

COMPUTING LOCAL CONSTANTS FOR CM ELLIPTIC CURVES

SUNIL CHETTY AND LUNG LI

ABSTRACT. Let E/k be an elliptic curve with CM by \mathcal{O} . We determine a formula for (a generalization of) the arithmetic local constant of [5] at almost all primes of good reduction. We apply this formula to the CM curves defined over \mathbf{Q} and are able to describe extensions F/\mathbf{Q} over which the \mathcal{O} -rank of E grows.

1. Introduction. Let p be an odd rational prime. Let $k \subset K \subset L$ be a tower of number fields, with K/k quadratic, L/K p -power cyclic and L/k Galois with a dihedral Galois group, i.e., a lift of $1 \neq c \in \text{Gal}(K/k)$ acts by conjugation on $g \in \text{Gal}(L/K)$ as $cgc^{-1} = g^{-1}$. In [5] Mazur and Rubin define arithmetic local constants δ_v , one for each prime v of K , which describe the growth in \mathbf{Z} -rank of E over the extension L/K . Specifically (cf., [5, Theorem 6.4]), for $\chi : \text{Gal}(L/K) \hookrightarrow \overline{\mathbf{Q}}^\times$ an injective character and S a set of primes of K containing all primes above p , all primes ramified in L/K and all primes where E has bad reduction,

$$(1.1) \quad \text{rank}_{\mathbf{Z}[\chi]} E(L)^\chi - \text{rank}_{\mathbf{Z}} E(K) \equiv \sum_{v \in S} \delta_v \pmod{2}.$$

To phrase their result this way, we must assume the Shafarevich-Tate conjecture¹, and we keep this assumption throughout.

In [1], the theory of arithmetic local constants is generalized to address the \mathcal{O} -rank of varieties with complex multiplication (CM) by an order \mathcal{O} , and we continue in that direction with specific attention to the elliptic curve case. Following [1], we assume that $\mathcal{O} \subset \text{End}_K(E)$ is the maximal order in a quadratic imaginary field \mathbf{K} , p is unramified in \mathcal{O} , and $\mathcal{O}^c = \mathcal{O}^\dagger = \mathcal{O}$ where \dagger indicates the action of the Rosati

This material is based upon work supported by the National Science Foundation under Grant No. DMS-0757807. A portion of this work was completed as part of the second author's undergraduate capstone research project at Colorado College.

Received by the editors on December 1, 2010, and in revised form on January 11, 2012.

involution (see [6, subsection I.14]). When $\mathbf{K} \not\subset k$, these assumptions imply $K = k\mathbf{K}$.

Our present aim is to provide a simple formula for the local constants δ_v (see Definition 2.2) for primes $v \nmid p$ of good reduction. We then will use a result [1, Section 6] which generalizes (1.1), with \mathbf{Z} replaced by \mathcal{O} , to determine conditions under which the \mathcal{O} -rank of E grows. In Section 3 we will describe, via class field theory, dihedral extensions F/\mathbf{Q} which satisfy those conditions, in order to give some concrete setting to the results of Section 2.

2. Computing the local constant. Suppose p splits² in \mathcal{O} , i.e., $p\mathcal{O} = \mathfrak{p}_1\mathfrak{p}_2$, with $\mathfrak{p}_1 \neq \mathfrak{p}_2$. We denote $R = \mathcal{O}/p\mathcal{O}$ and $R_i = \mathcal{O}/\mathfrak{p}_i$ for $i = 1, 2$, so that $R \cong R_1 \oplus R_2$.

Definition 2.1. If M is an \mathcal{O} -module of exponent p , define the R -rank of M by

$$\text{rank}_R M := (\text{rank}_{R_1} M \otimes_R R_1, \text{rank}_{R_2} M \otimes_R R_2).$$

The following definition is the same as in [1, 5]. Fix a prime v of K , and let u and w be primes of k below v and of L above v , respectively. Denote k_u, K_v and L_w for the completions of k, K and L at u, v and w , respectively. If $L_w \neq K_v$, let L'_w be the extension of K_v inside L_w with $[L_w : L'_w] = p$, and otherwise let $L'_w = L_w = K_v$.

Definition 2.2. Define the arithmetic local constant $\delta_v := \delta(v, E, L/K)$ by

$$\delta_v \equiv \text{rank}_R \frac{E(K_v)}{E(K_v) \cap N_{L_w/L'_w} E(L_w)} \pmod{2}.$$

Now, we will consider primes v of K such that E has good reduction at $v, v \nmid p, v^c = v$ and v ramifies in L/K (corresponding to [5, Lemma 6.6]). Under these conditions, [5, Theorem 5.6] shows that

$$(2.1) \quad \dim_{\mathbf{F}_p} \frac{E(K_v)}{E(K_v) \cap N_{L_w/L'_w} E(L_w)} \equiv \dim_{\mathbf{F}_p} E(K_v)[p] \pmod{2}.$$

Proposition 2.4 below shows that we are able to replace $\dim_{\mathbf{F}_p}$ by rank_R in (2.1). We first need Lemma 2.3, which follows Lemmas 5.4 and 5.5 of [5], and our proof is meant only to address the change from $\dim_{\mathbf{F}_p}$ to rank_R .

Let \mathcal{K} and \mathcal{L} be finite extensions of \mathbf{Q}_ℓ , with $\ell \neq p$, and suppose \mathcal{L}/\mathcal{K} is a finite extension.

Lemma 2.3. *Suppose \mathcal{L}/\mathcal{K} is cyclic of degree p , E is defined over \mathcal{K} and has good reduction. Then:*

- (i) $\text{rank}_R E(\mathcal{K})/pE(\mathcal{K}) = \text{rank}_R E(\mathcal{K})[p]$.
- (ii) *If \mathcal{L}/\mathcal{K} is ramified, then $E(\mathcal{K})/pE(\mathcal{K}) = E(\mathcal{L})/pE(\mathcal{L})$ and*

$$N_{\mathcal{L}/\mathcal{K}}E(\mathcal{L}) = pE(\mathcal{K}).$$

- (iii) *If \mathcal{L}/\mathcal{K} is unramified, then $N_{\mathcal{L}/\mathcal{K}}E(\mathcal{L}) = E(\mathcal{K})$.*

Proof. When $\ell \neq p$, we have $E(\mathcal{K})/pE(\mathcal{K}) = E(\mathcal{K})[p^\infty]/pE(\mathcal{K})[p^\infty]$. Since $E(\mathcal{K})[p^\infty]$ is finite, (i) follows from the exact sequence of \mathcal{O} -modules

$$0 \longrightarrow E(\mathcal{K})[p] \longrightarrow E(\mathcal{K})[p^\infty] \longrightarrow pE(\mathcal{K})[p^\infty] \longrightarrow 0.$$

The content of (ii) and (iii) is on the level of sets, so the proof is exactly as in [5, Lemma 5.5]. □

We return to the notation of Definition 2.2.

Proposition 2.4. *If $v \nmid p$ and L_w/K_v is nontrivial and totally ramified, then*

$$\delta_v \equiv \text{rank}_R E(K_v)[p] \pmod{2}.$$

Proof. As in [5, Proof of Theorem 5.6], Lemma 2.3 (ii) yields $N_{L_w/L'_w} E(L_w) = pE(L'_w)$, and hence $E(K_v) \cap pE(L'_w) = pE(K_v)$. So by Definition 2.2 and Lemma 2.3 (i),

$$\delta_v \equiv \text{rank}_R \frac{E(K_v)}{pE(K_v)} \equiv \text{rank}_R E(K_v)[p] \pmod{2}. \quad \square$$

Now, fix a prime v of K . We denote κ_u for the residue field of k_u , $q = \#\kappa_u$ for the size of finite field κ_u and \tilde{E} for the reduction of E to κ_u .

Proposition 2.5. *Suppose $v \nmid p$, v is ramified in L/K and $v^c = v$. If E has good reduction at v , then $\delta_v \equiv (1, 1)$ if and only if $p \mid \#\tilde{E}(\kappa_u)$.*

Proof. We follow the notation of [5, Lemma 6.6]. Since $v^c = v$, we know that K_v/k_u is quadratic, and it is unramified by [5, Lemma 6.5 (ii)]. Let Φ be the Frobenius generator of $\text{Gal}(K_v^{ur}/k_u)$, so Φ^2 is the Frobenius of $\text{Gal}(K_v^{ur}/K_v)$.

The proof of Lemma 6.6 [5] shows that the product of the eigenvalues α, β of Φ on $E[p]$ is -1 . Also, $E(K_v)[p] = E[p]^{\Phi^2=1}$ is equal (as a set) to $E[p]$ or is trivial depending on whether or not $\{\alpha, \beta\} = \{1, -1\}$, respectively. Since E has CM by \mathcal{O} , $E[p]$ is a rank 1 R -module (see, e.g., [10, Section II.1]), so the former case yields

$$\delta_v \equiv \text{rank}_R E(K_v)[p] = (1, 1) \pmod{2}.$$

By assumption, $v \nmid p$, so p is prime to the characteristic of κ_u , and therefore the reduction map restricted to p -torsion is injective [9, subsection VII.3]. We also know $E[p]$ is unramified [9, subsection VII.4], and so the eigenvalues of Φ acting on $E[p]$ coincide \pmod{p} with the eigenvalues of the q -power Frobenius map ϕ_q on $\tilde{E}[p]$. We know [9, Section V] that the characteristic polynomial of ϕ_q is $T^2 - aT + q$, where $a = q + 1 - \#\tilde{E}(\kappa_u)$, and from the above comments $q \equiv -1 \pmod{p}$. Therefore, Φ having eigenvalues ± 1 is equivalent to $a \equiv 0 \pmod{p}$ and in turn equivalent to $\#\tilde{E}(\kappa_u) \equiv 0 \pmod{p}$. \square

Corollary 2.6. *If $\mathbf{K} \not\subset k$, then $\delta_v \equiv (1, 1)$.*

Proof. To see that $p \mid \#\tilde{E}(\kappa_u)$, we show that $a = 0$ under our assumptions on v , where $a = q + 1 - \#\tilde{E}(\kappa_u)$ as above³. The theory of complex multiplication gives $a = \pi_u + \bar{\pi}_u$ for some $\pi_u \in \mathcal{O}$ such that $\pi_u \bar{\pi}_u = q$ (see, e.g., [3, Theorem 14.16], [10, subsection II.10] or [8] for a thorough discussion). As $\mathbf{K} \not\subset k$, we have $K = k\mathbf{K}$, and we let $\psi = \psi_{E/K}$ be the Grössencharacter associated to E and K (see

[10, subsection II.9] or [4]). By comparing their effect on $K(E[\ell])$, where ℓ is prime to v , we see that $\psi(v)^c = \psi(v^c)$, and since $v = v^c$, we have that $\psi(v)$ is fixed by c . It follows that $\psi(v)$ is rational, the corresponding $\pi_v \in \mathcal{O} \subset \text{End}_K(E)$ is integral, and in fact, $\pi_v = \pm q$ by degree arguments. In addition, $\pi_u^2 = \pi_v$, and we will see that $\pi_u = \sqrt{-q}$ is purely imaginary. Indeed, π_u having no real part implies $a = \pi_u + \bar{\pi}_u = 0$; hence,

$$\#\tilde{E}(\kappa_u) \equiv q + 1 \equiv 0 \pmod{p}$$

and $\delta_v \equiv (1, 1)$ by Proposition 2.5.

Suppose instead that $\pi_u = \sqrt{q}$ is real⁴. If, in addition, we suppose π_u is integral then the reduction $\phi_q \in \text{End}(\tilde{E})$ of π_u would commute with all endomorphisms of \tilde{E} . As $\mathbf{K} \not\subset k$, there is some $\rho \in \text{End}_K(E)$ such that $\rho \neq \rho^c$, and hence, $\tilde{\rho} \neq \tilde{\rho}^c$. Thus, for some $P \in \tilde{E}(\kappa_u)$, $P^c = P$ and $\tilde{\rho}(P^c) \neq \tilde{\rho}^c(P)$. As the action of c on κ_u coincides with that of Frobenius $\tilde{\Phi}$, it follows that $\tilde{\rho}$ does not commute with $\tilde{\Phi}$, and in turn $\tilde{\rho}$ does not commute with the Frobenius endomorphism $\phi_q \in \text{End}(\tilde{E})$ induced by $\tilde{\Phi}$.

If $\pi_u = \sqrt{q}$ is real and irrational, then $k \subsetneq \mathbf{Q}(\pi_u)k \subset K$ and so $c \in \text{Gal}(K/k)$ acts non-trivially on π_u , i.e., $\pi_u^c = -\sqrt{q}$. It follows that

$$q = N_{\mathbf{K}/\mathbf{Q}}(\pi_u) = \pi_u \pi_u^c = -q,$$

which is a contradiction, and we conclude π_u is purely imaginary as desired. \square

Define a set \mathfrak{S}_L of primes v of K by

$$\mathfrak{S}_L := \{v \mid p, \text{ or } v \text{ ramifies in } L/K, \text{ or where } E \text{ has bad reduction}\}.$$

Theorem 2.7 [1, Theorem 6.1]. *Let $\chi : \text{Gal}(L/K) \hookrightarrow \overline{\mathbf{Q}}^\times$ be an injective character and $\mathcal{O}[\chi]$ the extension of \mathcal{O} by the values of χ . Assuming the Shafarevich-Tate conjecture,*

$$\text{rank}_{\mathcal{O}[\chi]} E(L)^\chi - \text{rank}_{\mathcal{O}} E(K) \equiv \sum_{v \in \mathfrak{S}_L} \delta_v \pmod{2}.$$

We now consider a dihedral tower $k \subset K \subset F$ where F/K is p -power abelian. Following [5, Section 3], we note that there is a bijection between cyclic extensions L/K in F and irreducible rational representations ρ_L of $G = \text{Gal}(F/K)$. The semi-simple group ring $\mathbf{K}[G]$ decomposes as

$$\mathbf{K}[G] \cong \oplus_L \mathbf{K}[G]_L,$$

where $\mathbf{K}[G]_L$ is the ρ_L -isotypic component of $\mathbf{K}[G]$. For each L , it suffices to deal with an injective character $\chi : \text{Gal}(L/K) \hookrightarrow \overline{\mathbf{Q}}^\times$ appearing in the direct-sum decomposition of $\rho_L \otimes \overline{\mathbf{Q}}^\times$, and $\text{rank}_{\mathcal{O}[\chi]} E(F)^\times$ is independent⁵ of the choice of χ .

Theorem 2.8. *Assume $\mathbf{K} \not\subset k$.⁶ Suppose that, for every prime v satisfying $v^c = v$ and which ramifies in F/K , we have $v \nmid p$ and E has good reduction at v . For m equal to the number of such v , if $\text{rank}_{\mathcal{O}} E(K) + m$ is odd, then*

$$\text{rank}_{\mathcal{O}} E(F) \geq [F : K].$$

Proof. Fix a cyclic extension L/K inside F . If v is a prime of K and $v^c \neq v$, then $\delta_v \equiv \delta_{v^c}$ and hence $\delta_v + \delta_{v^c} \equiv (0, 0) \pmod{2}$ by [5, Lemma 5.1]. If $v^c = v$ and v is unramified in L/K , then v splits completely in L/K by [5, Lemma 6.5 (i)]. It follows that N_{L_w/L'_w} is surjective, and so $\delta_v \equiv (0, 0)$ by Definition 2.2. The remaining primes v are precisely those named in the assumption, so Proposition 2.6 gives $\sum_v \delta_v \equiv (m, m) \pmod{2}$. Thus,

$$\text{rank}_{\mathcal{O}[\chi]} E(L)^\times \equiv \text{rank}_{\mathcal{O}} E(K) + m \pmod{2},$$

and we have assumed that the right-hand side is odd.

From [5, Corollary 3.7], it follows that

$$\text{rank}_{\mathcal{O}} E(F) = \sum_L (\dim_{\mathbf{Q}} \rho_L) \cdot (\text{rank}_{\mathcal{O}[\chi]} E(L)^\times).$$

As the previous paragraph applies for every cyclic L/K in F we see from the decomposition of $\mathbf{K}[G]$ that $E(F) \otimes \mathbf{Q}$ contains a submodule isomorphic to $\mathbf{K}[G]$, and the claim follows. \square

3. CM elliptic curves defined over \mathbf{Q} . Here, we will consider the CM elliptic curves E defined over \mathbf{Q} (as in [10, A.3]). For each E , our aim is to determine⁷ examples of dihedral towers $\mathbf{Q} \subset K \subset F$ over which, according to Theorem 2.8, the \mathcal{O} -rank of E grows. As we have assumed $\mathcal{O} \subset \text{End}_K(E)$, we will consider towers in which $K = \mathbf{K}$ (see Section 1). All of our calculations will be done using Sage [11].

Let E_D/\mathbf{Q} be the elliptic curve of minimal conductor⁸ defined over \mathbf{Q} with CM by $K_D = \mathbf{Q}(\sqrt{-D})$. We determine computationally⁹ $\text{rank}_{\mathbf{Z}} E_D(K_D)$, and for $D = 3$, we see that this group is finite. For $D = 4, 7$, the situation is less certain, as Sage only tells us that $E_D(\mathbf{Q})$ is finite and $\text{rank}_{\mathbf{Z}} E_D(K_D) \leq 2$. For each of the remaining CM curves E_D defined over \mathbf{Q} , one can (provably) calculate that $\text{rank}_{\mathbf{Z}} E_D(\mathbf{Q}) = 1$. We also have that $\text{rank}_{\mathbf{Z}} E_D(K_D) \geq \text{rank}_{\mathbf{Z}} E_D(\mathbf{Q}) = 1$ and $\text{rank}_{\mathbf{Z}} E_D(K_D)$ cannot be even, so $\text{rank}_{\mathcal{O}} E_D(K_D) \geq 1$. For $D = 8, 11, 19, 43, 67$ and 163 , Sage gives an upper bound⁷ of 3 for $\text{rank}_{\mathbf{Z}} E_D(K_D)$ and so, for these D , we can conclude that in fact $\text{rank}_{\mathcal{O}} E_D(K_D) = 1$.

3.1. Dihedral extensions of \mathbf{Q} . Recall that p is a fixed odd rational prime. Presently, we also fix $D \in \{3, 4, 7, \dots, 163\}$, and let $E = E_D, K = K_D$. We are interested in abelian extensions F/K which are dihedral over \mathbf{Q} , and these are exactly the extensions contained in the ring class fields of K (see [3, Theorem 9.18]).

Let \mathcal{O}_f be an order in \mathcal{O}_K of conductor f . We have a simple formula for the class number $h(\mathcal{O}_f)$ of \mathcal{O}_f using, for example, [3, Theorem 7.24], and noting that, we have $h(\mathcal{O}_K) = 1$,

$$h(\mathcal{O}_f) = \frac{f}{[\mathcal{O}_K^\times : \mathcal{O}_f^\times]} \cdot \prod_{\text{primes } \ell | f} \left(1 - \left(\frac{-D}{\ell} \right) \frac{1}{\ell} \right).$$

For $D \neq 3, 4$, we have $\mathcal{O}_K^\times = \{\pm 1\}$ and, for $D = 4$, we have $\#\mathcal{O}_K^\times = 4$, so in both of these cases $[\mathcal{O}_K^\times : \mathcal{O}_f^\times]$ is prime to p . For $D = 3$, one can show that $[\mathcal{O}_K^\times : \mathcal{O}_f^\times] = 3$ when $f > 1$. The following paragraphs require only minor adjustments for the case $p = D = 3$.

Taking f to be an odd rational prime such that $(-D/f) = \pm 1$, the class number becomes $h(\mathcal{O}_f) = f \mp 1$, and so the ring class field $H_{\mathcal{O}_f}$ associated to \mathcal{O}_f is an abelian extension of K of degree $f \mp 1$. Thus, for $f \equiv \pm 1 \pmod{p}$, we have a (non-trivial) p -power subextension F/K which is dihedral over \mathbf{Q} .

Next, we need to understand the ramification in F/K . As K has class number 1, we know there are no unramified extensions of K , and so we must ensure that F satisfies the hypotheses of Theorem 2.8. A prime v of K ramifies in $H_{\mathcal{O}_f}/K$ if and only if $v \mid f\mathcal{O}_K$ (see, for example, [3, Exercise 9.20] and recall f is odd). If we choose f so that $-D$ is not a square $(\bmod f)$, f is inert in K/\mathbf{Q} , and so $f\mathcal{O}_K$ is prime and, moreover, the only prime that ramifies in $H_{\mathcal{O}_f}/K$. If $f\mathcal{O}_K$ does not ramify in F/K , then the p -extension F/K is contained in the Hilbert class field H_K of K . As $H_K = K$, this is impossible, so $f\mathcal{O}_K$ ramifies in F/K and no other primes ramify in F/K . Taking f such that $f \nmid D$ and $-D$ is a square $(\bmod f)$, we have that f is not inert and does not ramify in K/\mathbf{Q} . As in the previous case, the primes of K above f both ramify in the p -extension F/K contained in $H_{\mathcal{O}_f}$.

Now, suppose $\text{rank}_{\mathcal{O}}E(K)$ is odd¹⁰. To apply Theorem 2.8, we must have an even number m of primes v such that $v^c = v$, v ramifies in F/K , E has good reduction at v and for which $p \mid \#\tilde{E}(\mathbf{Z}/f\mathbf{Z})$. First, we can guarantee $m = 0$ if the only primes v which ramify in F/K do not satisfy $v^c = v$, e.g., taking $f \nmid D$ with $(-D/f) = 1$. Table 3.1 gives, for each D and for $p = 3, 5, 7$, the smallest prime f which gives an extension of degree p following this recipe. We note that we do not need Proposition 2.5 for this case.

If we wish to allow for primes v satisfying $v^c = v$, we choose two p -extensions F_1 and F_2 from two distinct rational primes f_i as above with $f_i \equiv -1 \pmod{p}$ and $(-D/f_i) = -1$, for $i = 1, 2$. The compositum $F = F_1F_2$ will satisfy our requirements. Indeed, firstly F is an abelian p -extension of K and is contained in the ring class field $H_{\mathcal{O}_{f_1f_2}}$, hence dihedral over \mathbf{Q} with only $f_1\mathcal{O}_K$ and $f_2\mathcal{O}_K$ ramifying in F/K . Secondly, as each f_i is inert in K/\mathbf{Q} , each is a supersingular prime for E (this follows from the arguments in Corollary 2.6) and hence p divides $\#\tilde{E}(\mathbf{Z}/f_i\mathbf{Z}) = f_i + 1$. Thus, E and the p -extension F/K satisfy the hypotheses of Theorem 2.8. Table 3.2 below gives, for each D and for $p = 3, 5, 7$, the smallest pair of distinct primes f_1, f_2 which give extensions of degree p^2 following this recipe.

Next, suppose $\text{rank}_{\mathcal{O}}E(K)$ is even.¹¹ In this case, we need m to be odd in order to apply Theorem 2.8. The same ideas as above still work, and in Table 3.3 we list, for each D and for $p = 3, 5, 7$, the smallest prime f for which Theorem 2.8 guarantees $\text{rank} \geq p$.

Remark 3.1. Though there are algorithms in the literature to compute the defining polynomial of a class field (e.g., [2, Section 6], [3, subsection 11-3]) and such computational problems are of interest independently, we make no attempt here to explicitly determine the ring class fields $H_{\mathcal{O}_f}$. As is apparent from Table 3.2, our method of determining a field to which Theorem 2.8 applies involves ring class fields of large degree in a computationally inefficient way.

TABLE 3.1. Case $m = 0$.

D	$p = 3$		$p = 5$		$p = 7$	
	f	$[F : K]$	f	$[F : K]$	f	$[F : K]$
4	13	3	41	5	29	7
7	43	3	11	5	29	7
8	43	3	11	5	43	7
11	31	3	31	5	71	7
19	7	3	11	5	43	7
43	13	3	11	5	127	7
67	103	3	71	5	29	7
163	43	3	41	5	43	7

TABLE 3.2. Case $m = 2$.

D	$p = 3$			$p = 5$			$p = 7$		
	f_1	f_2	$[F : K]$	f_1	f_2	$[F : K]$	f_1	f_2	$[F : K]$
4	11	23	9	19	59	25	83	139	49
7	5	41	9	19	59	25	13	41	49
8	5	23	9	29	79	25	13	167	49
11	2	29	9	29	79	25	13	41	49
19	2	29	9	29	59	25	13	41	49
43	2	5	9	19	29	25	223	349	49
67	2	5	9	79	109	25	13	41	49
163	2	5	9	19	29	25	13	139	49

TABLE 3.3. Case $m = 1$.

	$p = 3$		$p = 5$		$p = 7$	
D	f	$[F : K]$	f	$[F : K]$	f	$[F : K]$
3	17	3	29	5	41	7
4	11	3	19	5	83	7
7	5	3	19	5	13	7

Acknowledgments. The first author would like to thank Colorado College for support during his Riley-scholar post-doctoral fellowship. We would also like to thank the referee for the comments and suggestions, particularly regarding Proposition 2.5 and Corollary 2.6.

ENDNOTES

1. Without this assumption, all statements regarding \mathcal{O} -rank of E would be replaced by analogous statements regarding $\mathcal{O} \otimes \mathbf{Z}_p$ -corank of the p^∞ -Selmer group $\text{Sel}_{p^\infty}(E/K)$ of E .

2. The simpler case of p being inert in \mathcal{O} , i.e., $\mathcal{O}/p\mathcal{O}$ is a field, is treated similarly.

3. That $a = 0$ in this case is known (see [10, Exercise 2.30], [4, Section 4, Theorem 10] or [7, Theorem 7.46] for generalization to higher dimensional abelian varieties); we include an argument for completeness.

4. The case $\pi_u = -\sqrt{q}$ follows the same argument.

5. We could instead write that $\dim_{\overline{\mathbf{Q}}}(E(F) \otimes \overline{\mathbf{Q}})^\chi$ is independent of the choice of χ .

6. The case $\mathbf{K} \subset k$ is similar, with m equal to the number of v such that $p \mid \#\tilde{E}(\kappa_v)$.

7. Determined up to the correspondence of class field theory.

8. See [10, page 483], with $f = 1$ (in Silverman's notation), for a Weierstrass equation.

9. Specifically with Sage's interface to John Cremona's 'mwrnk' and Denis Simon's 'simon_two_descent.'

10. The cases $D = 8, 11, \dots, 163$, and possibly $D = 4, 7$.
11. The case $D = 3$, and possibly $D = 4, 7$.

REFERENCES

1. S. Chetty, *Arithmetic local constants for abelian varieties with complex multiplication*, <http://www.csbsju.edu/Mathematics/sunil-chetty.htm> (preprint).
2. H. Cohen, *Advanced topics in computational number theory*, Grad. Texts Math. **193**, Springer, New York, 2000.
3. D. Cox, *Primes of the form $x^2 + ny^2$: Fermat, class field theory, and complex multiplication*, Wiley Interscience, New York, 1989.
4. S. Lang, *Elliptic functions*, second edition, Grad. Texts Math. **112**, Springer, New York, 1987.
5. B. Mazur and K. Rubin, *Finding large Selmer rank via an arithmetic theory of local constants*, Ann. Math. **166** (2007), 581–614.
6. J.S. Milne, *Abelian varieties*, in *Arithmetic geometry*, G. Cornell and J. Silverman, eds., Springer-Verlag, New York, 1986, available at <http://www.jmilne.org/math/>.
7. G. Shimura, *Introduction to the arithmetic theory of automorphic functions*, Princeton University Press, Princeton, 1971.
8. A. Silverberg, *Group order formulas for reductions of CM elliptic curves*, in *Arithmetic, geometry, cryptography and coding theory*, Contemp. Math. **521**, American Mathematical Society, 2010.
9. J. Silverman, *Arithmetic of elliptic curves*, Grad. Texts Math. **106**, Springer, New York, 1986.
10. ———, *Advanced topics in the arithmetic of elliptic curves*, Grad. Texts Math. **151**, Springer, New York, 1994.
11. W.A. Stein, et al., *Sage mathematics software* (Version 4.2.1), The Sage Development Team, 2009, <http://www.sagemath.org>.

MATHEMATICS DEPARTMENT, COLLEGE OF ST. BENEDICT AND ST. JOHN'S UNIVERSITY, 37 SOUTH COLLEGE AVENUE, ST. JOSEPH, MN 56374

Email address: schetty@csbsju.edu

DEPARTMENT OF COMPUTER SCIENCE, RICE UNIVERSITY, 6100 MAIN STREET, HOUSTON TX 77005

Email address: lung.li@rice.edu