{\psi(z)}K_{\varphi(z)}$, we get for all $z\in\mathbf{D}$,

$$\begin{aligned}|\psi(z)|\|K_{\varphi(z)}\|_{(H^p(\beta))^*} &\le M\,\|K_z\|_{(H^p(\beta))^*} \Longrightarrow |\psi(z)|^q\,\|K_{\varphi(z)}\|_{H^q(\beta^{p/q})}^q\\ &\le M^q\|K_z\|_{H^q(\beta^{p/q})}^q \Longrightarrow |\psi(z)|^q\sum_{n=0}^{\infty}\frac{|\varphi(z)|^{nq}}{\beta^{pq}(n)}(\beta^{p/q}(n))^q\\ &\le M^q\sum_{n=0}^{\infty}\frac{|z|^{nq}}{\beta^{pq}(n)}(\beta^{p/q}(n))^q\end{aligned}$$

$$\Longrightarrow |\psi(z)|^q \sum_{n=0}^{\infty} \frac{|\varphi(z)|^{nq}}{\beta^q(n)} \leq M^q \sum_{n=0}^{\infty} \frac{|z|^{nq}}{\beta^q(n)}$$
$$\Longrightarrow |\psi(z)|^q \sum_{n=0}^{\infty} \frac{|\varphi(z)|^{nq}}{\beta^q(n)} \leq M^q \sum_{n=0}^{\infty} \frac{1}{\beta^q(n)} < \infty.$$

Hence, we conclude that if $W_{\psi,\varphi}$ is bounded, then

$$\sup_{z\in\mathbf{D}} |\psi(z)|^q \sum_{n=0}^{\infty} \frac{|\varphi(z)|^{nq}}{\beta^q(n)} < \infty. \qquad \square$$

The next theorem gives another necessary condition to the boundedness of $W_{\psi,\varphi}$ in $H^p(\beta)$, where the same hypothesis of Theorem 2.4 holds.

Theorem 2.5. *Let $1 < p < \infty$. Let $\beta(n)$ be a weight sequence such that $\sum_{n=0}^{\infty} \beta^{-q}(n) < \infty$ where $1/p + 1/q = 1$. Let φ be an analytic self-map of $\mathbf{D}$ and ψ any analytic function of $\mathbf{D}$. If $W_{\psi,\varphi}$ is bounded in $H^p(\beta)$, then*

$$\frac{\|\psi\|_{\infty}^q}{\sum_{n=0}^{\infty} \beta^{-q}(n)} \leq \|W_{\psi,\varphi}\|_{H^p(\beta)}^q.$$

Proof. For $z \in \mathbf{D}$, let $f_z = K_z/\|K_z\|_{(H^p(\beta))^*}$. Since $W_{\psi,\varphi}$ is bounded,

$$\|W_{\psi,\varphi}^*(f_z)\| \leq \|W_{\psi,\varphi}\|_{H^p(\beta)}.$$

Then, by Lemma 1.2,

$$\frac{\|\overline{\psi(z)} K_{\varphi(z)}\|_{(H^p(\beta))^*}}{\|K_z\|_{(H^p(\beta))^*}} \leq \|W_{\psi,\varphi}\|_{H^p(\beta)}$$
$$\Longrightarrow |\psi(z)| \frac{\|K_{\varphi(z)}\|_{(H^p(\beta))^*}}{\|K_z\|_{(H^p(\beta))^*}} \leq \|W_{\psi,\varphi}\|_{H^p(\beta)}.$$

Now, for any $z \in \mathbf{D}$,

$$\|K_z\|_{(H^p(\beta))^*}^q = \sum_{n=0}^{\infty} \frac{|z|^{nq}}{\beta^q(n)} \leq \sum_{n=0}^{\infty} \beta^{-q}(n)$$
$$\Longrightarrow \|K_z\|_{(H^p(\beta))^*} \leq \left(\sum_{n=0}^{\infty} \beta^{-q}(n)\right)^{1/q}.$$

Moreover,

$$\|K_{\varphi(z)}\|^q_{(H^p(\beta))^*} \geq \frac{1}{\beta(0)} \Longrightarrow \|K_{\varphi(z)}\|_{(H^p(\beta))^*} \geq 1.$$

Hence, from above argument we have for all $z \in \mathbf{D}$

$$\frac{|\psi(z)|}{\sqrt[q]{\sum_{n=0}^{\infty} \beta^{-q}(n)}} \leq \|W_{\psi,\varphi}\|_{H^p(\beta)}.$$

By taking the supremum over all $z \in \mathbf{D}$, we get the desired result. $\square$

If condition (4) in Theorem 2.4 is satisfied, then there is a positive constant M such that

$$\sup_{z\in\mathbf{D}} |\psi(z)|^q \sum_{n=0}^{\infty} \frac{|\varphi(z)|^{nq}}{\beta^q(n)} = M.$$

Now, let $f \in H^p(\beta)$, then for all $z \in \mathbf{D}$

$$\begin{aligned} |W_{\psi,\varphi}(f)(z)| &= |\psi(z)f(\varphi(z))| = |\psi(z)||\langle f, K_{\varphi(z)}\rangle_\beta| \\ &\leq |\psi(z)|\|K_{\varphi(z)}\|_{(H^p(\beta))^*}\|f\|_{H^p(\beta)} \\ &= |\psi(z)|\left(\sum_{n=0}^{\infty} \frac{|\psi(z)|^{nq}}{\beta^q(n)}\right)^{1/q} \|f\|_{H^p(\beta)}. \end{aligned}$$

By taking the supremum over $z \in \mathbf{D}$, we get

$$\|W_{\psi,\varphi}f\|_\infty \leq M^{1/q}\|f\|_{H^p(\beta)},$$

that is, $W_{\psi,\varphi}(f) \in H^\infty$. Therefore, that gives the sufficiency of the following theorem.

Theorem 2.6. *Suppose the hypothesis of the last theorem is satisfied. Then $W_{\psi,\varphi}$ is bounded from $H^p(\beta)$ into H^∞ if and only if*

$$\sup_{z\in\mathbf{D}} |\psi(z)|^q \sum_{n=0}^{\infty} \frac{|\varphi(z)|^{nq}}{\beta^q(n)} < \infty.$$

Proof. The sufficient condition can be seen from the previous argument. So we are going to prove that condition is necessary. Suppose that $W_{\psi,\varphi} : H^p(\beta) \to H^\infty$ is bounded, then there exists a constant

$M > 0$ such that $\|W^*_{\psi,\varphi}(K_z)\|_{(H^p(\beta))^*} \leq M\|K_z\|_{(H_\infty)^*} = M$, for all $z \in \mathbf{D}$. Since $W^*_{\psi,\varphi}(K_z) = \overline{\psi(z)}K_{\varphi(z)}$, we get for all $z \in \mathbf{D}$

$$|\psi(z)|\|K_{\varphi(z)}\|_{(H^p(\beta))^*} = |\psi(z)|\bigg(\sum_{n=0}^{\infty} \frac{|\varphi(z)|^{nq}}{\beta^q(n)}\bigg)^{1/q} \leq M,$$

which completes the proof. $\square$

3. Compactness of weighted composition operator. It is natural to consider next the compactness of weighted composition operators. Recall that a linear operator on a Banach space is compact if the image of a bounded set under the operator has compact closure. In this section we concentrate our attention on compactness, where we consider the case when the weight sequence $\{\beta(n)\}_{n=0}^{\infty}$ of $H^p(\beta)$ satisfied the condition

$$\sum_{n=0}^{\infty} \frac{1}{\beta^q(n)} \leq \infty, \quad \text{where } \frac{1}{p} + \frac{1}{q} = 1.$$

The next theorem gives a sufficient condition for the compactness of the weighted composition operator $W_{\psi,\varphi}$.

Theorem 3.1. *Let $1 < p < \infty$ and $1/p + 1/q = 1$. Let $\beta(n)$ be any weight sequence such that $\sum_{n=0}^{\infty} \beta^{-q}(n) \leq \infty$, let φ be any analytic self-map of $\mathbf{D}$ and $\psi \in H^\infty$. If*

$$\lim_{|\varphi(z)|\to 1} |\psi(z)|^q \sum_{n=0}^{\infty} \frac{|\varphi(z)|^{nq}}{\beta^q(n)} = 0,$$

then $W_{\psi,\varphi} : H^p(\beta) \to H^\infty$ is compact.

Proof. Suppose $|\psi(z)|^q \sum_{n=0}^{\infty}(|\varphi(z)|^{nq})/(\beta^q(n))$ tends to zero whenever $|\varphi(z)| \to 1$. Hence, for $\varepsilon > 0$, there exists $r < 1$ such that $|\varphi(z)| > r$, and $|\psi(z)|^q \sum_{n=0}^{\infty}(|\varphi(z)|^{nq})/(\beta^q(n)) \leq \varepsilon^q$.

Now, let $\{f_n\}$ be a bounded sequence in $H^p(\beta)$ that converges to zero on compact subsets of $\mathbf{D}$. Then there exists a positive integer n_0 such that, for $n \geq n_0$, we have

$$\sup_{|z|\leq r} |f_n(z)| \leq \frac{\varepsilon}{\|\psi\|_\infty}.$$

Hence, for $n \geq n_0$, we have

$$\begin{aligned}
\|W_{\psi,\varphi}(f_n)\|_\infty &\leq \sup_{|\varphi(z)|<r} |\psi(z) f_n(\varphi(z))| + \sup_{|\varphi(z)|>r} |\psi(z) f_n(\varphi(z))| \\
&\leq \sup_{|\varphi(z)|<r} |\psi(z) f_n(\varphi(z))| + \sup_{|\varphi(z)|>r} |\psi(z)| |\langle f_n, K_{\varphi(z)}\rangle_\beta| \\
&\leq \|\psi\|_\infty \sup_{|w|<r} |f_n(w)| \\
&\quad + \sup_{|\varphi(z)|>r} |\psi(z)| \|K_{\varphi(z)}\|_{(H^p(\beta))^*} \|f_n\|_{H^p(\beta)} \\
&\leq \|\psi\|_\infty \frac{\varepsilon}{\|\psi\|_\infty} + |\psi(z)| \bigg(\sum_{n=0}^\infty \frac{|\varphi(z)|^{nq}}{\beta^q(n)} \bigg)^{1/q} \|f_n\|_{H^p(\beta)} \\
&\leq \varepsilon + (\varepsilon^q)^{1/q} \|f_n\|_{H^p(\beta)}.
\end{aligned}$$

By taking ε sufficiently small, we get the desired result. $\square$

By using Lemma 2.1 and Theorem 3.1, we get the following sufficient condition for the compactness of $W_{\psi,\varphi}$ acting between different weighted Hardy spaces with different weight sequences.

Corollary 3.2. *Let $1 < q \leq p < \infty$. Let φ be an analytic self-map of $\mathbf{D}$ and $\psi \in H^\infty$. Let $\beta_1(n)$ and $\beta_2(n)$ be weight sequences such that $\beta_1(n) \leq \beta_2(n)$, $\beta_2(n)$ is a bounded sequence, $\sum_{n=0}^\infty \beta_1^{-r} < \infty$ where $1/p + 1/r = 1$, and*

$$\lim_{|\varphi(z)|\to 1} |\psi(z)|^r \sum_{n=0}^\infty \frac{|\varphi(z)|^{nr}}{\beta_1^r(n)} = 0.$$

If, for all $f \in H^p(\beta_1)$, $f \circ \varphi \in H^\infty$, then $W_{\psi,\varphi} : H^p(\beta_1) \to H^q(\beta_2)$ is compact.

Proof. Assume that $\beta_2(n)$ is bounded, then by using Lemma 2.2, we get $H^\infty \subseteq H^q(\beta_2)$ for $1 < q < \infty$. Since $\psi \in H^\infty$ and, for all $f \in H^p(\beta_1)$, $f \circ \varphi \in H^\infty$, we get $\psi f \circ \varphi \in H^\infty \subseteq H^q(\beta_2)$. Then there exists a constant $c > 0$ such that

$$\|W_{\psi,\varphi}(f)\|_{H^q(\beta_2)} \leq c \|W_{\psi,\varphi}(f)\|_\infty.$$

Now, by using Theorem 3.1 we get for any bounded sequence $\{f_n\}$ in $H^p(\beta_1)$ that converges uniformly to zero on compact subsets of $\mathbf{D}$

$$\|W_{\psi,\varphi}(f_n)\|_{H^q(\beta_2)} \le c\|W_{\psi,\varphi}(f_n)\|_\infty \longrightarrow 0.$$

Hence, $W_{\psi,\varphi}$ is compact from $H^p(\beta_1)$ into $H^q(\beta_2)$. $\square$

4. Compact composition operators and fixed points. In this section we discuss the relationship between compactness of C_φ on $H^p(\beta)$ and fixed points of the induced map in the unit disk. In the next lemma, the authors of **[7]** proved that the necessary condition for C_φ to be compact is φ has exactly one fixed point in the closed unit disk.

Lemma 4.1. ([**7,** Theorem 1]). *Suppose that φ is an analytic self-map of the open unit disk $\mathbf{D}$ and the composition operator $C_\varphi : H^p(\beta_1) \to H^q(\beta_2)$ is bounded where $1 < q \le p < \infty$ and $\beta_1(n) \le \beta_2(n)$ for all n. Let $\sum_{n=0}^{\infty} \beta_1^{-r}(n) < \infty$, where $1/p + 1/r = 1$. If C_φ is compact, then φ has exactly one fixed point in the closed unit disk $\overline{\mathbf{D}}$.*

To get the fixed point in the open unit disk, we add one more condition, that is $H^p(\beta)$ is a disk-automorphism invariant. We say $H^p(\beta)$ is disk-automorphism invariant if any disk-automorphism map induces a bounded composition operator on $H^p(\beta)$.

Theorem 4.2. *Suppose $H^p(\beta)$ is disk-automorphism invariant and $\sum_{n=0}^{\infty} \beta^{-r}(n) < \infty$, where $1/p + 1/r = 1$. If C_φ is compact on $H^p(\beta)$, then φ has exactly one fixed point in $\mathbf{D}$ and $\|\varphi\|_\infty < 1$.*

Theorem 4.2 can be seen as a special case of Theorem 4.3, take $p = q$ and $\beta_1(n) = \beta_2(n)$ for all n. The Denjoy-Wolff point of φ can be described as the unique fixed point of φ in $\overline{\mathbf{D}}$ at which the modulus of the angular derivative is less than or equal to 1.

Theorem 4.3. *Suppose $1 < q \le p < \infty$, $\beta_1(n)$ and $\beta_2(n)$ satisfy the conditions in Lemma 4.1. Suppose, for any disk-automorphism ψ, C_ψ is bounded from $H^p(\beta_1)$ into $H^q(\beta_2)$. If $C_\varphi : H^p(\beta_1) \to H^q(\beta_2)$ is compact, then φ has exactly one fixed point in $\mathbf{D}$ and $\|\varphi\|_\infty < 1$.*

Proof. Suppose that $1 < q \le p < \infty$, $\beta_1(n) \le \beta_2(n)$ for all n, and $\sum_{n=0}^{\infty} \beta_1^{-r}(n) < \infty$ where $1/p + 1/r = 1$. Then the functions in $H^p(\beta_1)$ and $H^q(\beta_2)$ are continuous on $\overline{\mathbf{D}}$ (see Section 2); therefore, those functions can be evaluated on the boundary of $\mathbf{D}$, $\partial\mathbf{D}$. Suppose that C_φ is compact, then by Lemma 4.1, φ has exactly one fixed point in $\overline{\mathbf{D}}$.

Now we are going to show the fixed point cannot be on the boundary of $\mathbf{D}$. Suppose that φ has a Denjoy-Wolff point λ on $\partial\mathbf{D}$. Let ψ be a disk-automorphism of $\mathbf{D}$ such that $\psi(\lambda) = \lambda$ and $\psi(\varphi(0)) = 0$. Hence, we get a new function $\varphi_1 = \psi \circ \varphi$ with two fixed points 0 and λ in $\overline{\mathbf{D}}$, i.e., $\varphi_1(0) = 0$ and $\varphi_1(\lambda) = \lambda$. Now $C_{\varphi_1} = C_{\psi\circ\varphi} = C_\varphi C_\psi$. Since ψ is disk-automorphism, by the hypothesis, C_ψ is bounded from $H^p(\beta_1)$ into $H^q(\beta_2)$. But C_φ is compact; then C_{φ_1} is compact from $H^p(\beta_1)$ into $H^q(\beta_2)$ and φ_1 has two fixed points in $\overline{\mathbf{D}}$, which is a contradiction to Lemma 4.1. Thus, φ has exactly one fixed point in $\mathbf{D}$.

Finally, we are going to prove $\|\varphi\|_\infty < 1$. Suppose that $\|\varphi\|_\infty = 1$. Then there exist $\alpha, \beta \in [0, 2\pi]$ such that $\varphi(e^{i\alpha}) = e^{i\beta}$. Let ψ be a rotation map such that $\psi(z) = \xi z$ where $\xi = e^{i(\alpha-\beta)}$. Now, $\psi \circ \varphi(e^{i\alpha}) = \psi(e^{i\beta}) = \xi e^{i\beta} = e^{i\alpha}$, that is, $\psi \circ \varphi$ has a fixed point $e^{i\alpha}$ on $\partial\mathbf{D}$. Since ψ is a rotation and C_φ is compact, we get $C_{\psi\circ\varphi} = C_\varphi C_\psi$ is compact from $H^p(\beta_1)$ into $H^q(\beta_2)$ with a fixed point on $\partial\mathbf{D}$, which contradicts the first part of the proof. Hence, $\|\varphi\|_\infty < 1$. $\square$

The next theorem gives a necessary condition for the compactness of the weighted composition operator $W_{\psi,\varphi}$, where we assume that ψ is bounded away from zero on the unit circle, that is, $\liminf_{r\to 1^-} |\psi(r\zeta)| > 0$ for any ζ on the unit circle. As we will see, such ψ forces φ to have a fixed point in the unit disk.

Theorem 4.4. *Let $1 < p < \infty$, and let $\beta(n)$ be a weight sequence such that $\sum_{n=0}^{\infty} \beta^{-q}(n) = \infty$ where $1/p + 1/q = 1$. Let φ be an analytic self-map of $\mathbf{D}$ and ψ a function bounded away from zero on the unit circle. If $W_{\psi,\varphi}$ is compact on $H^p(\beta)$, then the map φ must have its Denjoy-Wolff point in the open unit disk.*

Proof. Suppose that φ has no fixed points in $\mathbf{D}$. Then, by Wolff's lemma, there is a unique fixed point ζ of φ on the unit circle with $|\varphi'(\zeta)| \le 1$. By the Julia-Carathéodory theorem, for all $r > 0$, we have $(1-|\varphi(r\zeta)|)/(1-|r\zeta|) \leq 1$. Then, for $r \in (0,1)$, we have $|\varphi(r\zeta)| \geq |r\zeta|$. Now consider the normalized kernel function $f_r(z) = (K_{r\zeta}(z))/\|K_{r\zeta}\|_{(H^p(\beta))^*}$. By using Lemmas 1.1 and 1.2, we get

$$\|W^*_{\psi,\varphi}(f_r)\|_{(H^p(\beta))^*} = |\psi(r\zeta)| \frac{\|K_{\varphi(r\zeta)}\|}{\|K_{r\zeta}\|} \geq |\psi(r\zeta)|,$$

where the last inequality can be seen by using Lemma 1.1 and $|\varphi(r\zeta)| \geq |r\zeta|$. From Lemma 1.3, we have the normalized kernel function f_r converges weakly to zero as r tends to 1. Since $W^*_{\psi,\varphi}$ is compact $\|W^*_{\psi,\varphi}(f_r)\|_{(H^p(\beta))^*} \to 0$ as r tends to 1, which contradicts the hypothesis that ψ is bounded away from zero on the unit circle. Hence, φ must have its Denjoy-Wolff point in the open unit disk. $\square$

5. Compact composition operators and angular derivatives. In this section we give necessary conditions, in terms of the angular derivative $\varphi'(\zeta)$ where $\zeta \in \partial\mathbf{D}$, for C_φ to be compact in the weighted Hardy spaces $H^p(\beta)$. Recall that any analytic map $\varphi : \mathbf{D} \to \mathbf{D}$ with no fixed point in $\mathbf{D}$ has a unique boundary fixed point (in the sense of radial limit), called the Denjoy-Wolff point, at which the modulus of the angular derivative is less than or equal to 1 (Wolff's lemma). If φ is not the identity nor an elliptic automorphism of $\mathbf{D}$, then its nth iterates $\varphi_n = \varphi \circ \varphi \cdots \circ \varphi$ (n times) converges uniformly on compact subsets of $\mathbf{D}$ to an interior fixed point of φ (if there is one) or to its Denjoy-Wolff boundary point (Denjoy-Wolff theorem). The first two theorems, in this section, are generalizations of the ideas presented in **[2]**.

Notice that Wolff's lemma can be seen as a direct analogue of the Schwarz lemma, where the role of the fixed point at the origin is taken over to a boundary point of $\mathbf{D}$. Moreover, a map with no interior fixed point has a Denjoy-Wolff boundary fixed point which acts very much like an interior fixed point for the map.

Theorem 5.1. *Let $1 < q \leq p < \infty$, let $\beta_1(n)$ and $\beta_2(n)$ be weight sequences such that $\beta_1(n) \leq \beta_2(n)$ for all n, and let $\sum_{n=0}^{\infty} \beta_1^{-r}(n) < \infty$ where $1/p + 1/r = 1$. If φ is an analytic self-map of $\mathbf{D}$ with $|\varphi'(\zeta)| > 1$ for some $\zeta \in \partial\mathbf{D}$ satisfying $|\varphi(\zeta)| = 1$, then C_φ is not compact from $H^p(\beta_1)$ into $H^q(\beta_2)$.*

Proof. Suppose $\varphi(\zeta) = \eta$, for some $\eta \in \partial\mathbf{D}$. Let $\psi(z) = \zeta\overline{\eta}\varphi(z)$. Then $\psi(\zeta) = \zeta\overline{\eta}\varphi(\zeta) = \zeta$ and $\psi'(\zeta) = \zeta\overline{\eta}\varphi'(\zeta)$ (in a radial limit sense). By the Julia-Carathéodory theorem, we have

$$\begin{aligned}\lim_{r\to 1}\varphi'(r\zeta) = \varphi'(\zeta) = d(\zeta)\overline{\zeta}\eta &\Longrightarrow \psi'(\zeta) = \zeta\overline{\eta}\, d(\zeta)\overline{\zeta}\eta = d(\zeta)\\ &= |\varphi'(\zeta)| \Longrightarrow \psi'(\zeta) = |\psi'(\zeta)| = |\varphi'(\zeta)| > 1.\end{aligned}$$

Therefore, by Wolff's lemma, ψ has either an interior fixed point or Denjoy-Wolff point on $\partial\mathbf{D}$ with angular derivative less than or equal to 1, which cannot be ζ. Denote the other fixed point by a. Hence, ψ has two fixed points in $\overline{\mathbf{D}}$, namely, ζ and a. So by Lemma 4.1, C_ψ cannot be compact. Thus, C_φ cannot be compact from $H^p(\beta_1)$ into $H^q(\beta_2)$, as desired. $\square$

For Theorem 5.1 we need to require that the functions in the space have derivatives which extend continuously to $\overline{\mathbf{D}}$. So it will be helpful to present the following observations regarding the reproducing kernel functions for evaluation of the first derivative in $H^p(\beta)$. Recall that the point evaluation function of $H^p(\beta)$ at w is given by

$$K_w(z) = \sum_{n=0}^{\infty} \frac{(\overline{w})^n}{\beta^p(n)} z^n.$$

We know that, for each point $w \in \mathbf{D}$ and a positive integer m, the evaluation of the mth derivative of functions in $H^p(\beta)$ at w is a bounded linear functional and $f^{(m)}(w) = \langle f, K_w^{(m)}\rangle_\beta$, where

$$K_w^{(m)}(z) = \frac{d^m}{d\overline{w}^m} K_w(z) = \sum_{n=m}^{\infty} \frac{n!}{(n-m)!}\frac{(\overline{w})^{n-m}}{\beta^p(n)} z^n.$$

For $m = 1$ and $1/p + 1/q = 1$,

$$\begin{aligned}\|K_w^{(1)}\|^q_{(H^p(\beta))^*} &= \|K_w^{(1)}\|^q_{H^q(\beta^{p/q})}\\ &= \sum_{n=1}^{\infty} \frac{n^q |w|^{(n-1)q}}{\beta^{pq}(n)} (\beta^{p/q}(n))^q\\ &= \sum_{n=1}^{\infty} n^q \frac{|w|^{(n-1)q}}{\beta^q(n)} < \infty,\end{aligned}$$

the convergence of the series follows from the assumption $\liminf_{n\to\infty} \beta(n)^{1/n} = 1$ and root test. Hence, $K_w^{(1)}(z)$ is a bounded point evaluation of $f' \in H^p(\beta)$ whenever $w \in \mathbf{D}$. Moreover, it is clear that $C_\varphi^* K_w^{(1)} = \overline{\varphi'(w)} K_{\varphi(w)}^{(1)}$ since, for $f \in H^p(\beta)$,

$$\begin{aligned}\langle f, C_\varphi^* K_w^{(1)}\rangle_\beta &= \langle C_\varphi(f), K_w^{(1)}\rangle_\beta = \langle f\circ\varphi, K_w^{(1)}\rangle_\beta \\ &= (f\circ\varphi)'(w) = \varphi'(w)\langle f, K_{\varphi(w)}^{(1)}\rangle_\beta \\ &= \langle f, \overline{\varphi'(w)} K_{\varphi(w)}^{(1)}\rangle_\beta .\end{aligned}$$

Theorem 5.2. *Let $1 < p < \infty$, and let $\beta(n)$ be a weight sequence such that $\sum_{n=0}^\infty (n^q)/(\beta^q(n)) < \infty$, where $1/p + 1/q = 1$. Suppose that φ is an analytic self-map of $\mathbf{D}$ with $|\varphi'(\zeta)| = 1$ for some $\zeta \in \partial\mathbf{D}$ satisfying $|\varphi(\zeta)| = 1$. Then C_φ is not compact on $H^p(\beta)$.*

Proof. Suppose $\varphi(\zeta) = \eta$, for some $\eta \in \partial\mathbf{D}$. Let $\psi(z) = \zeta\overline{\eta}\varphi(z)$. Then $\psi(\zeta) = \zeta$ and $\psi'(\zeta) = |\psi'(\zeta)| = |\varphi'(\zeta)| = 1$. Hence, $C_\psi^* K_\zeta = K_{\psi(\zeta)} = K_\zeta$, and $C_\psi^*(K_\zeta^{(1)}) = \overline{\psi'(\zeta)} K_{\psi(\zeta)}^{(1)} = K_\zeta^{(1)}$. Thus, 1 is an eigenvalue of C_ψ^* with multiplicity at least 2. Now if C_φ is compact, and hence C_ψ is compact, then $\dim \operatorname{Ker}(C_\psi - 1) = \dim \operatorname{Ker}(C_\psi^*) \geq 2$. By Wolff's lemma, either ψ has no interior fixed point and ζ is a unique Denjoy-Wolff point of ψ, or ψ has an interior fixed point $a \in \mathbf{D}$.

Let $f \in \operatorname{Ker}(C_\psi - 1)$. Then $f(\psi_n) = f$, where ψ_n is the nth iterate of ψ. If $\zeta \in \partial\mathbf{D}$ is a Denjoy-Wolff point, then by the continuity of f on $\overline{\mathbf{D}}$, we have for all $z \in \mathbf{D}$ $f(z) = f(\psi_n(z)) \to f(\zeta)$, that is, f is constant. If $a \in \mathbf{D}$ is an interior fixed point, since ψ is not an elliptic automorphism of $\mathbf{D}$, we get for all $z \in \mathbf{D}$, $f(z) = f(\psi_n(z)) \to f(a)$. Thus, in both cases, f is a constant. Hence, C_ψ cannot be compact, and therefore C_φ cannot be compact either. $\square$

The essential norm of an operator, denoted by $\|.\|_e$, is its distance in the operator norm from the ideal of compact operators. So, for any weighted composition operator $W_{\psi,\varphi} : H^p(\beta_1) \to H^q(\beta_2)$, we define

$$\|W_{\psi,\varphi}\|_e = \inf_{K\in\mathbf{K}} \|W_{\psi,\varphi} - K\|$$

where $\mathbf{K} = \mathbf{K}(H^p(\beta_1), H^q(\beta_2)$ is the set of compact operators acting from $H^p(\beta_1)$ into $H^q(\beta_2)$. The following lemma is a slight modification of [**1,** Proposition 3.13], which gives the lower bound of the essential norm of weighted composition operator.

Lemma 5.3. *Let $1 < p < \infty$. Let $\beta(n)$ be a weight sequence such that $\sum_{n=0}^{\infty} \beta^{-q}(n) = \infty$, where $1/p + 1/q = 1$. Let K_w be the point evaluation of $H^p(\beta)$. Then*

$$\|W_{\psi,\varphi}\|_e \geq \limsup_{|w|\to 1} \frac{|\psi(w)| \|K_{\varphi(w)}\|_{(H^p(\beta))^*}}{\|K_w\|_{(H^p(\beta))^*}}.$$

In Theorem 5.2, where $\beta(n)$ is a weight sequence satisfying $\sum_{n=0}^{\infty} (n^q)/(\beta^q(n)) < \infty$, we give a condition under which C_φ is not compact. For the case $\sum_{n=0}^{\infty}(1/\beta^q(n)) = \infty$, where $1/p + 1/q = 1$, we have the following theorem.

Theorem 5.4. *Let $1 < p < \infty$, and let $\beta(n)$ be a weight sequence such that $\sum_{n=0}^{\infty} \beta^{-q}(n) = \infty$ where $1/p+1/q = 1$. Let φ be an analytic self-map of $\mathbf{D}$. Suppose C_φ is bounded on $H^p(\beta)$. If there exists $\zeta \in \partial\mathbf{D}$ with $|\varphi'(\zeta)| \leq 1$, then $\|C_\varphi\|_e \geq 1$.*

Proof. Suppose $|\varphi'(\zeta)| \leq 1$ for some $\zeta \in \partial\mathbf{D}$. By the Julia-Carathéodory theorem, for all $r > 0$, we have $(1 - |\varphi(r\zeta)|)/(1 - |r\zeta|) \leq 1$. Then, by an argument similar to one in Theorem 4.4, we have for $r \in (0, 1)$, $|\varphi(r\zeta)| \geq r$. Now, by using Lemma 1.1 and Lemma 5.3 we have

$$\|C_\varphi\|_e \geq \limsup_{r\to 1} \frac{\|K_{\varphi(r\zeta)}\|}{\|K_{r\zeta}\|} \geq 1,$$

where the last inequality can be seen by using Lemma 1.1 and $|\varphi(r\zeta)| \geq r$. Hence, we get the desired result. □

By using Theorem 5.4 and the fact C_φ is compact if and only if $\|C_\varphi\|_e = 0$, we get the following corollary.

Corollary 5.5. *Let $1 < p < \infty$, and let $\beta(n)$ be a weight sequence such that $\sum_{n=0}^{\infty} \beta^{-q}(n) = \infty$ where $1/p + 1/q = 1$. If C_φ is compact on $H^p(\beta)$, then $|\varphi'(\zeta)| > 1$ for all $\zeta \in \partial\mathbf{D}$ whenever $\varphi'(\zeta)$ exists.*

Recall that, in the case $\sum_{n=0}^{\infty}\beta^{-q}(n)<\infty$, we needed $H^p(\beta)$ to be disk-automorphism invariant to prove that compactness of C_φ on $H^p(\beta)$ implies $\|\varphi\|_\infty<1$, and φ has exactly one fixed point in the open unit disk (see Corollary 4.3). In the other case $\sum_{n=0}^{\infty}\beta^{-q}(n)=\infty$ we have the following corollary which can be proved using Corollary 5.5 and Wolff's lemma.

Corollary 5.6. *Let $1<p<\infty$, let φ be an analytic self-map of* $\mathbf{D}$ *and let $\beta(n)$ be a weight sequence such that $\sum_{n=0}^{\infty}\beta^{-q}(n)=\infty$ where $1/p+1/q=1$. If C_φ is compact on $H^p(\beta)$, then φ has a unique fixed point in the open unit disk* $\mathbf{D}$.

Proof. Assume that φ has no fixed point in $\mathbf{D}$, then by using Wolff's lemma there is a unique fixed point ζ in $\partial\mathbf{D}$ with $|\varphi'(\zeta)|\le 1$. By using Corollary 5.5 we get C_φ cannot be compact, which is our desired contradiction. The uniqueness of the interior fixed point is obvious, since the compactness of C_φ implies that φ is not the identity map. $\square$

In Corollary 5.5 if we replace the condition on the weight sequence by a weaker hypothesis $\sum_{n=0}^{\infty}(n^{qj})/(\beta^q(n))=\infty$ for some non-negative integer j and $1/p+1/q=1$ we get Theorem 1 in [**6**]. Moreover, in Corollary 5.6, if we replace the condition on the weight sequence by the hypothesis $\sum_{n=0}^{\infty}(n^{qj})/(\beta^q(n))=\infty$ for some non-negative integer j and $1/p+1/q=1$, we get Corollary 2 in [**6**].

6. Schatten p-class $S_p(H^s(\beta_1),H^q(\beta_2))$. A positive operator T on $H^q(\beta)$ is in the trace class if

$$\mathrm{tr}\,(T)=\sum_{n=1}^{\infty}\langle Te_n,e_n\rangle_\beta<\infty,$$

for some orthonormal basis $\{e_n\}$ of $H^q(\beta)$. More generally, if $0<p<\infty$ and T is a compact operator on $H^q(\beta)$, then we say that T belongs to the Schatten $p-$class S_p if $(T^*T)^{p/2}$ is in the trace class. Also, the S_p norm of T is given by

$$\|T\|_{S_p}=\left[tr(T^*T)^{p/2}\right]^{1/p}.$$

For more information, one can consult Schatten [**4**] and Ringrose [**3**].

In this section, we present sufficient conditions for the membership of Schatten p-class of composition operators C_φ, where these operators C_φ are induced by some special type of function, namely, the functions that have supremum norm strictly smaller than 1. Why are we interested in these functions? We know that in the case of Hardy and Bergman spaces, as well, if $\|\varphi\|_\infty < 1$, then C_φ is compact. It would be interesting to know for which weighted Hardy spaces this result holds. Shapiro **[5]** gave a weighted Hardy space for which this result is not true. Moreover, in the same paper, Shapiro proved a striking compactness theorem, namely, if the sequence $\beta(n)$ is such that the functions in $H^2(\beta)$ are continuous on the close unit disk $\overline{\mathbf{D}}$ and $H^2(\beta)$ is disk-automorphism invariant space, then the compactness of C_φ on $H^2(\beta)$ implies that $\|\varphi\|_\infty < 1$; for the general cases, see Theorems 4.2 and 4.3. In this section, we generalize the ideas presented in **[8]**. If we take $p = s = 2$ and $\beta_1(n) = \beta_2(n)$, we get the results in **[8]**.

Theorem 6.1. *Let $1 < s,\ q < \infty$. Let $\beta_1(n)$ and $\beta_2(n)$ be weight sequences such that $\beta_2(n) \leq \beta_1(n)$ for all n. Suppose that every analytic self-map of $\mathbf{D}$ with supremum norm strictly smaller than 1 induces a bounded composition operator from $H^s(\beta_1)$ into $H^q(\beta_2)$. If φ is an analytic self-map of $\mathbf{D}$ and $\|\varphi\|_\infty < 1$, then C_φ belongs to $S_p(H^s(\beta_1), H^q(\beta_2))$, for $p > 0$.*

Proof. Let $\|\varphi\|_\infty < 1$, from the hypothesis $C_\varphi : H^s(\beta_1) \to H^q(\beta_2)$ be bounded. Let r_1 be such that $\|\varphi\|_\infty < r_1 < 1$. Define $\psi(z) = r_1 z$. Then $\|\psi\|_\infty = r_1 < 1$, and therefore $C_\psi : H^s(\beta_1) \to H^q(\beta_2)$ is bounded. Let $h = 1/(r_1)\varphi$. Then $\|h\|_\infty = 1/(r_1)\|\varphi\|_\infty < 1$. Therefore, $C_h : H^s(\beta_1) \to H^q(\beta_2)$ is bounded. Since $C_\varphi = C_{\psi\circ h} = C_h C_\psi$, it is enough to show C_ψ in $S_p(H^s(\beta_1), H^q(\beta_2))$.

Now, by using the orthonormal basis $\{z^n/(\|z^n\|_{H^s(\beta_1)})\}$ of $H^s(\beta_1)$, we get

$$\begin{aligned}\|C_\psi\|_{S_p}^p &= \sum_{n=0}^{\infty} \left\| C_\psi \frac{z^n}{\|z^n\|_{H^s(\beta_1)}} \right\|_{H^q(\beta_2)}^p \\ &= \sum_{n=0}^{\infty} \frac{1}{\|z^n\|_{H^s(\beta_1)}^p} \|r_1^n z^n\|_{H^q(\beta_2)}^p\end{aligned}$$

$$= \sum_{n=0}^{\infty} r_1^{np} \frac{\|z^n\|^p_{H^s(\beta_2)}}{\|z^n\|^p_{H^s(\beta_1)}} = \sum_{n=0}^{\infty} r_1^{np} \left(\frac{\beta_2(n)}{\beta_1(n)}\right)^p.$$

Since $r_1 < 1$ and $\beta_2(n)/\beta_1(n) \leq 1$ for all n, we get $\|C_\psi\|_{S_p} < \infty$, that is, $C_\psi \in S_p(H^s(\beta_1), H^q(\beta_2))$. Since C_h is bounded, we get the desired result. $\square$

As a special case of Theorem 6.1, we have the next corollary, which will be useful for the rest of this section.

Corollary 6.2. *Let $1 < q < \infty$. Let $\beta(n)$ be a weight sequence. Suppose that every analytic self-map of $\mathbf{D}$ with supremum norm strictly smaller than 1 induces a bounded composition operator on $H^q(\beta)$. If φ is an analytic self-map of $\mathbf{D}$ and $\|\varphi\|_\infty < 1$, then C_φ belongs to the Schatten p-class of $H^q(\beta)$, for $p > 0$.*

Recall that the sufficient condition for $H^\infty \subset H^p(\beta)$ is the boundedness of the weight sequence $\{\beta(n)\}$ (see Lemma 2.2). In the next theorem, by looking only at the maps φ with $\|\varphi\|_\infty < 1$, we show that the boundedness of the weight sequence $\{\beta(n)\}$ is the sufficient condition for C_φ to belong to the Schatten $p-$class of $H^q(\beta)$.

Theorem 6.3. *Let $1 < q < \infty$. Let $\beta(n)$ be a bounded sequence. Then, for any analytic self-map φ of $\mathbf{D}$ with $\|\varphi\|_\infty < 1$, C_φ belongs to $S_p(H^q(\beta))$, for $p > 0$.*

Proof. By using Corollary 6.2, it is enough to show that C_φ is bounded on $H^q(\beta)$. Let f be in $H^q(\beta)$. Since $\|\varphi\|_\infty < 1$, $\overline{\varphi(\mathbf{D})} \subset \mathbf{D}$. Hence, f is continuous on $\overline{\varphi(\mathbf{D})} \subset \mathbf{D}$, that is, $C_\varphi(f) = f \circ \varphi \in H^\infty$. Since β is bounded, by Lemma 2.2, we get $f \circ \varphi \in H^q(\beta)$. Hence, by the closed graph theorem, C_φ is bounded on $H^q(\beta)$, as desired. $\square$

When the weight sequence is unbounded, the question arising from the above theorem is: if $\|\varphi\|_\infty < 1$ and $\varphi \in H^q(\beta)$, is C_φ in the Schatten $p-$class of $H^q(\beta)$? We do not know the answer to this question, but if we put more restriction on the space $H^q(\beta)$ and on the function φ, namely, we assume that $H^q(\beta)$ contains all analytic functions in a neighborhood of the closed unit disk and φ analytic in a neighborhood of the closed unit disk, we have the next theorem.

Theorem 6.4. *Let $1 < q < \infty$. Let $\beta(n)$ be a weight sequence. Suppose $H^q(\beta)$ contains all functions analytic in a neighborhood of $\overline{\mathbf{D}}$, and let φ be analytic in a neighborhood of $\overline{\mathbf{D}}$ with $\|\varphi\|_\infty < 1$. Then C_φ belongs to $S_p(H^q(\beta))$, for $p > 0$.*

Proof. By using Corollary 6.2, we only need to show C_φ is bounded on $H^q(\beta)$ for any analytic function φ in a neighborhood of $\overline{\mathbf{D}}$ with $\|\varphi\|_\infty < 1$. For that φ, we can find a disk $\mathbf{D}_1$ such that $\overline{\mathbf{D}} \subset \mathbf{D}_1$, such that φ analytic on $\mathbf{D}_1$ and $\varphi(\mathbf{D}_1) \subset \mathbf{D}$. Hence, for any $f \in H^q(\beta)$, $f \circ \varphi$ is analytic on $\mathbf{D}_1$, that is, $f \circ \varphi$ analytic in the neighborhood of the closed disk $\overline{\mathbf{D}}$. Then, from the hypothesis, $f \circ \varphi$ is in $H^q(\beta)$. Hence, by the closed graph theorem, we get C_φ is bounded on $H^q(\beta)$, which completes the proof. $\square$

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