ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 44, Number 3, 2014

A DESCENT HOMOMORPHISM FOR SEMIMULTIPLICATIVE SETS

BERNHARD BURGSTALLER

ABSTRACT. We define and provide some basic analysis of various types of crossed products by semimultiplicative sets, and then prove a KK-theoretical descent homomorphisms for semimultiplicative sets in accord with the descent homomorphism for discrete groups.

1. Introduction. An associative semimultiplicative set is a set G together with a partially defined associative multiplication. For instance, categories, groupoids, semigroups, inverse semigroups and groups are associative semimultiplicative sets. An equivariant KK-theory for semimultiplicative sets is defined in **[5]**, and in this theory the G-action is realized by linear (non-adjointable) partial isometries on C^*-algebras and Hilbert modules. In this paper we prove a descent homomorphism for KK^G and various types of crossed products,

$$KK^{H\times G}(A,B) \longrightarrow KK^H(A \rtimes G, B \rtimes G),$$

see Theorem 13.4, parallel to Kasparov's descent homomorphism for groups (**[9]**). We consider four types of crossed products: the reduced one, the full one, the full strong one and another one for so-called inversely generated semigroups.

This work originated in an attempt to generalize the Baum-Connes map for discrete groups **[1]** to discrete semimultiplicative sets. If G is an inverse semigroup, then this seems conceptually (and at least partially) to work, see **[3, 4]**. If G is not an inverse semigroup, then still certain reduced crossed products $A \rtimes_r G$ are isomorphic to inverse semigroup crossed products $A \rtimes S$, see Corollary 7.11, and so for these crossed products one has potentially a Baum-Connes theory.

2010 AMS *Mathematics subject classification.* Primary 19K35, 20N02, 46L55.

The author was supported by Czech MEYS Grant LC06002.

Received by the editors on August 18, 2011, and in revised form on February 11, 2012.

DOI:10.1216/RMJ-2014-44-3-809

In the full crossed product of a semimultiplicative set, however, one usually has non-commuting source and range projections of the underlying partial isometries, and this turns out to be an obstacle in constructing a Baum-Connes map similarly as for groups and groupoids: these Baum-Connes maps can be constructed by a combination of a descent homomorphism and an averaging map. Averaging, however, fails for semimultiplicative sets and their induced non-commuting projections on modules. (But even for inverse semigroups one cannot directly average but needs to slice modules at first (see **[3]**)).

Roughly speaking, the theory of crossed products by semimultiplicative sets is a theory of C^*-algebras generated by partial isometries. Hence, we generalize this point of view by also considering inversely generated semigroups, which are $*$-semigroups that are generated by their invertible elements.

We give a brief overview of this paper. In Sections 2–3 we recall the basic definitions of equivariant KK-theory for semimultiplicative sets from **[5]**. In Section 4 we prove some facts about partial isometries in connection with G-actions. Sections 5–8 and Section 10 are dedicated to the definition of the various crossed products; Section 10 also includes the definition of equivariant KK-theory for inversely generated semigroups. In Section 9 we compare semimultiplicative set G-equivariant KK-theory with Kasparov's G-equivariant KK-theory when G is a group. Sections 11–13 occupy the proof of the descent homomorphism, which is an adaption of Kasparov's proof in **[9]**.

2. Semimultiplicative sets.

Definition 2.1. A (general) *semimultiplicative set* G is a set endowed with a subset $G^{(2)} \subseteq G \times G$ and a map (written as a multiplication)

$$G^{(2)} \longrightarrow G : (s,t) \longmapsto st$$

satisfying the following weak associativity condition: $s(tu) = (st)u$ whenever both expressions are defined ($s,t,u \in G$).

Definition 2.2. A semimultiplicative set G is called *associative* if whenever $(st)u$ or $s(tu)$ is defined, then both $(st)u$ and $s(tu)$ are defined ($s,t,u \in G$).

There is a similar notion called a *semigroupoid* [**7**]. A semigroupoid is an associative semimultiplicative set with the property that $(st)u$ is defined if and only if st and tu are defined. For instance, groupoids and small categories are semigroupoids. In general, however, an associative semimultiplicative set is not a semigroupoid, a typical example being a ring R without the zero element, so the semimultiplicative set $G = R\backslash\{0\}$ under the multiplication inherited from R. Examples for associative semimultiplicative sets include groups, groupoids, small categories, inverse semigroups, semigroups and semigroupoids. An associative semimultiplicative set is also called a partial semigroup in the literature (see [**2**]).

We remark that the weak associativity condition for a general semimultiplicative set is not essential in this paper. A general semimultiplicative set is always realized by associative actions, so we require the weak associativity without essential loss of generality. However, for instance, an arbitrary subset of a group is a general but not necessarily an associative semimultiplicative set. Now the point is that general and associative semimultiplicative sets G yield different classes of actions, since G has to be realized by partial isometries.

If an associative semimultiplicative set G has left cancelation, that is, for all $s, t_1, t_2 \in G$, $st_1 = st_2$ implies $t_1 = t_2$, then we are able to define a left reduced C^*-algebra for G. Write $(e_g)_{g\in G}$ for the canonical base in $\ell^2(G)$.

Definition 2.3. Let G be an associative semimultiplicative set with left cancelation. The *left regular representation* of G is the map $\lambda : G \to B(\ell^2(G))$ given by

$$\lambda_g\left(\sum_{h\in G} \alpha_h e_h\right) = \sum_{h\in G,\, gh \text{ is defined}} \alpha_h e_{gh},$$

where $\alpha_h \in \mathbf{C}$. The C^*-subalgebra of $B(\ell^2(G))$ generated by $\lambda(G)$ is called the *reduced C^*-algebra* of G and denoted by $C^*_r(G)$.

Definition 2.4. A *morphism* $\phi : G \to H$ between two semimultiplicative sets G and H is a map satisfying $\phi(gh) = \phi(g)\phi(h)$ whenever gh is defined $(g, h \in G)$.

Definition 2.5. An *anti-morphism* $\varphi : G \to H$ between semimultiplicative sets G and H is a map satisfying $\varphi(gh) = \varphi(h)\varphi(g)$ whenever gh is defined $(g, h \in G)$.

Definition 2.6. A *left action* of a semimultiplicative set G on a set X is given by a subset $Y \subseteq G \times X$ and a map

$$Y \longrightarrow X,\ (g, x) \mapsto gx$$

such that, if gh is defined, then $(gh)x$ is defined if and only if $g(hx)$ is defined, and in this case $(gh)x = g(hx)$ $(g, h \in G, x \in X)$.

By the last definition, we see that a G-action on a set is a morphism $\phi : G \to \mathrm{PartFunc}\,(X)$ from G into the set of partial functions on X. (That is, if gh is defined, then $\phi(gh) = \phi(g) \circ \phi(h)$ and the domain of both sides coincide.) The domain of the composition of two partial functions is understood to be the maximal possible one. The identity $\phi_1 = \phi_2$ of partial functions is understood to imply that both sides of the identity must have the same domain.

Definition 2.7. A left G-action ϕ on X is called *injective* if the maps $\phi(g) \in \mathrm{PartFunc}\,(X)$ are injective on their domain for all $g \in G$.

A *linear action* of G on a vector space X is a morphism $\phi : G \to \mathrm{LinMap}\,(X)$ from G into the linear maps on X. The map λ of Definition 2.3 may be checked to be a linear action on $\ell^2(G)$. Left G-actions correspond to morphisms, and right G-actions to anti-morphisms. That is, a right linear action on a vector space X is an anti-morphism $\varphi : G \to \mathrm{LinMap}\,(X)$.

Definition 2.8. An injective left G-action ϕ on a Hausdorff space X is *continuous* if all maps $\phi(g) \in \mathrm{PartFunc}\,(X)$ are continuous and have clopen domains and ranges for all $g \in G$.

3. G-Hilbert C^*-algebras and -modules. In this section we recall the basic definitions for G-equivariant KK-theory for a general semimultiplicative set G ([**5**]). All C^*-algebras and Hilbert modules are assumed to be $\mathbf{Z}_2$-graded [**8, 9**]. If ε is a grading on a linear space X, then $\varepsilon(T) = \varepsilon T \varepsilon$ is a grading on the space of linear maps T on X.

All $*$-homomorphisms between C^*-algebras are supposed to respect the grading. We let $[x, y] = xy - (-1)^{\partial x \partial y} yx$ be the graded commutator.

At first we shall define an action by a general semimultiplicative set G on a C^*-algebra. This is the next definition (from [**5**], Definitions 11, 12, 20 and the remark thereafter).

Definition 3.1. A *G-Hilbert C^*-algebra* A is a $(\mathbf{Z}/2)$-graded C^*-algebra A which is also regarded as a Hilbert module over itself under the inner product $\langle x, y\rangle = x^* y$, and which is equipped with a semimultiplicative set morphism

$$\alpha : G \longrightarrow \mathrm{End}\,(A)$$

and a semimultiplicative set anti-morphism

$$\alpha^* : G \longrightarrow \mathrm{End}\,(A)$$

such that α_g and α_g^* are zero-graded for all $g \in G$,

$$\begin{aligned} \alpha_g &= \alpha_g \alpha_g^* \alpha_g, \\ \alpha_g^* &= \alpha_g^* \alpha_g \alpha_g^*, \end{aligned}$$

and $\alpha_g^* \alpha_g$ and $\alpha_g \alpha_g^*$ are self-adjoint for all $g \in G$, and

$$\begin{aligned} \langle \alpha_g(x), y\rangle &= \alpha_g(\langle x, \alpha_g^*(y)\rangle), \\ \langle \alpha_g^*(x), y\rangle &= \alpha_g^*(\langle x, \alpha_g(y)\rangle) \end{aligned}$$

holds for all $x, y \in A$ and all $g \in G$.

We usually simply write $g(x)$ rather than $\alpha_g(x)$, and $g^*(x)$ rather than $\alpha_g^*(x)$. Instead of G-Hilbert C^*-algebra we often say just Hilbert C^*-algebra if G is clear from the context or unimportant.

Definition 3.2. A *G-equivariant homomorphism* $\tau : A \to B$ between two Hilbert C^*-algebras A and B is a $*$-homomorphism intertwining both the left and the right G-action, i.e., $\tau(g(x)) = g(\tau(x))$ and $\tau(g^*(x)) = g^*(\tau(x))$ for all $x \in A$ and $g \in G$.

Definition 3.3. A G-Hilbert module $\mathcal{E}$ is a $(\mathbf{Z}/2)$-graded Hilbert module $\mathcal{E}$ over a Hilbert C^*-algebra B, such that $\mathcal{E}$ is equipped with a semimultiplicative set morphism

$$U : G \longrightarrow \mathrm{LinMap}\,(\mathcal{E})$$

and a semimultiplicative set anti-morphism

$$U^* : G \longrightarrow \mathrm{LinMap}\,(\mathcal{E})$$

such that U_g and U_g^* are zero-graded for all $g \in G$,

$$\begin{aligned} U_g &= U_g U_g^* U_g, \\ U_g^* &= U_g^* U_g U_g^*, \end{aligned}$$

and $U_g^* U_g$ and $U_g U_g^*$ are self-adjoint for all $g \in G$, and

$$U_g(\xi b) = U_g(\xi) g(b), \tag{3.1}$$

$$U_g^*(\xi b) = U_g^*(\xi) g^*(b), \tag{3.2}$$

$$\langle U_g(\xi), \eta \rangle = g(\langle \xi, U_g^*(\eta) \rangle), \tag{3.3}$$

$$\langle U_g^*(\xi), \eta \rangle = g^*(\langle \xi, U_g(\eta) \rangle) \tag{3.4}$$

holds for all $\xi, \eta \in \mathcal{E}$, $b \in B$ and $g \in G$.

Definition 3.4. Let A and B be G-Hilbert C^*-algebras and $\mathcal{E}$ a G-Hilbert module over B. A $*$-homomorphism $\pi : A \to \mathcal{L}(\mathcal{E})$ is called *G-equivariant* if

$$[U_g U_g^*, \pi(a)] = 0, \tag{3.5}$$

$$[U_g^* U_g, \pi(a)] = 0, \tag{3.6}$$

$$U_g \pi(a) U_g^* = \pi(g(a)) U_g U_g^*, \tag{3.7}$$

$$U_g^* \pi(a) U_g = \pi(g^*(a)) U_g^* U_g \tag{3.8}$$

for all $a \in A$ and $g \in G$.

Definition 3.5. Let A and B be G-Hilbert C^*-algebras. A *G-Hilbert (A,B)-bimodule* $\mathcal{E}$ is a G-Hilbert B-module $\mathcal{E}$ together with a G-equivariant $*$-homomorphism $\pi : A \to \mathcal{L}(\mathcal{E})$. The homomorphism π is often regarded as a left module multiplication of A on $\mathcal{E}$.

We also write $g(T) = U_g T U_g^*$ and $g^*(T) = U_g^* T U_g$ for $g \in G$ and adjoint-able operators $T \in \mathcal{L}(\mathcal{E})$. Note that in general $\mathcal{L}(\mathcal{E})$ is not a G-Hilbert C^*-algebra, as usually the action $g(\cdot)$ is not multiplicative, i.e.,

$g(TS) \neq g(T)g(S)$. The *trivial* G-action on an object X of a category is the action $\tau_g(x) = x$ for all $x \in X$ and $g \in G$.

For a subset $C \subseteq \mathcal{L}(\mathcal{E})$ we set

$$Q_C = \{T \in \mathcal{L}(\mathcal{E}) |\, [T, c] \in \mathcal{K}(\mathcal{E}),\, \forall c \in C\},$$
$$I_C = \{T \in \mathcal{L}(\mathcal{E}) |\, cT \text{ and } Tc \text{ are in } \mathcal{K}(\mathcal{E}),\, \forall c \in C\}.$$

Here, $\mathcal{K}(\mathcal{E})$ denotes the set of compact operators in the sense of Kasparov ([**9**]).

Definition 3.6. Let A, B be G-Hilbert C^*-algebras. Cycles in $\mathbf{E}^G(A, B)$ are Kasparov's cycles $(\pi, \mathcal{E}, T)$ in $\mathbf{E}(A, B)$ ([**9**]) with the following addition: $\mathcal{E}$ is a G-Hilbert module (Definition 3.3) and $\pi : A \to \mathcal{L}(\mathcal{E})$ is a G-equivariant (Definition 3.4), and the elements

$$g(T) - g(1)T, \quad [g(1), T], \quad [g^*(1), T] \tag{3.9}$$

are in $I_A(\mathcal{E})$. Parallel to Kasparov's theory, $KK^G(A, B)$ is defined to be $\mathbf{E}^G(A, B)$ divided by homotopy induced by $\mathbf{E}^G(A, B[0, 1])$.

$KK^G(A, B)$ is functorial in A and B and allows an associative Kasparov product [**5**].

We recall that we have a diagonal G-action on tensor products, see [**5,** Lemmas 4 and 5]. If $\mathcal{E}_1$ and $\mathcal{E}_2$ are G-Hilbert modules, then $\mathcal{E}_1 \otimes \mathcal{E}_2$ is a G-Hilbert module, and $\mathcal{E}_1 \otimes_{B_1} \mathcal{E}_2$ is a G-Hilbert module if $B_1 \to \mathcal{L}(\mathcal{E}_1)$ is a G-equivariant representation (Definition 3.4), both under the diagonal action $U^{(1)} \otimes U^{(2)}$.

4. Partial isometries. In this section we shall show that an action of a semimultiplicative set on a Hilbert module is realized by partial isometries (Corollary 4.3), where inverse elements go over to adjoint partial isometries (Corollary 4.6).

A *projection* on a Hilbert module $\mathcal{E}$ is a self-adjoint idempotent map P on $\mathcal{E}$. Recall that the identity $P(\mathcal{E}) = \mathcal{H}$ links complemented subspaces $\mathcal{H}$ of $\mathcal{E}$ with projections P on $\mathcal{E}$ in a bijective way.

Definition 4.1. A *partial isometry* T on a Hilbert-module $\mathcal{E}$ is a linear map $T : \mathcal{E} \to \mathcal{E}$ for which there exist two complemented subspaces $\mathcal{H}_0$

and $\mathcal{H}_1$ in $\mathcal{E}$ such that T maps $\mathcal{H}_0$ norm-isometrically onto $\mathcal{H}_1$ and vanishes on $\mathcal{H}_0^\perp$.

Notice that we do not require that a partial isometry T be adjoint-able. (For instance, in Lance's book **[12]**, partial isometries are supposed to be adjoint-able.) The projections P and Q of a partial isometry T as in Definition 4.1 projecting onto $\mathcal{H}_0$ and $\mathcal{H}_1$, respectively, are called the *source* and *range projections* of T. Since $\mathcal{H}_0^\perp = \ker(T)$ and $\mathcal{H}_1 = \mathrm{range}(T)$, P and Q are uniquely determined by T. The *inverse partial isometry* S of T, also denoted by $S = T^*$, is the unique partial isometry S on $\mathcal{E}$ which vanishes on $\mathcal{H}_1^\perp$ and satisfies $S|_{\mathcal{H}_1} = (T|_{\mathcal{H}_0})^{-1}$. If T happens to be adjoint-able, then the notation T^* cannot cause confusion as in this case the inverse partial isometry is the adjoint of T, see **[12]**. The set of partial isometries of $\mathcal{E}$ is denoted by $\mathrm{PartIso}(\mathcal{E})$.

Lemma 4.2. *T is a partial isometry if and only if T is a norm contractive linear map and there exists a norm contractive linear map $S : \mathcal{E} \to \mathcal{E}$ such that ST and TS are projections, $T = TST$ and $S = STS$. In this case $S = T^*$.*

Proof. Since S and T are contractive, we have $\|Tx\| = \|TSTx\| \leq \|STx\| \leq \|Tx\|$ and $\|Sy\| = \|TSy\|$ for all $x, y \in \mathcal{E}$. Thus, T is a partial isometry with source and range projections ST and TS, respectively, and $S = T^*$. $\square$

Corollary 4.3. *If U is a G-action on a Hilbert module, then U_g is a partial isometry with inverse partial isometry U_g^* ($g \in G$).*

Proof. The boundedness of U_g follows from $\|\langle U_g x, U_g x\rangle\| = \|g(\langle x, U_g^* U_g x\rangle)\| \leq \|x\|^2$, and then one applies Lemma 4.3. $\square$

Lemma 4.4. *A partial isometry T satisfying $T = TT$ and $T^* = T^*T^*$ is a projection.*

Proof. Let $x \in \mathcal{E}$. Set $y = Tx$. Then $Ty = TTy = Tx = y$. Let $y = y_0 + y_1$ with $y_0 = T^*Ty$ and $y_1 = (1 - T^*T)y$ be the orthogonal decomposition. Then $T^*y = T^*Ty = y_0$. Hence, $y_0 = T^*y = T^*T^*y =$

T^*y_0, thus $T^*(y_0+y_1) = y_0 = T^*y_0$, and so $T^*y_1 = 0$. We thus have

$$\begin{aligned} 0 &= \langle TT^*y_1, y_0\rangle = \langle y_1, TT^*y_0\rangle = \langle y_1, Ty_0\rangle = \langle y_1, TT^*Ty\rangle \\ &= \langle y_1, Ty\rangle = \langle y_1, y\rangle = \langle y_1, y_1\rangle. \end{aligned}$$

Thus, $y_1 = 0$ and so $T^*Ty = y_0 = y = Ty$. Hence, $T^*TTx = TTx$, and so $T^*Tx = Tx$. Since x was arbitrary, $T^*T = T$, and thus T is a projection. □

Definition 4.5. An element g of a semimultiplicative set G is called *invertible* if there exists an element $h \in G$ such that $ghg = g$ and $hgh = h$.

Even if the inverse element h may not be unique, we occasionally denote a given choice by $h = g^{-1}$.

Corollary 4.6. *Assume that $\mathcal{E}$ is a G-Hilbert module and $g \in G$ is invertible. Then $U_g^* = U_{g^{-1}}$ and $U_{g^{-1}}^* = U_g$.*

Proof. Set $T = U_{gg^{-1}} = U_gU_{g^{-1}}$. Then $TT = T$ and $T^*T^* = T^*$. Hence, T is a projection by Lemma 4.4. Similarly, $U_{g^{-1}}U_g$ is a projection. By Lemma 4.2 (for $S := U_g$ and $T := U_{g^{-1}}$), $U_g^* = U_{g^{-1}}$. □

5. Algebraic crossed products. In this section G denotes a discrete general semimultiplicative set (if nothing else is said). For the work with crossed products we shall also need to consider free products of elements of G and their adjoints, and for that purpose we shall introduce G^* below.

Definition 5.1. An *involution* on a semigroup S is a map $* : S \to S : s \mapsto s^*$ such that $(s^*)^* = s$ and $(st)^* = t^*s^*$ for all $s, t \in S$.

Definition 5.2. Define $F(G)$ to be the free semigroup generated by two copies of G. The elements of the second copy of G are denoted by g^* for $g \in G$ and stand for adjoint elements. In other words, element γ of $F(G)$ consists formally of $\gamma = x_1^{\epsilon_1} \cdots x_n^{\epsilon_n}$ with $x_i \in G$ and $\epsilon_i \in \{1, *\}$.

We shall occasionally denote the multiplication in G by $g \odot h$ $(g, h \in G)$ to distinguish it from the multiplication in $F(G)$.

Definition 5.3. Define G^* to be the semigroup which is the quotient semigroup of $F(G)$ by the following *elementary equivalences* defined for all $g, h \in G$.

$$\begin{aligned} g \odot h &= gh, \qquad & (g \odot h)^* &= h^* g^* \quad \text{if } g \odot h \text{ is defined} \\ g &= gg^*g, & g^* &= g^*gg^*. \end{aligned}$$

In other words, elements of G^* consist of representatives living in $F(G)$, and two representatives $\gamma, \delta \in F(G)$ are equivalent, if there is a finite sequence of representatives in $F(G)$ starting with γ and ending with δ, where two representatives in this sequence differ only by a single elementary equivalence (within a word).

G^* is an involutive semigroup by concatenation and taking the formal adjoints of representatives of $F(G)$. For simplicity, we shall omit the class brackets and write g rather than the class $[g]$ for elements in G^*, where $g \in F(G)$ is a representative. Note that an element in G^* need not be invertible: if $g, h \in G$ are indecomposable in G, then usually $gh(gh)^*gh \neq gh$ in G^*.

Lemma 5.4. *A morphism* (*respectively, anti-morphism*) $\varphi : G \to H$ *between semimultiplicative sets* G *and* H *extends canonically to a* $*$-*morphism* (*respectively,* $*$-*anti-morphism*) $G^* \to H^*$.

Proof. A morphism $\varphi : G \to H$ induces a canonical $*$-morphism $F(G) \to F(H)$ which respects the elementary equivalences of Definition 5.3. $\square$

For the work with crossed products it is useful to extend a G-action to a G^*-action, and this is what the next couple of lemmas will be about.

Lemma 5.5. *If* ϕ *is an injective* G-*action on a set* X *and* $g \in G$ *is invertible in* G, *then* $\phi(g)^{-1} = \phi(g^{-1})$.

Proof. Let h be an inverse element for g. If gx is defined, then $(ghg)x = g(h(gx))$ is defined, so $h(gx)$ is defined; and conversely, if $hx = hghx$ is defined, then $x = ghx$ by injectivity of the G-action. We have checked that the range of $\phi(g)$ is the domain of $\phi(h)$. From $ghgx = gx$ it follows $ghx = x$ by injectivity of the G-action, and similarly $hgx = x$. Thus, $\phi(g)$ and $\phi(h)$ are inverses to each other. $\square$

Lemma 5.6. *A continuous injective left G-action on a Hausdorff space X can be extended to a continuous injective left G^*-action on X.*

Proof. Let $\phi : G \to \mathrm{PartFunc}\,(X)$ be the G-action on X. For $g = g_1^{\epsilon_1} \cdots g_n^{\epsilon_n} \in F(G)$ $(g_i \in G, \epsilon_i \in \{1, *\})$ define

$$\widehat{\phi}(g) = \phi(g_1)^{\epsilon_1} \circ \cdots \circ \phi(g_n)^{\epsilon_n}. \tag{5.1}$$

Here, $\phi(g)^*$ denotes the inverse partial function for $\phi(g)$. We have to show that (5.1) factors through G^*, in other words, we must show that ϕ is invariant under the elementary equivalences of Definition 5.3.

Let $s, t \in F(G)$, $g, h \in G$ and $g \odot h \in G$ be defined. Then $s(g \odot h)^* t = sh^* g^* t$ in G^*. By (5.1) and the definition of an action ϕ we have

$$\begin{aligned}\widehat{\phi}(s(g \odot h)^* t) &= \phi(s)\big(\phi(g \odot h)\big)^* \phi(t)\\ &= \phi(s)\big(\phi(g)\phi(h)\big)^* \phi(t) = \phi(s)\phi(h)^*\phi(g)^*\phi(t) = \widehat{\phi}(sh^* g^* t).\end{aligned}$$

The other elementary equivalences are checked similarly. It is easy to see that the extended ϕ is also a continuous action (the inverse partial functions and composition of partial functions have clopen domains and ranges again). $\square$

Lemma 5.7. *Every G-Hilbert B-module $\mathcal{E}$ induces a morphism $\widehat{U} : G^* \to \mathrm{LinMap}\,(\mathcal{E})$ extending the G-action U on $\mathcal{E}$. The relations* (3.1)–(3.4) *also hold for all $g \in G^*$.*

Proof. For $g_1^{\epsilon_1} \cdots g_n^{\epsilon_n} \in F(G)$ $(g_i \in G, \epsilon_i \in \{1, *\})$ define

$$\widehat{U}_{g_1^{\epsilon_1} \cdots g_n^{\epsilon_n}} = U_{g_1}^{\epsilon_1} \cdots U_{g_n}^{\epsilon_n}.$$

This map respects the elementary equivalences of Definition 5.3 since U and U^* are a morphism and anti-morphism, respectively, by Definition 3.3. Consequently, $\widehat{U}$ factors through G^*. Relations (3.1)–(3.4) are checked by induction (recall [**5,** Lemma 3]). $\square$

We emphasize that $\widehat{U}$ of the last lemma is a morphism but not a $*$-morphism. Usually $\mathcal{E}$ is not a G^*-Hilbert module as $\widehat{U}_g$ need not be a partial isometry for $g \in G^*$. It may thus be suggestive to write $\widehat{U}_g^*$ for U_{g^*} ($g \in G^*$), but one should be aware that this star might not be a (well defined) operator on the sets of U_g's. There is no (obvious) involution in the image of $\widehat{U}$.

We shall usually write U rather than $\widehat{U}$.

Lemma 5.8.

(i) *Every G-Hilbert C^*-algebra A is also a G^*-Hilbert C^*-algebra. In particular, there is a $*$-morphism $\widehat{\alpha} : G^* \to \mathrm{PartIso}\,(A) \cap \mathrm{End}\,(A)$ extending the G-action α.*

(ii) *Every G-equivariant representation $\pi : A \to \mathcal{L}(\mathcal{E})$ of A on a G-Hilbert module $\mathcal{E}$ is G^*-equivariant in the sense that the identities* (3.5)–(3.8) *also hold for $g \in G^*$ (where U_g^* has to be interpreted as U_{g^*}).*

(iii) *For all $a, b \in A$ and $g \in G^*$, one has $gg^*(ab) = gg^*(a)b = agg^*(b)$.*

Proof. We extend the G-action α to a morphism $\widehat{\alpha}$ on A according to Lemma 5.7. Let $g, h \in G^*$ and $a, b \in A$. We may write $\alpha_g\alpha_g^*(a)b = \langle \widehat{\alpha}_g\widehat{\alpha}_g^*(a^*), b\rangle$ for all $a \in A$ and $g \in G^*$. Writing $\widehat{\alpha}_g(a) = g(a)$, by identity (3.7) (Lemma 5.7), we have

$$gg^*(a)b = \langle gg^*(a^*), b\rangle = g(g^*(a)g^*(b)) = gg^*(a)gg^*(b),$$

and similarly, $agg^*(b) = gg^*(a)gg^*(b)$. Hence, $gg^*(a)b = agg^*(b)$, that is, $gg^* \equiv \widehat{\alpha}_g\widehat{\alpha}_g^*$ is self-adjoint, since $gg^*gg^*(a)b = gg^*(a)gg^*(b) = gg^*(a)b$, gg^* is a projection. These identities already prove (iii). Now

$$gg^*hh^*(a)b = gg^*(hh^*(a)b) = gg^*(ahh^*(b)) = gg^*(a)hh^*(b) = hh^*gg^*(a)b,$$

that is, gg^* and hh^* commute. Hence, $g \equiv \widehat{\alpha}_g$ is the product of partial isometries α_i, α_j^* ($i, j \in G$) with commuting range and source projections and thus by a standard inductive proof and Lemma 4.2 a partial isometry with inverse partial isometry $\widehat{\alpha}_g^* = \widehat{\alpha}_{g^*}$. This shows that $\widehat{\alpha}$ maps into the partial isometries, and is thus a G^*-action, which proves (i). The G^*-equivariance claimed in (ii) (meaning that the formulas of Definition 3.4 hold) follows by induction; see also [**5,** Lemma 9]. $\square$

Lemma 5.9. *Let X be a Hausdorff space equipped with an injective continuous right G-action τ. Then $C_0(X)$ is a G-Hilbert C^*-algebra under the action $\alpha_g(f)x = 1_{\{\tau_g(x) \text{ is defined}\}} f(\tau_g(x))$ $(\alpha_g^* := \alpha_g^{-1})$ for $f \in C_0(X)$, $g \in G$ and $x \in X$.*

Proof. By definition of a continuous action τ on X, the domain and range, respectively, of τ_g is a clopen subset D_g and R_g, respectively, of X. So $\alpha_g(f)$ is indeed a continuous function. α_g projects onto $1_{D_g} C_0(X)$, and α_g moves $1_{R_g} C_0(X)$ onto $1_{D_g} C_0(X)$. α_g^* is the inverse map. It is straightforward to verify Definition 3.1, and this is left to the reader. $\square$

We give another characterization of a Hilbert C^*-algebra.

Lemma 5.10. *Let A be a C^*-algebra. Then A is a Hilbert C^*-algebra with G-action α if and only if α is a morphism $\alpha : G \to \mathrm{PartIso}\,(A) \cap \mathrm{End}\,(A)$, and for every $g \in G$, the source and range projections $\alpha_g^* \alpha_g, \alpha_g \alpha_g^*$ are in $Z\mathcal{M}(A)$ (center of the multiplier algebra of A).*

Proof. If A is a Hilbert C^*-algebra, then source and range projections of α_g are in $Z\mathcal{M}(A)$ as remarked in [**5,** Section 7]. Conversely, assume the condition. Then $A \subseteq \mathcal{L}(A)$ by left multiplication. Since gg^* is in $Z\mathcal{M}(A)$, gg^* commutes with the left multiplication operator $L_a(b) = ab$ $(a, b \in A)$, and so $gg^*(ab) = agg^*(b)$. Moreover, $gg^*(ab) = gg^*(a)b$ (since $gg^* \in \mathcal{L}(\mathcal{E})$). In particular, $gg^*(a)b = gg^*(ab) = agg^*(b)$. With this, one easily gets $\langle g(a), b\rangle = g\langle a, g^*(b)\rangle$. $\square$

We shall now come to crossed products by G.

Definition 5.11. Let A be a G-Hilbert C^*-algebra. Write $\mathbf{F}(G, A)$ for the universal $*$-algebra generated by A and G subject to the following relations: The $*$-algebraic relations of A are respected and the identities

(5.2) $\quad g \odot h = gh \quad$ if $g \odot h$ is defined,

(5.3) $\quad gg^*g = g, \qquad gg^*a = agg^*,$

(5.4) $\quad g^*ga = ag^*g, \qquad gag^* = g(a)gg^*, \qquad g^*ag = g^*(a)g^*g$

hold true for all $g, h \in G$ and $a \in A$.

Definition 5.12. Let A be a G-Hilbert C^*-algebra. The *algebraic crossed product* $A \rtimes_{\text{alg}} G$ of A by G is the $*$-subalgebra of $\mathbf{F}(G, A)$ generated by the set

$$\{ag \in \mathbf{F}(G, A) | \, a \in A, g \in G\}.$$

Let A be a G-Hilbert C^*-algebra. Write

$$A_g = gg^*(A)$$

for $g \in G^*$. A_g is a two-sided closed ideal in A by Lemma 5.8 (iii).

Lemma 5.13. *$A \rtimes_{\text{alg}} G$ is canonically isomorphic to the $*$-algebra $C_c(G^*, A)$ consisting of formal finite sums $\sum_{g \in G^*} a_g g$ ($a_g \in A_g$) with involution*

$$\Big(\sum_{g \in G^*} a_g g\Big)^* = \sum_{g \in G^*} g^*(a_g^*) g^*$$

and convolution product

$$\sum_{g \in G^*} a_g g \sum_{h \in G^*} b_h h = \sum_{g,h \in G^*} a_g g(b_h) gh.$$

Proof. By induction on the length of a word in G^*, one checks that $ga = g(a)g$ holds in $\mathbf{F}(G, A)$ for all $g \in G^*$. Note that $g(a) = gg^*g(a) \in A_g$ since the G^*-action on a Hilbert C^*-algebra is realized by partial isometries (Lemma 5.8). One has

$$ag = (g^*a^*)^* = (g^*(a^*)g^*)^* = gg^*(a)g = a_g g \tag{5.5}$$

for all $a \in A$ and $g \in G^*$, where $a_g := gg^*(a) \in A_g$. It follows that

$$gg^*a = gg^*(a)gg^* = agg^* \tag{5.6}$$

$$gag^* = g(a)gg^* \tag{5.7}$$

for all $a \in A$ and $g \in G^*$. Define $D = A \oplus C_c(G^*, A) \oplus G^*$. Endow D with the algebraic structure on the summands as given, and between the summands as we have it in $\mathbf{F}(G, A)$, for instance, $g \cdot a = g(a)g \in C_c(G^*, A)$ for $a \in A$ and $g \in G^*$. By universality of $\mathbf{F}(G, A)$, there is a $*$-homomorphism $\phi : \mathbf{F}(G, A) \to D$ such that $\phi(a) = a$ and $\phi(g) = g$ for all $a \in A$ and $g \in G^*$ (using (5.6)–(5.7)). It is obviously injective, as D, and particularly $C_c(G^*, A)$, is a direct sum. The restriction ϕ' of ϕ to $A \rtimes_{\mathrm{alg}} G$ yields $C_c(G^*, A)$. The surjectivity of ϕ' follows by induction from the factorization

$$agh = (a^{1/2}g)(g^*(a^{1/2})h)$$

for $a \in A_+$ and $g, h \in G$. □

Lemma 5.14. (i) *There is a linear isomorphism*

$$\mathbf{F}(G, A) \cong A \oplus C_c(G^*, A) \oplus G^*.$$

(ii) *The identities* (5.3)–(5.4) *hold for all $a \in A$ and $g \in G^*$.*

Proof. This was proved in Lemma 5.13. □

One usually has no cancelation in G^*, even if G has it. Assume for instance that $g, h \in G$ are not invertible and not decomposable in G. Then usually $h \neq g^*gh$ in G^*. For this reason, we need not have a transformation like '$x = gh \Leftrightarrow g^*x = h$' in the convolution product of Lemma 5.13.

Definition 5.15. By a *covariant representation* of a G-Hilbert C^*-algebra A, we mean a G-equivariant representation $\pi : A \to B(H)$ on a G-Hilbert space H (Definition 3.3 with trivial G-action on $\mathbf{C}$) in the sense of Definition 3.4.

Lemma 5.16. *Restricting a $*$-homomorphism $\phi : \mathbf{F}(G, A) \to B(H)$ of $\mathbf{F}(G, A)$ to A and G gives a covariant representation $(\phi|_A, \phi|_G, H)$ of A. Conversely, a covariant representation (π, u, H) of A extends canonically to a representation $\phi : \mathbf{F}(G, A) \to B(H)$ of $\mathbf{F}(G, A)$ determined by $\phi|_A = \pi$ and $\phi|_G = u$. This correspondence between representations of $\mathbf{F}(G, A)$ and covariant representations of A is a bijection.*

By the last lemma, it is often comfortable to work with *one* homomorphism ϕ rather than an equivariant representation. A covariant representation of $A \rtimes_{\mathrm{alg}} G$ is then just a restriction of ϕ. We have the following diagram (where ι denotes the canonical embedding).

$$\begin{tikzcd} \mathbf{F}(G,A) \arrow[drr, "\phi"] & & \\ A \rtimes_{\mathrm{alg}} G \arrow[u, "\iota"] \arrow[rr, "(\phi|_A, \phi|_G, H)", "\phi|_{A \rtimes_{\mathrm{alg}} G}"'] & & B(H) \end{tikzcd}$$

6. Full crossed products.

Definition 6.1. Let (π, u, H) be a G-covariant representation of a G-Hilbert C^*-algebra A and ϕ its induced representation on $\mathbf{F}(G,A)$. The C^*-algebra $A \rtimes_{(\pi,u,H)} G$ *induced by this covariant representation* is the norm closure of $\phi(A \rtimes_{\mathrm{alg}} G)$.

Definition 6.2. For $x = \sum_{g \in G^*} a_g g \in A \rtimes_{\mathrm{alg}} G$, let $s(x)$ denote the supremum of $\|(\pi \times u)(x)\|$ taken over all G-covariant representations (π, u, H) of A. This supremum is finite as we have

$$s(x) \le \sup_{(\pi,U,H)} \sum_{g \in G^*} \|\pi(a_g)\| \cdot \|u_g\| \le \sum_{g \in G^*} \|a_g\| < \infty.$$

The *full crossed product* $A \rtimes G$ is the completion of the quotient of $A \rtimes_{\mathrm{alg}} G$ divided by the kernel of the seminorm s.

Definition 6.3. Similarly as in Definition 6.2, we define a C^*-algebra $B \subseteq B(H)$ which is the completion of the quotient of $\mathbf{F}(G,A)$ divided by the kernel of the seminorm s' which arises by taking the supremum of the norms over all representations of $\mathbf{F}(G,A)$. The canonical homomorphism $\phi^\infty : \mathbf{F}(G,A) \to B \subseteq B(H)$ is called the *universal representation* of $\mathbf{F}(G,A)$, and the covariant representation $(\phi^\infty|A, \phi^\infty|_G, H)$ of Lemma 5.16 the *universal G-covariant representation* of A.

The correspondence between G-covariant representations of A and representations of $\mathbf{F}(G,A)$ by Lemma 5.16 shows that the seminorm s is the restriction of the seminorm s' to $A \rtimes_{\mathrm{alg}} G$, and note that the kernels of s and s' are automatically ideals. Hence, it is easy to see that $A \rtimes G$ can be canonically isometrically embedded in B. Thus, we may alternatively regard $A \rtimes G$ as the norm closure of $\phi^\infty(A \rtimes_{\mathrm{alg}} G)$. In particular, $A \rtimes G$ is the C^*-algebra induced by the universal covariant representation $(\phi^\infty|A, \phi^\infty|_G)$. Keeping Lemma 5.16 in mind, by an abuse of language, we may also call ϕ^∞ a covariant representation of A.

Lemma 6.4. *Let ϕ^∞ be the universal covariant representation of A. If ϕ is another covariant representation of A, then there is a homomorphism $\sigma : A \rtimes G \to A \rtimes_\phi G$ such that $\sigma\phi^\infty(x) = \phi(x)$ for all $x \in A \rtimes_{\mathrm{alg}} G$.*

$$\begin{tikzcd}
A \rtimes_{\mathrm{alg}} G \arrow[r, "\phi^\infty"] \arrow[dr, "\phi"'] & A \rtimes G \arrow[d, "\sigma"] \\
& A \rtimes_\phi G
\end{tikzcd}$$

Proof. This is clear as $\|\phi^\infty(x)\| = \sup_\varphi \|\varphi(x)\| \geq \|\phi(x)\|$, where $x \in A \rtimes_{\mathrm{alg}} G$ and the supremum is taken over all representations φ of $\mathbf{F}(G, A)$. $\square$

Note that the above full crossed product is for proper semimultiplicative sets, and so there are differences to existing crossed products if one considers special categories. Let (π, U, H) be a covariant representation of a G-Hilbert C^*-algebra A. If G is a discrete group, then $U_g U_g^* = U_g^* U_g = U_e$ for all $g \in G$ by Lemma 4.6, but this need not be a unit (we may resolve this difference by requiring $U_e = 1$, as optionally done in Sections 11–13). If G is an inverse semigroup, our crossed product differs from Sieben's crossed product [**?**] which is based on strictly covariant representations in the sense that $U_g \pi(a) U_g^* = \pi(g(a))$. We are, however, consistent with Khoshkam and Skandalis's definition [**10**], see Lemma 8.4. The precise difference between the latter two crossed products is clarified in [**10**]. If G is a semigroup, then in the existing definitions a semigroup covariant representation consists of isometries U_g which strictly covariantly intertwine the G-action, see Stacey [**19**], Murphy [**14**], Laca [**11**] and Larsen [**13**]. Stacey even allows a family of isometries for representations of different multiplicities. The crossed product of $\mathbf{N}$ by surjective shift maps on $\{0,1\}^{\mathbf{N}}$

degenerates to 0 according to Stacey in [**19,** Example 2.1(a)] (this affects any crossed product construction induced by strictly covariantly intertwining isometries) but there is an obvious non-degenerate covariant representation on $B(\ell^2(\{0,1\}^{\mathbf{N}}))$ in our sense. In all constructions of this paragraph the full crossed product is (roughly speaking) the enveloping C^*-algebra of the respective class of equivariant representations.

If $\mathcal{G}$ is a discrete groupoid then $gh = 0$ in the groupoid C^*-algebra if g and h are indecomposable ($g, h \in \mathcal{G}$). Taking into account such an approach to the crossed product, we consider such a variant also for semimultiplicative sets.

Definition 6.5. Let G be a general semimultiplicative set. A covariant representation (π, u, H) is called *strong* if $u_g u_h = 0$ for all indecomposable pairs $g, h \in G$. The *full strong crossed product* $A \rtimes_{\mathrm{s}} G$ is the C^*-algebra induced by the class of strong G-covariant representations of A by a similar construction as in Definitions 6.2 and 6.3 and the remark thereafter.

Similarly as in Definition 6.3 we define the *universal strong representation* and the *universal strong G-covariant representation*. A similar lemma as Lemma 6.4 also holds for the strong crossed product and the strong covariant representations.

7. Reduced crossed products. In this section we shall assume that G is an associative semimultiplicative set with left cancelation. Let ρ be the injective G-action on G given by left multiplication ($\rho_g(h) = gh$ in G). It can be extended to an injective G^*-action on G (also denoted by ρ) by Lemma 5.6. ρ induces an action $\lambda : G \to B(\ell^2(G))$ (Definition 2.3). This action is an action under which $\ell^2(G)$ becomes a G-Hilbert space (i.e., a G-Hilbert module over $\mathbf{C}$). We shall regard $\ell^2(G)$ as a G-Hilbert module (if nothing else is said). We may extend this action to a G^*-action, and denote this extension also by λ (and it is the same action as the extended ρ would induce). For arbitrary g in G^* and arbitrary h in G, we use the abbreviation

$$e_{gh} := \lambda_g(e_h).$$

Definition 7.1. If G has left cancelation, then a G-action U on a G-Hilbert module $\mathcal{E}$ is said to have *transferred left cancelation* if $U_g^* U_g U_h = U_h$ for all $g, h \in G$ for which gh is defined.

The last definition is understood to include G-Hilbert C^*-algebras (which are special G-Hilbert modules). By sloppy terminology, we shall also say that a G-Hilbert module has transferred left cancelation (rather than the G-action itself).

If G is a semigroupoid, then λ has transferred left cancelation. Indeed, assume gh is defined and $x \in G$. Since G is a semigroupoid and gh is defined, $(gh)x$ is defined if and only if hx is defined. Thus, $\lambda_g^* \lambda_g \lambda_h(e_x) = \lambda_h(e_x)$.

Lemma 7.2. *A G-action U has transferred left cancelation if and only if for all $g \in G^*$ and all $h \in G$ one has $U_{gh} = U_{\rho_g(h)}$ whenever $\rho_g(h)$ is defined (note that $gh \in G^*$ but $\rho_g(h) \in G$).*

Proof. Assume the condition holds true. If $\rho_g(h)$ exists for $g, h \in G$, then $\rho_g^* \rho_g(h) = h$ (Lemma 5.6). Consequently, $U_h = U_{\rho_{g^*g}(h)} = U_{g^*gh}$ by assumption. Thus, U has transferred left cancelation. Assume that U has transferred left cancelation and by induction hypothesis on the length of g that $U_{\rho_g(h)} = U_{gh}$, where $g \in G^*, h \in G$ and $\rho_g(h)$ is defined. Suppose that $t \in G$ and $\rho_{t^*g}(h)$ are defined. Then $gh = \rho_{tt^*g}(h) = \rho_t(\rho_{t^*g}(h)) = \rho_t(x)$ for $x := \rho_{t^*g}(h)$. Since U has transferred left cancelation, $U_t^* U_t U_x = U_x$. Hence, $U_{\rho_{t^*g}(h)} = U_x = U_{t^*tx} = U_{t^*gh}$. This proves the inductive step. On the other hand, if $\rho_{tg}(h)$ is defined, then $U_{\rho_{tg}(h)} = U_{\rho_t(\rho_g(h))} = U_{t(\rho_g(h))} = U_t U_{\rho_g(h)} = U_t U_{gh} = U_{tgh}$, proving the inductive step again. $\square$

Definition 7.3. Suppose that A is a G-Hilbert C^*-algebra, G is associative with left cancelation, and A has transferred left cancelation. Let $\sigma : A \to B(H)$ be a faithful nondegenerate representation (without G-action) of A on a Hilbert space H. The *left reduced crossed product* $A \rtimes_r G$ is the C^*-algebra induced by the *left regular* covariant representation $(\pi, u, H \otimes \ell^2(G))$ of A given by

$$\pi(a)(\xi_h \otimes e_h) = \sigma\big(h^*(a)\big)\xi_h \otimes e_h, \quad u(g)(\xi_h \otimes e_h) = \xi_h \otimes \lambda_g(e_h)$$

for all $a \in A, \xi_h \in H$ and $g, h \in G$.

Lemma 7.4. *The left regular representation* (*Definition* 7.3) *is indeed covariant.*

Proof. We need to check Definition 3.4 and demonstrate only (3.7). Let $\widehat{\alpha}$ denote the G^*-action on A. By Lemma 5.8 (i) and Lemma 7.2, we have

$$\begin{aligned}
u_g \pi(a) u_g^* (\xi \otimes e_h) &= u_g \pi(a)(\xi \otimes e_{\rho_{g^*}(h)}) \\
&= u_g \left(\sigma\big(\widehat{\alpha}^*_{\rho_{g^*}(h)}(a)\big) \xi \otimes e_{\rho_{g^*}(h)} \right) \\
&= u_g \left(\sigma\big(\widehat{\alpha}^*_{g^* h}(a)\big) \xi \otimes e_{\rho_{g^*}(h)} \right) \\
&= \sigma\big(\widehat{\alpha}_{h^* g}(a)\big) \xi \otimes e_{\rho_{g g^*}(h)} \\
&= \sigma\big(\widehat{\alpha}_{h^* g g^* g}(a)\big) \xi \otimes e_{\rho_{g g^*}(h)} \\
&= \pi(g(a)) u_g u_g^* (\xi \otimes e_h)
\end{aligned}$$

for all $g \in G^*$ and $h \in G$. $\square$

Obviously, u of Definition 7.3 is the diagonal G-action $1 \otimes \lambda$. We are going to show that the definition of $A \rtimes_r G$ is actually independent of σ.

We shall recall three lemmas which can all be found in Kasparov [**8,** pages 522–523]. Only Lemma 7.5 is somewhat extended (cf., Lance [**12,** Proposition 2.1]).

Lemma 7.5. *Let X be a Hilbert module, A a C^*-algebra and $\pi : A \to \mathcal{L}(X)$ a non-degenerate homomorphism. Then there is an isomorphism*

$$\rho : A \otimes_A X \longrightarrow X : \rho(a \otimes x) = \pi(a)x.$$

If $T \in \mathcal{L}(A)$ then $T \otimes 1 = \rho^{-1} \widehat{\pi}(T) \rho$, where $\widehat{\pi} : \mathcal{L}(A) \to \mathcal{L}(X)$ denotes the strictly continuous extension of π.

Lemma 7.6. *If X and H are Hilbert modules over C^*-algebras B_1 and B_2, respectively, and $B_1 \to \mathcal{L}(H)$ is an injective homomorphism, then $\mu : \mathcal{L}(X) \to \mathcal{L}(X \otimes_{B_1} H)$, $\mu(T) = T \otimes 1$ is an injective homomorphism.*

Lemma 7.7. *If $E_1, \ldots, E_4$ are Hilbert B_i-modules and $B_1 \to \mathcal{L}(E_3)$, $B_2 \to \mathcal{L}(E_4)$ are homomorphisms, then*

$$(E_1 \otimes E_2) \otimes_{B_1 \otimes B_2} (E_3 \otimes E_4) \cong (E_1 \otimes_{B_1} E_3) \otimes (E_2 \otimes_{B_2} E_4).$$

For a G-Hilbert C^*-algebra A, let $A \otimes \ell^2(G)$ denote the skew tensor product of G-Hilbert modules. We make it a G-Hilbert module over $A \otimes \mathbf{C} \cong A$ under the diagonal action $1 \otimes \lambda$.

Lemma 7.8. *Consider the setting of Definition* 7.3. *There is an injective* $*$*-homomorphism*

$$\zeta : A \rtimes_r G \longrightarrow \mathcal{L}(A \otimes \ell^2(G))$$

induced by the covariant representation $\phi : A \rtimes_{\mathrm{alg}} G \to \mathcal{L}(A \otimes \ell^2(G))$ *given by*

$$\begin{gathered} \phi(a)(x_h \otimes e_h) = h^*(a) x_h \otimes e_h, \\ \phi(g) = 1 \otimes \lambda_g, \end{gathered}$$

for all $a, x_h \in A$ *and* $g, h \in G$.

Proof. Let ϕ_r be the representation of $A \rtimes_{\mathrm{alg}} G$ induced by the left regular representation (Definition 7.3). Let $\sigma : A \to B(H)$ be a faithful and non-degenerate representation (without G-action) of A on a Hilbert space H. We aim to show that there is a commutative diagram

$$\begin{tikzcd} A \rtimes_{\mathrm{alg}} G \arrow[r, "\phi"] \arrow[dr, "\phi_r"'] & \mathcal{L}\big(A \otimes \ell^2(G)\big) \arrow[r, "\mu"] \arrow[d, "\kappa"] & \mathcal{L}\big((A \otimes \ell^2(G)) \otimes_{A \otimes \mathbf{C}} (H \otimes \mathbf{C})\big) \arrow[d, "\mu_1"] \\ & \mathcal{L}\big(H \otimes \ell^2(G)\big) & \mathcal{L}\big((A \otimes_A H) \otimes (\ell^2(G) \otimes_{\mathbf{C}} \mathbf{C})\big) \arrow[l, "\mu_2"] \end{tikzcd}$$

Here, μ is the injective homomorphism of Lemma 7.6, and μ_1 and μ_2 denote the isomorphisms induced by the isomorphisms of Lemmas 7.7 and 7.5, respectively. Define $\kappa := \mu_2 \mu_1 \mu$, which is injective. We are going to analyze $\kappa(\phi(a \rtimes g))$. We write an element $\xi \in H$ as $\sigma(a_0)\xi_0$ for $a_0 \in A$ and $\xi_0 \in H$ by Lemma 7.5. We shall write down, step by step, how $\phi(a \rtimes g)$ transforms under κ. Let $g \in G^*$, $h \in G$, $a \in A_g$,

$x_h \in A$ and $\xi \in H$. We have

$$\begin{aligned} \phi\big(a \rtimes g\big)\big(x_h \otimes e_h\big) &= (gh)^*(a)x_h \otimes e_{gh} \\ \mu\phi\big(a \rtimes g\big)\big((x_h \otimes e_h) \otimes (\xi \otimes 1_{\mathbf{C}})\big) &= ((gh)^*(a)x_h \otimes e_{gh}) \otimes (\xi \otimes 1_{\mathbf{C}}) \\ \kappa\phi\big(a \rtimes g\big)\big(\sigma(x_h)\xi \otimes e_h\big) &= \sigma\big((gh)^*(a)\big)\sigma(x_h)\xi \otimes e_{gh} \\ \kappa\phi\big(a \rtimes g\big)\big(\overline{\xi} \otimes e_h\big) &= \sigma\big((gh)^*(a)\big)\overline{\xi} \otimes e_{gh} \\ &= \phi_r\big(a \rtimes g\big)\big(\overline{\xi} \otimes e_h\big) \end{aligned}$$

In the last step, we have set $\overline{\xi} := \sigma(x_h)\xi$ (Lemma 7.5). We have checked that $\phi_r = \kappa\phi$. This shows that $\overline{\phi(A \rtimes_{\mathrm{alg}} G)}$ is isomorphic to $A \rtimes_r G$, and we set $\zeta := \kappa^{-1}$. □

Corollary 7.9. *The definition of the left reduced crossed product in Definition* 7.3 *does not depend on σ.*

For the rest of this section we consider the following assumptions. Let $L : \mathbf{F}(G, A) \to B(H \otimes \ell^2(G))$ be the left regular representation. Then $L(G^*)$ is an inverse semigroup. Suppose that the G^*-action on A factors through $L(G^*)$ via an inverse semigroup homomorphism μ.

$$\begin{tikzcd} G^* \arrow[r, "L"] \arrow[dr, "\hat{\alpha}"'] & L(G^*) \arrow[d, "\mu"] \\ & \operatorname{End}(A) \end{tikzcd}$$

(For instance, when the G-action on A is trivial.) Then μ defines a $L(G^*)$-action on A. Suppose further that L is injective on A.

Lemma 7.10. *There is an isomorphism*

(7.1)
$$\gamma : L\big(\mathbf{F}(G, A)\big) \longrightarrow \mathbf{F}\big(L(G^*), A\big) : \quad \gamma(L(a)) = a, \ \gamma(L(g)) = L(g),$$

where $a \in A$ and $g \in G^*$, which restricts to an isomorphism

$$L(A \rtimes_{\mathrm{alg}} G) \longrightarrow A \rtimes_{\mathrm{alg}} L(G^*). \tag{7.2}$$

Proof. Note that in $\mathbf{F}(L(G^*), A)$ we have $L(g)a = \mu_{L(g)}(a)L(g) = \widehat{\alpha}_g(a)L(g) = g(a)L(g)$. At first we shall show that $\gamma \circ L$ is a representation of $\mathbf{F}(G, A)$. To this end, we need to check that the relations

(5.2)–(5.4) are respected by $\gamma \circ L$. We only show (5.4),

$$\begin{aligned}\gamma L(g)\gamma L(a)(\gamma L(g))^* &= L(g)aL(g)^* = g(a)L(g)L(g)^* \\ &= \gamma L(g(a))\gamma L(g)(\gamma L(g))^*.\end{aligned}$$

Since L and $\gamma \circ L$ are homomorphisms, γ is a homomorphism.

We need to show that there is an inverse map σ for γ, where $\sigma(a) = L(a)$ and $\sigma(L(g)) = L(g)$. Again, we have to check that the relations (5.2)–(5.4) are respected by σ. For instance,

$$\sigma(L(g))(\sigma(L(g)))^*\sigma(L(g)) = L(g)L(g)^*L(g) = L(g) = \sigma(L(g)),$$

since $L(g)$ is a partial isometry. □

Corollary 7.11. *If the given C^*-norm on $L(A \rtimes_{\mathrm{alg}} G)$ is the maximal* (*covariant*) *one, then*

$$A \rtimes_r G \cong A \rtimes L(G^*). \tag{7.3}$$

Proof. Let γ_0 be the isomorphism (7.2) and endow domain and range with the norms from $A \rtimes_r G$ and $A \rtimes L(G^*)$, respectively. Since γ_0^{-1} is the restriction of γ^{-1}, (7.1), by Lemma 5.16 it is a covariant representation of $A \rtimes_{\mathrm{alg}} L(G^*)$. Thus, γ_0^{-1} is norm-decreasing. On the other hand, γ_0 is a (covariant) representation of $L(A \rtimes_{\mathrm{alg}} G)$, which by assumption must decrease in norm. Thus, γ_0 is an isometry and extends continuously to (7.3). □

The last corollary may be useful to translate reduced crossed products to inverse semigroup crossed products, for which there exist more Baum-Connes theory (see for instance **[3, 4]**). For example, some Toeplitz graph C^*-algebras for graphs Λ are reduced C^*-algebras $\mathbf{C} \rtimes_r \Lambda^*$ (via the so-called path space representation, see for instance **[17]**). By a Cuntz-Krieger uniqueness theorem (the C^*-norm on $L(\mathbf{C} \rtimes_{\mathrm{alg}} \Lambda^*)$ is unique), Corollary 7.11 applies immediately.

8. Representations of $\ell^1(G)$. Write $\ell^1(G, A)$ for the completion of $C_c(G^*, A)$ under the norm $\|\sum_{g \in G^*} a_g g\|_1 = \sum_{g \in G^*} \|a_g\|$. For $a, b \in C_c(G^*, A)$, the estimate $\|ab\|_1 \le \|a\|_1 \|b\|_1$ is easy.

Lemma 8.1. *$\ell^1(G, A)$ is a Banach $*$-algebra.*

A *representation* of $\ell^1(G,A)$ is a norm bounded $*$-homomorphism $\pi : \ell^1(G,A) \to B(H)$, where H is a Hilbert space.

Proposition 8.2. *If $\ell^1(G,A)$ has an approximate unit, then a representation of $\ell^1(G,A)$ is realized by a covariant representation of A, and vice versa.* (*It need not be a bijection, see* [**10,** Remark, page 271].)

Consequently, if $\ell^1(G,A)$ has an approximate unit, then a representation of $A \rtimes_{\mathrm{alg}} G$ extends to $\mathbf{F}(G,A)$ if and only if it is covariant if and only if it is bounded in $\ell^1(G,A)$-norm.

Proof. We essentially follow Pedersen's book [**15,** Proposition 7.6.4]. Let $\pi : \ell^1(G,A) \to B(H)$ be a representation on a Hilbert space H. It is a direct sum of a non-degenerate representation and the null-representation. We may ignore the null-part, which we can then add to the covariant representation of A again, and vice versa, and assume that π is non-degenerate. The left and right multiplication of elements $z \in A \rtimes_{\mathrm{alg}} G$ by elements $a \in A, g \in G$ in the algebra $\mathbf{F}(G,A)$, that is, $z \mapsto az$ would be the operator given by left multiplication by a, induce bounded linear maps (even centralizers) L_a, L_g, R_a, R_g from $\ell^1(G,A)$ into itself. Let $(y_i) \subseteq \ell^1(G,A)$ be a given approximate unit. Since π is non-degenerate, $\pi(\ell^1(G,A))H$ is dense in H. Since, for each $\eta = \pi(x)\xi$ ($x \in \ell^1(G,A), \xi \in H$) one has $\|\eta - \pi(y_i)\eta\| \leq \|\pi(x - y_i x)\xi\| \leq \|x - y_i x\|_1 \|\xi\| \to 0$ for $i \to \infty$, $\pi(y_i)$ converges strongly to the unit of $B(H)$. Similarly, for all $a \in A$ and $x \in \ell^1(G,A)$, the Cauchy criterium $\|\pi(ay_i - ay_j)\pi(x)\xi\| \leq \varepsilon$ for all $i,j \geq i_0$ shows that $\pi(ay_i) = \pi(L_a(y_i))$ has a strong limit point $\sigma(a)$. Hence, $\pi(ax) = \lim_i \pi(ay_i x) = \lim_i \pi(ay_i)\pi(x) = \sigma(a)\pi(x)$. Since $\|\pi(y_i a - ay_i)\pi(x)\xi\| \to 0$ for $i \to \infty$, $\sigma(a) = \lim_i \pi(L_a(y_i)) = \lim_i \pi(R_a(y_i))$ (strong limits). In the same manner we define $U_g = \lim_i \pi(L_g(y_i)) = \lim_i \pi(R_g(y_i))$ (strong limits), and one has $\pi(gx) = U_g \pi(x)$ for $g \in G$. Analogously, we define U_g^* for $g \in G$. A direct check shows that (σ, U, H) is a G-covariant representation of A. For instance,

$$\begin{aligned} U_g \sigma(a) U_g^* \pi(x) &= U_g \sigma(a) \pi(g^* x) = \pi(gag^* x) = \pi(g(a) gg^* x) \\ &= \sigma(g(a)) U_g U_g^* \pi(x), \end{aligned}$$

and replacing x by y_i and taking the limit yields (3.7). In particular, we have $\pi(a_g g) = \sigma(a_g) U_g$, which extends by norm continuity to $\ell^1(G,A)$. This shows that π will be assigned to (σ, U, H). On the other hand, starting with a representation (σ, U, H), we define a representation π of $\ell^1(G,A)$ by $\pi(a_g g) = \sigma(a_g)U_g$. $\square$

Corollary 8.3. *If $\ell^1(G,A)$ has an approximate unit, then $A \rtimes G$ (respectively $A \rtimes_s G$) is the C^*-algebra generated by the universal (respectively universal strong) representation of $\ell^1(G,A)$.*

Lemma 8.4. *If G is an inverse semigroup, then $A \rtimes G$ coincides with Khoshkam and Skandalis's definition in* **[10]**, *so is the enveloping C^*-algebra of $\ell^1(G,A)$.*

Proof. Let α be any bounded representation of $\ell^1(G,A)$ on Hilbert space. Then it factors through Khoshkam-Skandalis's crossed product $A \rtimes G$. Any C^*-representation of $A \rtimes G$ is realized as a covariant representation of A by [**10,** Theorem 5.7.(b)], so the same must be true for α.

Hence, a C^*-representation of $\ell^1(G,A)$ is G-covariant. But then, since every G-covariant C^*-representation of $A \rtimes_{\mathrm{alg}} G$ is obviously bounded in $\ell^1(G,A)$-norm, $A \rtimes_{\mathrm{alg}} G$ and $\ell^1(G,A)$ have the same universal G-covariant representation (which induces the C^*-crossed products). $\square$

9. KK^G **for unital** G**.** In this section we will compare Kasparov's equivariant KK-theory with semimultiplicative sets equivariant KK-theory when G happens to be a group. We shall then also introduce a unital version of KK^G-theory for unital semimultiplicative sets G, where we let the unit of G act as the identity on Hilbert modules and C^*-algebras.

Recall that two cycles $(\mathcal{E}, T)$ and $(\mathcal{E}, T')$ in $\mathbf{E}^G(A,B)$ are *compact perturbations* of each other if $a(T - T') \in \mathcal{K}(\mathcal{E})$ for all $a \in A$, and that then the straight line segment from T to T' is an operator homotopy; in particular, $(\mathcal{E}, T)$ and $(\mathcal{E}, T')$ are homotopic in the sense of KK^G-theory (see **[5]**). We will denote Kasparov's equivariant KK-theory for groups G (**[8, 9]**) by $\widetilde{KK^G}(A,B)$.

Proposition 9.1. *Let G be a group (or a unital semimultiplicative set, see Remark 9.2). Let A and B be Hilbert C^*-algebras where the unit of*

G acts identically on A and B, respectively. Then

$$KK^G(A,B) \cong \widetilde{KK^G}(A,B) \oplus \widetilde{KK}(A,B).$$

Proof. The proof of this proposition (which had also been suspected by the author) was indicated by an unknown referee. Let $(\mathcal{E},T)$ be a cycle in $\mathbf{E}^G(A,B)$. By Lemma 4.4 and Corollary 4.6, U_e is a projection and a unit for all U_g, and $U_{g^{-1}} = U_g^*$, and so $U_g U_g^* = U_g^* U_g = U_e$ for all $g \in G$. Hence, $KK^G(A,B)$ and $\widetilde{KK^G}(A,B)$ differ only by the fact that $\widetilde{KK^G}(A,B)$ is build up by cycles $(\mathcal{E},T) \in \widetilde{\mathbf{E}^G}(A,B)$ where U_e acts identically on $\mathcal{E}$.

Denote $u = U_e$. We aim to show that the map

$$\Phi_{A,B} : \mathbf{E}^G(A,B) \longrightarrow \widetilde{\mathbf{E}^G}(A,B) \oplus \widetilde{\mathbf{E}}(A,B)$$
$$\Phi_{A,B}(\mathcal{E},T) = (u\mathcal{E}, uTu) \oplus ((1-u)\mathcal{E}, (1-u)T(1-u))$$

induces an isomorphism in KK-theory. Homotopic elements in $\mathbf{E}^G(A, B)$ become homotopic elements in the image of $\Phi_{A,B}$ via the map $\Phi_{A,B[0,1]}$ (because $U_e \otimes \alpha_e = U_e \otimes 1$ on $\mathcal{E} \otimes_{B[0,1]} B$). The map $\Phi_{A,B}$ has an obvious canonical inverse map $\Phi_{A,B}^{-1}$, which also respects homotopy. Obviously we have $\Phi_{A,B}\Phi_{A,B}^{-1} = 1$. On the other hand,

$$\Phi_{A,B}^{-1}\Phi_{A,B}(\mathcal{E},T) = (\mathcal{E}, uTu + (1-u)T(1-u))$$

is just a compact perturbation of $(\mathcal{E},T)$. Hence, also $\Phi_{A,B}^{-1}\Phi_{A,B} \sim 1$. □

***Remark* 9.2.** The above revealed difference between Kasparov's theory and ours seems natural as usually lacking an identity in G, G-actions are allowed to act degenerate on C^*-algebras or Hilbert modules. This is reflected in the KK^G-theory. If, however, one considers unital G's one can neutralize the difference to Kasparov's theory by assuming that the unit 1_G of G always acts as the identity on Hilbert modules and Hilbert C^*-algebras. Then the whole KK^G-theory of **[5]** goes through under this modification (so one also has an associative Kasparov product). This is clear as we only have to take care that all used constructions of G-Hilbert modules respect the unitization, and these are the tensor products and the direct sum where it is obvious. Furthermore, one has to ensure that under modified KK^G-theory the class 1 in $KK^G(\mathbf{C},\mathbf{C})$ associated to the cycle $(\mathbf{C},0)$ (as used in

Section 7 of **[5]**) exists; but this is also clear. Actually, the proof of Proposition 9.1 works (without essential modification) for any unital semimultiplicative set G, that is, KK^G is the direct sum of the unital version of KK^G, where the unit of G acts fully on Hilbert C^*-algebras and Hilbert bimodules, and Kasparov's $\widetilde{KK}$.

10. Inversely generated semigroups.

Definition 10.1. We call an element g of an involutive semigroup $\overline{G}$ a *partial isometry* if it is invertible with respect to the involution, that is, if $gg^*g = g$.

Note that if s is a partial isometry then s^* is also one. Consequently, the set of partial isometries of an involutive semigroup is self-adjoint.

Definition 10.2. An *inversely generated semigroup* is an involutive semigroup $\overline{G}$ which is generated by its partial isometries. In other words, for every $g \in \overline{G}$ there exist partial isometries $s_1, \ldots, s_n \in \overline{G}$ such that $g = s_1 \ldots s_n$.

The standard example for an inversely generated semigroup is the involutive semigroup G^* for a semimultiplicative set G (Definition 5.3). (The set of partial isometries of G^* might differ from G, since there could exist more partial isometries.)

Definition 10.3. A *$*$-morphism* between involutive semigroups is a map with respect to multiplication and involution. A *$*$-antimorphism* between involutive semigroups is an involution respecting semigroup antimorphism.

We shall write G for the set of partial isometries of an inversely generated semigroup $\overline{G}$. G is a semimultiplicative set which usually is not associative. (One can easily construct examples where $st \in G$ and $(st)u \in G$ are partial isometries, but $tu \notin G$ is not one; this contradicts the associativity condition.)

Definition 10.4. A $\overline{G}$-Hilbert C^*-algebra is a semimultiplicative set G-Hilbert C^*-algebra A where the action maps $\alpha, \alpha^* : G \to \mathrm{End}\,(A)$ extend to a map $\overline{\alpha} : \overline{G} \to \mathrm{End}\,(A)$

$$\overline{\alpha}(g) = \alpha(g), \tag{10.1}$$

$$\overline{\alpha}(g^*) = \alpha^*(g), \tag{10.2}$$

$$\overline{\alpha}(hk) = \overline{\alpha}(h)\overline{\alpha}(k) \tag{10.3}$$

for all $g \in G$ and $h, k \in \overline{G}$.

Since $\overline{\alpha}$ maps into the partial isometries of A which have commuting source and range projections (in the center of the multiplier algebra), $\overline{\alpha}$ is actually a $*$-morphism.

Definition 10.5. A $\overline{G}$-Hilbert module is a Hilbert module which is endowed with a general semimultiplicative set G-action α that extends to a map $\overline{\alpha}$ via the formulas (10.1)–(10.3).

Note that the G-action $\overline{\alpha}$ on a Hilbert module is usually not realized by partial isometries; only the partial isometries of $\overline{G}$, that is the elements of G, go over to partial isometries (because a semimultiplicative set G-action is always realized by partial isometries). These partial isometries determine how we have to define the other elements of $\overline{G}$, as they can be written as products of elements of G. These products, however, need not be partial isometries on the Hilbert module.

We may equivalently reformulate Definition 10.4 (and similarly Definition 10.5) by saying that the G^*-action $\widehat{\alpha}$ on A factors through $\overline{G}$.

$$\begin{tikzcd} G^* \arrow[r, "\hat{\alpha}"] \arrow[d, "p"] & A \\ \overline{G} \arrow[ur, "\overline{\alpha}"'] & \end{tikzcd}$$

Here, p is the quotient $*$-morphism determined by $p(g) = g$ for all $g \in G$. Indeed, if α allows an extension $\overline{\alpha}$ given by (10.1)–(10.3), then the above diagram commutes. On the other hand, if the above diagram exists, $\overline{\alpha}$ is an extension of α satisfying (10.1)–(10.3).

Because of this fact, we view a $\overline{G}$-Hilbert module also as a G-Hilbert module with the property that the induced G^*-map factors through G. We sloppily say that the G-Hilbert module factors through $\overline{G}$.

Lemma 10.6. *Identities* (3.9) *also hold for all $g \in G^*$.*

Proof. We leave the inductive proof to the reader and sketch only one identity modulo $I_A(\mathcal{E})$; note that $g(\mathcal{K}(\mathcal{E})), g^*(\mathcal{K}(\mathcal{E})) \subseteq \mathcal{K}(\mathcal{E})$ for all $g \in G$. For $g \in G$ and some $h \in G^*$ (given by inductive hypothesis) we have

$$U_g U_h T U_h^* U_g^* \equiv U_g T U_h U_h^* U_g^* \equiv U_g T U_g^* U_g U_h U_h^* U_g^* \equiv T U_g U_h U_h^* U_g^*. \quad \square$$

A G-equivariant homomorphism $\pi : A \to \mathcal{L}(\mathcal{E})$ (Definition 3.4) is automatically G^*-equivariant by Lemma 5.8 (ii). Thus, it is also $\overline{G}$-equivariant when the G-Hilbert module $\mathcal{E}$ and G-Hilbert C^*-algebra A which appear factor through $\overline{G}$. Such a similar fact can also be said for a cycle $(\mathcal{E}, T) \in \mathbf{E}^G(A, B)$. By Lemma 10.6, identities (3.9) also hold for $g \in \overline{G}$ if all Hilbert modules $\mathcal{E}, A$ and B factor through $\overline{G}$. The following definition thus seems natural.

Definition 10.7. We define $\overline{G}$-equivariant KK-theory in the same way as KK^G-theory but with the addition that all G-Hilbert modules and G-Hilbert C^*-algebras which appear factor through $\overline{G}$.

In other words, $KK^{\overline{G}}$-theory is built up by $\overline{G}$-Hilbert modules rather than by G-Hilbert modules as in KK^G-theory.

It is easy to see that the category of $\overline{G}$-Hilbert modules is stable under tensor products and direct sums. Also, any Hilbert module is a $\overline{G}$-Hilbert module under the trivial $\overline{G}$-action. We have thus checked that all discussion and theorems like the Kasparov product in **[5]** carry over from KK^G to $KK^{\overline{G}}$ (compare with Remark 9.2).

We say a representation $\phi : \mathbf{F}(G, A) \to B(H)$ factors through $\overline{G}$ if the restriction map $\phi|_{G^*}$ factors through $\overline{G}$. (Analogously and equivalently, the G-equivariant representation $(\phi|_A, \phi|_G, H)$ is said to factor through H). We prefer to view a crossed product of A by $\overline{G}$ as a special crossed product of A by G and introduce the following definition.

Definition 10.8. The full crossed product $A \rtimes \overline{G}$ is the norm closure of $\phi^{\overline{G}}(A \rtimes_{\mathrm{alg}} G)$, where $\phi^{\overline{G}}$ denotes the universal representation of $\mathbf{F}(G, A)$ which factors through $\overline{G}$.

11. Hilbert bimodules over full crossed products. In the remainder of this paper we are going to prove the descent homomorphism. In this and the remaining sections H and G denote discrete countable semimultiplicative sets. We may either assume that H and G have units 1_H and 1_G and treat everything in the unital world of KK-theory (see Remark 9.2), and define the product of H and G by $H \times G$; or we consider the non-unital version, in this case defining the product of H and G as the semimultiplicative set $H \sqcup G \sqcup H \times G$ with multiplication

$$h \cdot g := (h, g), \ h \cdot (h', g') := (hh', g'), \ (g, h) \cdot (g', h') := (gg', hh'),$$

and so on for $h, h' \in H$ and $g, g' \in G$, and denote this product, by sloppy but suggestive notation, still as $H \times G$. In any case, a morphism $H \times G \to K$ is determined by its restriction to H and G, where H and G are identified with $H \times 1_G$ and $1_H \times G$, respectively, in the unital case.

For all $H \times G$-actions on Hilbert modules or C^*-algebras, we require that the induced H^*-actions and G^*-actions (in the sense of Lemmas 5.7 and 5.8) commute: the point is that h^* may not commute with g otherwise ($h \in H, g \in G$). This requirement also affects the definition of $KK^{H \times G}$, and in this sense the notion $KK^{H \times G}$ is suggestive but sloppy. (See the discussion in Remark 9.2 why we can slightly adjust equivariant KK-theory. Actually, we only need stability under tensor products, direct sums, and the existence of $1 = (\mathbf{C}, 0)$ in $KK^G(\mathbf{C}, \mathbf{C})$.)

Let $l \in \{\emptyset, s, r, i\}$ and D be a G-Hilbert C^*-algebra. Let $\phi_{D,G,l}$ be the representation of $\mathbf{F}(G, D)$ induced by the universal G-covariant representation (in case that $l = \emptyset$), or the universal strong G-covariant representation (when $l = s$), or the reduced representation of D (when $l = r$).

The case $l = i$ requires that we are given an inversely generated semigroup denoted by $\overline{G}$ and $\overline{H}$, and G and H, respectively, denote their subsets of partial isometries. In this case all G-Hilbert modules and G-Hilbert C^*-algebras which appear are supposed to factor through $\overline{G}$ (and similarly so for H and $G \times H$) in accordance with Definition 10.7.

If $l = i$, then we need to work with $\overline{G}$-equivariant KK-theory, that is, $KK^{G \rtimes H}$ means then actually Moreover, $\phi_{D,G,i}$ denotes the universal $\overline{G}$-factorizing G-covariant representation of D, and $D \rtimes_i G$ will stand for $D \rtimes \overline{G}$ (Definition 10.8).

We shall sometimes write ϕ_l rather than $\phi_{D,G,l}$ if D and G are clear from the context. Recall that

$$D \rtimes_l G \cong \overline{\phi_{D,G,l}(D \rtimes_{\mathrm{alg}} G)}.$$

We denote

$$G' = \{g, g^* \in G^* \mid g \in G\}.$$

If $l = r$, then we deal with the reduced crossed product, and in this case we assume that G is an associative semimultiplicative set with left cancelation, and all G-Hilbert modules and G-Hilbert C^*-algebras have transferred left cancelation. So, in this sense, we also have a modified KK^G-theory as we adapt it in the sense that it is build up by modules with left transferred cancelation (confer Remark 9.2 why we can easily slightly adapt KK-theory). However, we do not require cancelation for H or its actions. If $l = r$, then we assume that $B = \mathbf{C}$ equipped with the trivial G-action.

We will assume that G has a unit, partially because of non-degenerateness concerns as in Lemma 13.1. Nevertheless, we shall sometimes try to avoid using a unit.

Assume that A, B are $(H \times G)$-Hilbert C^*-algebras and $\mathcal{E}$ is a $(H \times G)$-Hilbert B-module. The G-action on $\mathcal{E}$ is denoted by U.

Lemma 11.1.

(i) *$B \rtimes_l G$ is an $H \times G$-Hilbert C^*-algebra (where the G-action is trivial).*

(ii) *Under a different $H \times G$-action denoted by V, $B \rtimes_l G$ is a $H \times G$-Hilbert module over the $H \times G$-Hilbert C^*-algebra $B \rtimes_l G$. This Hilbert module is denoted by $B \rtimes_l^{\mathrm{Mod}} G$.*

Proof. (i) Let $\phi_l = \phi_{B,G,l}$. We endow $B \rtimes_l G$ with the $H \times G$-Hilbert C^*-action

$$\alpha_{h \times g}\big(\phi_l(b_k k)\big) = \phi_l\big(h(b_k)k\big) =: \psi(b_k k) \tag{11.1}$$

for $k \in G^*$, $b_k \in B_k$ and $h \times g \in (H \times G)'$. (So the G-action is trivial.) We claim that $\psi : \mathbf{F}(G,B) \to B \rtimes_l G$ is a representation. We need to show that $(\psi|_B, \psi|_G)$ is G-covariant, where $\psi(b) = \phi_l(h(b))$ and $\psi(g) = \phi_l(g)$. Let us check (3.5). In $\phi_l(\mathbf{F}(G,B))$ we have

$$\begin{aligned}\psi(g)\psi(g)^*\psi(b) &= \phi_l(g)\phi_l(g)^*\phi_l(h(b)) \\ &= \phi_l(gg^*h(b)) = \phi_l(gg^*(h(b))gg^*) \\ &= \phi_l(h(b)gg^*) = \psi(b)\psi(g)\psi(g)^*,\end{aligned}$$

where $gg^*(b)gg^* = bgg^*$ is identity (5.4) (Lemma 5.14 (ii)).

In the case where l indicates the full or full strong crossed product, the map $\alpha_{h\times g}$ extends to a well-defined endomorphism of $B \rtimes_l G$ by Lemma 6.4. For the reduced crossed product we see the boundedness of $\alpha_{h\times g}$ by direct evaluation of the left regular representation of Definition 7.3: one computes

$$\Big\|\phi_r\Big(\sum_{k\in G^*} h(b_k)k\Big)\xi\Big\| \le \Big\|\phi_r\Big(\sum_{k\in G^*} b_k k\Big)\xi\Big\|$$

for all $\xi \in H \otimes \ell^2(G)$.

It remains to check the identities of Definition 3.3 to see that α is a $G \times H$-action on $B \rtimes_l G$. For instance, by Lemma 5.8 (iii), one has

$$\begin{aligned}\big\langle \alpha_{h\times g}\,\phi_l(b_k k), \phi_l(c_m m)\big\rangle &= \phi_l\big(k^*\,h(b_k^*)c_m\,m\big) \\ &= \phi_l\big(k^*\,h(b_k^*\,h^*(c_m))\,m\big) \\ &= \alpha_{h\times g}\,\big\langle \phi_l(b_k k), \alpha^*_{h\times g}\,\phi_l(c_m m)\big\rangle.\end{aligned}$$

(ii) We make $B \rtimes_l G$ a Hilbert $B \rtimes_l G$-module $B \rtimes_l^{\mathrm{Mod}} G$ with inner product $\langle x,y\rangle = x^*y$and $(H \times G)$-Hilbert $B \rtimes_l G$-module action

$$V_{h\times g}\big(\phi_l(b_k k)\big) = \phi_l\big(g\big(h(b_k)\big)gk\big) \tag{11.2}$$

for all $k \in G^*$, $b_k \in B_k$ and $h \times g \in (H \times G)'$. Note that

$$V_{h\times g}\big(\phi_l(x)\big) = \phi_l(g)\,\alpha_h(\phi_l(x)) \tag{11.3}$$

$(x \in A \rtimes_{\mathrm{alg}} G)$, which shows the boundedness of $V_{h\times g}$. Then

V is an action, and we shall demonstrate only one rule:

$$\begin{aligned}\big\langle V_g\phi_l(x), \phi_l(y)\big\rangle &= \phi_l(x^*)\phi_l(g^*)\phi_l(y)\\ &= \big\langle \phi_l(x), V_g^*\phi_l(y)\big\rangle\\ &= \alpha_g\big\langle \phi_l(x), V_g^*\phi_l(y)\big\rangle.\end{aligned}$$

□

Lemma 11.2. *There is an $H\times G$-equivariant homomorphism $\tau : B \to \mathcal{L}(B \rtimes_l^{\mathrm{Mod}} G)$ given by left multiplication, i.e.,*

$$\tau(b)\big(\phi_l(x)\big) = \phi_l(b)\phi_l(x)$$

for $b \in B$ and $x \in B \rtimes_{\mathrm{alg}} G$.

Proof. We only check (3.7)–(3.8). Let $k \in G^*$, $g \times h \in (G \times H)'$, $b \in B$ and $c_k \in B_k$. Then we have

$$\begin{aligned}V_{g\times h}\tau(b)\, V^*_{g\times h}\phi_l(c_k k) &= V_{g\times h}\tau(b)\phi_l(g^*)\,\phi_l\big(h^*(c_k)k\big)\\ &= \phi_l(g)\,\phi_l\big(h(bg^*h^*(c_k))g^*k\big)\\ &= \phi_l\big(gh(bg^*h^*(c_k))gg^*k\big)\\ &= \tau(gh(b))\,V_{h\times g}V^*_{h\times g}\phi_l(c_k k).\end{aligned}$$

Notice that here we used the requirement that the G- and H-actions (and their adjoint actions) commute. □

Definition 11.3. Define an $H \times G$-Hilbert module over $B \rtimes_l G$ by

$$\mathcal{E} \rtimes_l G = \mathcal{E} \otimes_B (B \rtimes_l^{\mathrm{Mod}} G)$$

(internal tensor product of $H \times G$-Hilbert modules), where B acts on $B \rtimes_l^{\mathrm{Mod}} G$ by left multiplication (Lemma 11.2).

By definition, $\mathcal{E} \rtimes_l G$ is an $H \times G$-Hilbert module over the $H \times G$-Hilbert C^*-algebra $B \rtimes_l G$ under the diagonal action $U \otimes V$ (see **[5,** Lemma 4]). Here, V denotes the $H \times G$-action on $B \rtimes_l G$, see (11.2). Note that, if $l = i$, then both $B \rtimes_i G$ and $B \rtimes_i^{\mathrm{Mod}} G$ factor through $\overline{H} \times \overline{G}$ under their actions α and V ((11.1), (11.3)), respectively. Consequently, the tensor product $\mathcal{E} \rtimes_i G$ factors through $\overline{H} \times \overline{G}$.

Proposition 11.4. *If l indicates one of the full crossed products, i.e., $l \in \{\emptyset, s, i\}$, then $\mathcal{E} \rtimes_l G$ is an H-Hilbert $(A \rtimes_l G, B \rtimes_l G)$-bimodule.*

Proof. $A \rtimes_l G$ is an H-Hilbert C^*-algebra by Lemma 11.1. Let $U \otimes V$ be the diagonal $H \times G$-action on $\mathcal{E} \otimes_B (B \rtimes_l^{\mathrm{Mod}} G)$. Note that $U_g \otimes V_g$ is an adjoint-able operator as the G-action on $B \rtimes_l G$ is trivial (see (11.1)). Let $\phi_l = \phi_{A,G,l}$. We define a $*$-homomorphism $\Theta_l : A \rtimes_l G \to \mathcal{L}(\mathcal{E} \rtimes_l G)$ by

$$\Theta_l(\phi_l(a_g g)) = (a_g \otimes 1)(U_g \otimes V_g), \tag{11.4}$$

where $a_g \in A_g, g \in G^*$. It is induced by the G-covariant representation $a \mapsto a \otimes 1$ and $g \mapsto U_g \otimes V_g$ (Lemma 6.4), because $U_g \otimes V_g$ is partial isometry in the C^*-algebra $\mathcal{L}(\mathcal{E} \rtimes_l G) \subseteq B(\mathcal{H})$ ($\mathcal{H}$ a Hilbert space). When $l = i$, then Θ_l is also well defined as $g \mapsto U_g \otimes V_g$ factors through $\overline{G}$ (see (11.3)). For the H-equivariance of Θ, we compute

$$U_h \otimes V_h \Theta(\phi_l(a_g g))\, U_h^* \otimes V_h^* = \Theta\big(\phi_l\big(h(a_g) g\big)\big)\, U_h U_h^* \otimes V_h V_h^*. \tag{11.5}$$ $\square$

12. Hilbert bimodules over reduced crossed products. The discussion in this section is only related to the reduced crossed product, that is, when $l = r$. Recall that in this case we only allow $B = \mathbf{C}$ with the trivial G-action. (Nevertheless we shall write B rather than $\mathbf{C}$ in this section.) Consequently, the operator U_g ($g \in G$) on a B-Hilbert module $\mathcal{E}$ is adjoint-able by (3.3). For the boundedness of the action of $A \rtimes_r G$ on $\mathcal{E} \rtimes_r G$ in Proposition 12.4 below, we will need a standard intertwining trick for covariant representations tensored by the left regular representation, see for instance [**6,** Appendix A, Lemma A.18.(ii)].

Let $\mathcal{E} \otimes \ell^2(G)$ be the skew tensor product of G-Hilbert modules. By Lemma 7.7, there is an isomorphism

$$\mathcal{E} \otimes \ell^2(G) \cong (\mathcal{E} \otimes_B B) \otimes (\mathbf{C} \otimes_{\mathbf{C}} \ell^2(G)) \cong \mathcal{E} \otimes_B \big(B \otimes \ell^2(G)\big). \tag{12.1}$$

Define a partial isometry W on $\mathcal{E} \otimes \ell^2(G)$ by

$$W(x_t \otimes e_t) = U_t(x_t) \otimes e_t$$

for all $t \in G$ and $x_t \in \mathcal{E}$ (Lemma 4.2). Let

$$\Gamma : A \rtimes_{\mathrm{alg}} G \longrightarrow \mathcal{L}(\mathcal{E} \otimes \ell^2(G)) \tag{12.2}$$

be induced by the covariant representation

$$\Gamma(a) = (a \otimes 1), \qquad \Gamma(g) = U_g \otimes \lambda_g \tag{12.3}$$

for all $a \in A, g \in G$. Recall that we write

$$A \rtimes_\Gamma G = \overline{\Gamma(A \rtimes_{\mathrm{alg}} G)}.$$

Lemma 12.1. *WW^* commutes with the G-action $U \otimes V$, with $A \otimes 1$ and with $A \rtimes_\Gamma G$.*

Proof. One checks that the projection WW^* commutes with the adjoint-able partial isometry $U_g \otimes \lambda_g$ (and so with $U_g^* \otimes \lambda_g^*$) and $a \otimes 1$ for all $g \in G$ and $a \in A$. (One uses $U_{\rho_g(t)} U_{\rho_g(t)}^* U_g = U_{gtt^*g^*g} = U_{gt(g^*gt)^*} = U_{gtt^*}$ by transferred left cancelation and Lemma 7.2.) $\square$

Definition 12.2. G is called *non-degenerate* if for all Hilbert (A,B)-bimodules and all $x \in A \rtimes_\Gamma G$, $xWW^* = 0$ implies $x = 0$.

If G is a groupoid, then WW^* is an identity for $A \rtimes_\Gamma G$ and so G is non-degenerate. Indeed, every $y \in \Gamma(A \rtimes G)$ can be written as a product of elements of the form $x = (a_g \otimes 1)(U_g \otimes \lambda_g) \in A \rtimes_\Gamma G$ for $g \in G'$. Let $\eta := \xi_t \otimes e_t \in \mathcal{E} \otimes \ell^2(G)$. Then

$$xWW^*\eta = a_g U_g U_t U_t^* \xi_t \otimes \lambda_g e_t = a_g U_g \xi_t \otimes \lambda_g e_t = x\eta$$

by Lemma 4.6.

Our motivating examples for reduced crossed products were semi-multiplicative sets like directed graphs. A prototype-example is $G = \mathbf{N}_0$. By showing in the next lemma that $\mathbf{N}_0$ is non-degenerate, we would like to demonstrate that non-degenerateness may not be a too restrictive condition.

Lemma 12.3. *$\mathbf{N}_0$ is non-degenerate.*

Proof. Let S denote the $\mathbf{N}_0$-action on a Hilbert module $\mathcal{E}$ with transferred left cancelation. We claim that every word S_g for $g \in \mathbf{N}_0^*$ allows a representation as $S_g = S_n S_k^* = S_1^n (S_1^k)^*$ for $n, k \in \mathbf{N}_0$. Indeed, S_0 is a unit for every word, as in particular S_0 is self-adjoint by Lemma 4.4. Also, $S_0 = S_1^* S_1 S_0 = S_1^* S_1$ by transferred left cancelation. The claim then follows by induction on the length of a word.

Let $X \subseteq A \rtimes_{\mathrm{alg}} G \subseteq \mathbf{F}(G, A)$ denote the set of elements of the form $a = \sum_{n,k \in \mathbf{N}_0} a_{n,k} nk^*$ for $a_{n,k} \in A$ (recall identity (5.5) which holds in

$\mathbf{F}(G,A))$. By the above claim, $\Gamma(X) = \Gamma(A \rtimes_{\mathrm{alg}} G)$. Write $p = WW^*$. To check Definition 12.2, assume that $T \in A \rtimes_\Gamma G$ satisfies $Tp = 0$. Then there is a sequence $T^i = \sum_{n,k\in\mathbf{N}_0} a^i_{n,k} nk^*$ in X such that $\Gamma(T^i)$ converges in norm to T.

In $\mathcal{E} \otimes \ell^2(\mathbf{N}_0)$ and by (12.3) we have

$$\begin{aligned} \Gamma(T^i)(x_0 \otimes e_0) &= \sum_{n,k\in\mathbf{N}_0} a^i_{n,k} S_{nk^*}(x_0) \otimes \lambda_{nk^*}(e_0) \\ &= \sum_{n\in\mathbf{N}_0} a^i_{n,0} S_n x_0 \otimes e_n \\ &= \Gamma(T^i)p(x_0 \otimes e_0) \longrightarrow Tp(x_0 \otimes e_0) = 0 \qquad (12.4) \end{aligned}$$

when $i \to \infty$, since $Tp = 0$, for all $x_0 \in \mathcal{E}$. Similarly, we have

$$\begin{aligned} \Gamma(T^i)(x_1 \otimes e_1) &= \sum_{n\in\mathbf{N}_0} a^i_{n,0} S_n x_1 \otimes e_{n+1} \qquad (12.5) \\ &\quad + \sum_{n\in\mathbf{N}_0} a^i_{n,1} S_n(S_1^* x_1) \otimes e_n, \\ \Gamma(T^i)p(x_1 \otimes e_1) &= (1 \otimes \lambda) \sum_{n\in\mathbf{N}_0} a^i_{n,0} S_n(S_1 S_1^* x_1) \otimes e_n \qquad (12.6) \\ &\quad + \sum_{n\in\mathbf{N}_0} a^i_{n,1} S_n(S_1^* x_1) \otimes e_n \longrightarrow 0. \qquad (12.7) \end{aligned}$$

The convergence is here because of $Tp = 0$. Entering convergence (12.4) in convergence (12.6)–(12.7) shows that (12.5) converges to zero (using convergence (12.4) again). One can proceed in this way further by considering $\Gamma(T_i)(x_2 \otimes e_2)$ and showing that it converges to zero, and so on. In this way, we get $T(x) = \lim_{i\to\infty} \Gamma(T_i)(x) = 0$ for all $x \in \mathcal{E} \odot \ell^2(\mathbf{N}_0)$. Hence $T = 0$. $\square$

We now come to the main result of this section.

Proposition 12.4. *$\mathcal{E} \rtimes_r G$ is an H-Hilbert $(A \rtimes_r G, B \rtimes_r G)$-bimodule.*

Proof. We want to define the action Θ_r of $A \rtimes_r G$ on $\mathcal{E} \rtimes_r G$ as in (11.4). Thus, we aim to define Θ_r on $\phi_r(A \rtimes_{\mathrm{alg}} G)$ by $\Theta_r \phi_r = \varphi$, where $\varphi : A \rtimes_{\mathrm{alg}} G \to \mathcal{L}(\mathcal{E} \rtimes_r G)$ is determined by

$$\varphi(a_g g) = (a_g \otimes 1)(U_g \otimes V_g).$$

We have a commutative diagram:

$$\begin{tikzcd}
A \rtimes_{\mathrm{alg}} G \arrow[r, "\varphi"] \arrow[dr, "\Gamma"'] & \mathcal{L}(\mathcal{E} \otimes_B (B \rtimes_r G)) \arrow[r, "\mu"] \arrow[d, "f"] & \mathcal{L}(\mathcal{E} \otimes_B (B \rtimes_r G) \otimes_{B \rtimes_r G} (B \otimes \ell^2(G))) \arrow[d, "\mu_1"] \\
& \mathcal{L}(\mathcal{E} \otimes \ell^2(G)) & \mathcal{L}(\mathcal{E} \otimes_B (B \otimes \ell^2(G))) \arrow[l, "\mu_2"]
\end{tikzcd}$$

Here, $B \rtimes_r G$ acts on $B \otimes \ell^2(G)$ by ζ of Lemma 7.8, μ is the injective map of Lemma 7.6, μ_1 the isomorphism induced by the isomorphism of Lemma 7.5 and μ_2 the isomorphism induced by the isomorphism (12.1). It is important here that G acts trivially on B. Hence, in the right bottom corner of the above diagram, B acts on $B \otimes \ell^2(G)$ by left multiplication (so acts only on B). Let $f := \mu_2 \mu_1 \mu$, which is injective. A tedious computation (similar to that of Lemma 7.8) yields

$$f\big(\varphi(a_g g)\big)(x_t \otimes e_t) = a_g U_g x_t \otimes \lambda_g e_t = \Gamma(a_g g)$$

for $g \in G^*, t \in G, x_t \in \mathcal{E}$ and $a_g \in A_g$. Hence, $f\varphi = \Gamma$ on $A \rtimes_{\mathrm{alg}} G$.

In order that Θ_r is evidently a well-defined continuous map we need to show that

$$\|\Theta_r(\phi_r(x))\| = \|\varphi(x)\| = \|f(\varphi(x))\| = \|\Gamma(x)\| \le \|\phi_r(x)\|_{A \rtimes_r G}$$

for all $x \in A \rtimes_{\mathrm{alg}} G$. Only the last inequality needs a discussion; the other identities are clear.

Since G is non-degenerate (Definition 12.2), the homomorphism

$$\nu : A \rtimes_\Gamma G \longrightarrow (A \rtimes_\Gamma G)WW^*$$

given by $\nu(x) = xWW^*$ (see Lemma 12.1) is an isometry. Thus, $\|WW^*\Gamma(x)\| = \|\Gamma(x)\|$ for all $x \in A \rtimes_{\mathrm{alg}} G$.

By Lemma 7.2 and the fact that U has transferred left cancelation, we thus have

$$\begin{aligned}
\Gamma(a_g g)WW^*(\xi_t \otimes e_t) &= a_g U_g U_t U_t^* \xi_t \otimes \lambda_g(e_t) \\
&= a_g U_{\rho_g(t)} U_t^* \xi_t \otimes e_{\rho_g(t)} \\
&= U_{\rho_g(t)} U_{\rho_g(t)}^* a_g U_{\rho_g(t)} U_t^* \xi_t \otimes e_{\rho_g(t)} \\
&= U_{\rho_g(t)} \big((\rho_g(t))^*(a_g)\big) U_t^* \xi_t \otimes e_{\rho_g(t)} \\
&= \big(W \phi_r(a_g g) W^*\big)(\xi_t \otimes e_t)
\end{aligned}$$

for $t \in G$, $g \in G^*$, $a_g \in A_g$ and $\xi_t \in \mathcal{E}$, and when $\rho_g(t)$ is defined. (Note that $\mathcal{E}$ is actually a Hilbert space.) This thus shows

$$\|\Gamma(x)\| = \|\Gamma(x)WW^*\| = \|W\phi_r(x)W^*\| \leq \|\phi_r(x)\|. \qquad \square$$

13. The descent homomorphism. Let B_1 and B_2 be $H \times G$-Hilbert modules. Let $(\mathcal{E}_1, T_1) \in \mathbf{E}^G(A, B_1)$ and $(\mathcal{E}_2, T_2) \in \mathbf{E}^G(B_1, B_2)$. Write $\mathcal{E}_{12} = \mathcal{E}_1 \otimes_{B_1} \mathcal{E}_2$.

Lemma 13.1. *There is an H-Hilbert module isomorphism*

$$\mathcal{E}_{12} \rtimes_l G \cong (\mathcal{E}_1 \rtimes_l G) \otimes_{B_1 \rtimes_l G} (\mathcal{E}_2 \rtimes_l G).$$

Proof. In the category of H-Hilbert modules $B_2 \rtimes_l G$ and $B_2 \rtimes_l^{\mathrm{Mod}} G$ are identic, as they differ only in their G-action (see Lemma 11.1). The map $\varphi : B_1 \to B_1 \rtimes_l G$ given by $\varphi(b) = b1_G$ is an H-equivariant homomorphism of H-Hilbert C^*-algebras (Definition 3.2). By **[5,** Lemma 14], there is an isomorphism of H-Hilbert modules

$$\mathcal{E}_1 \otimes_{B_1} (B_1 \rtimes_l G) \otimes_{B_1 \rtimes_l G} \big(\mathcal{E}_2 \otimes_{B_2} (B_2 \rtimes_l G)\big) \cong \mathcal{E}_1 \otimes_{B_1} \mathcal{E}_2 \otimes_{B_2} (B_2 \rtimes_l G). \qquad \square$$

Lemma 13.2. *If $(\mathcal{E}_{12}, T_{12})$ is a Kasparov product, then $R = [T_1 \otimes 1, T_{12}]$ belongs to $Q_A(\mathcal{E}_{12})$, further $R \geq 0$ modulo $I_A(\mathcal{E}_{12})$, and the elements*

$$\begin{aligned} g(R) - g(1)R &= U_g R U_g^* - U_g U_g^* R, \\ g(1)R - Rg(1) &= U_g U_g^* R - R U_g U_g^* \end{aligned}$$

are in $I_A(\mathcal{E}_{12})$ for all $g \in G'$.

Proof. The first two assertions follows from the Remark below Definition 2.10 in **[9]**, applied to the trivial group $G = \{e\}$. Let $a \in A$, $a' = g^*(a)$ and $T_1' = T_1 \otimes 1$. For simplicity, we only compute the case when $\partial a = 0$. Modulo $\mathcal{K}(\mathcal{E}_{12})$, we have

$$\begin{aligned} ag(T_{12}T_1') &= ag(g^*(1)T_{12}T_1') = g(a'g^*(1)T_{12}T_1') \\ &\equiv g(a'T_{12}g^*(1)T_1') = ag(T_{12})g(T_1') \\ &\equiv aT_{12}g(1)g(T_1') \equiv T_{12}ag(T_1') \\ &= T_{12}(k \otimes g(1)) + T_{12}T_1'ag(1), \end{aligned}$$

where $k = ag(T_1) - T_1 g(1)a \in \mathcal{K}(\mathcal{E}_1)$. Similarly, we compute

$$ag(T_1' T_{12}) = (k \otimes g(1))T_{12} + T_1' T_{12} ag(1).$$

Hence,

$$ag([T_1', T_{12}]) - [T_1', T_{12}]g(1)a \equiv [k \otimes g(1), T_{12}] \equiv 0$$

by [**5,** Lemma 10.(1)]. Also, one has $[a, [T_1', T]] \equiv 0$ by this lemma. A similar computation yields the last claim. □

The following lemma is a standard result for crossed products.

Lemma 13.3. *If D is a C^*-algebra with trivial G-action, then $(A \otimes_{\max} D) \rtimes G \cong (A \rtimes G) \otimes_{\max} D$ (also for the strong crossed product) and $(A \otimes_{\min} D) \rtimes_r G \cong (A \rtimes_r G) \otimes_{\min} D$ canonically.*

Theorem 13.4. *Let A and B be $H \times G$-Hilbert C^*-algebras and $l \in \{\emptyset, s, r, i\}$. Assume that G is unital. For all $G \times H$-actions which appear on Hilbert modules and C^*-algebras we require that the induced H^*-actions and G^*-actions commute. If $l = r$, then we assume that G is non-degenerate and associative and has left cancelation, all G-Hilbert modules and G-Hilbert C^*-algebras have transferred left cancelation, and $B = \mathbf{C}$ with the trivial G-action. Then there exists a descent homomorphism*

$$j_l^G : KK^{H \times G}(A, B) \longrightarrow KK^H(A \rtimes_l G, B \rtimes_l G)$$

given by

$$j_l^G(\mathcal{E}, T) = (\mathcal{E} \rtimes_l G, T \otimes 1)$$

for all $(\mathcal{E}, T) \in \mathbf{E}^{H \times G}(A, B)$. Moreover, the following two points hold true:

(a) *If $x_1 \in KK^{H \times G}(A, B_1)$, $x_2 \in KK^{H \times G}(B_1, B_2)$ and the intersection product $x_1 \otimes_{B_1} x_2$ exists, then*

$$j_l^G(x_1 \otimes_{B_1} x_2) = j_l^G(x_1) \otimes_{B_1 \rtimes_l G} j_l^G(x_2).$$

(b) *If $A = B$ is σ-unital, then $j_l^G(1_A) = 1_{A \rtimes_l G}$.*

Proof. In our proof we essentially follow Kasparov [**9**]. We define compact operators $\theta_{\xi,\eta} \in \mathcal{K}(\mathcal{F})$ by $\theta_{\xi,\eta}(x) = \xi\langle \eta, x\rangle$, where $\xi, \eta, x \in \mathcal{F}$ and $\mathcal{F}$ is any Hilbert module. Write Z for the diagonal G-Hilbert action

$U \otimes V$ on $\mathcal{E} \otimes_B (B \rtimes_l^{\mathrm{Mod}} G)$. Let $\phi_l = \phi_{B,G,l}$. Let (a_i) be an approximate unit in B. Let $E \in \mathcal{E}$ and $F \in B \rtimes_l G$. Let $x, y \in G^*$. Then one has (in $\mathcal{E} \otimes_B (B \rtimes_l^{\mathrm{Mod}} G)$)

$$\begin{aligned}
& \theta_{U_{xy^*}(\xi) \otimes \phi_l(xy^*(a_i)x),\, \eta \otimes \phi_l(yy^*(a_i)y)}(E \otimes F) \\
= \;& U_{xy^*}(\xi) \otimes \phi_l\big(xy^*(a_i)x\big) \big\langle \eta \otimes \phi_l\big(yy^*(a_i)y\big), E \otimes F \big\rangle \\
= \;& U_{xy^*}(\xi) \otimes \phi_l\big(xy^*(a_i)x\big)\, \phi_l\big(yy^*(a_i)y\big)^* \, \phi_l\big(\langle \eta, E\rangle\big)\, F \\
= \;& U_{xy^*}(\xi) \otimes \phi_l\big(xy^*(a_i)x\, y^* yy^*(a_i^*)y^* \,\langle \eta, E\rangle\big) F \\
= \;& U_{xy^*}(\xi) \otimes \phi_l\big(xy^*(a_i)\, xy^*(a_i^*)\, xy^*(\langle \eta, E\rangle)\big)\, \phi_l(xy^*) F \\
= \;& U_{xy^*}\big(\xi\, a_i\, a_i^* \,\langle \eta, E\rangle\big) \otimes \phi_l(xy^*) F \\
= \;& U_{xy^*} \otimes V_{xy^*}\, \big(\theta_{\xi a_i a_i^*, \eta} \otimes 1 \,\big(E \otimes F\big)\big).
\end{aligned}$$

Omitting here $E \otimes F$ and then taking the limit $i \to \infty$ yields

$$Z_{xy^*}\big(\mathcal{K}(\mathcal{E}) \otimes 1\big) \subseteq \mathcal{K}\big(\mathcal{E} \otimes_B (B \rtimes_l^{\mathrm{Mod}} G)\big).$$

For $x \in G'$, we have $Z_x = Z_x Z_x^* Z_x$, and since $Z_x(\mathcal{K}) \subseteq \mathcal{K}$, we obtain

$$Z_x\big(\mathcal{K}(\mathcal{E}) \otimes 1\big) \subseteq \mathcal{K}\big(\mathcal{E} \otimes_B (B \rtimes_l^{\mathrm{Mod}} G)\big). \tag{13.1}$$

Let Θ be the action of $A \rtimes_l G$ on $\mathcal{E} \rtimes_l G$, see (11.4). By (13.1), it is straightforward to compute that

$$[\Theta\big(\phi_l(a_g g)\big), T \otimes 1] \in \mathcal{K}(\mathcal{E} \rtimes_l G)$$

for all $g \in G'$, where ϕ_l denotes $\phi_{A,G,l}$ (use $aU_g = U_g U_g^* a U_g = U_g g(a)$). This result extends by induction to all g in G^* by using products: write $\Theta\big(\phi_l(agh)\big)$ as

$$\Theta\big(\phi_l(agh)\big) = \Theta\big(\phi_l(a^{1/2} g)\big)\Theta\big(\phi_l(g^*(a^{1/2})h)\big)$$

for $g \in G^*$, $h \in G'$ and positive $a \in A_{gh}$ by (5.5) and Lemma 5.8 (iii). By similar computations, one easily checks all other requirements showing that $(\mathcal{E} \rtimes_l G, T \otimes 1)$ is a cycle.

The map j^G is well defined, as a homotopy $(\mathcal{F}, S) \in \mathbf{E}^{H \times G}\big(A, B[0,1]\big)$ gives a homotopy $j^G(\mathcal{F}, S) \in \mathbf{E}^G\big(A \rtimes_l G, B[0,1] \rtimes_l G\big)$, as

$$\begin{aligned}
B[0,1] \rtimes_l G \;&\cong\; \big(B \rtimes_l G\big) \otimes C[0,1], \\
\mathcal{F} \otimes_{B[0,1]} \big(B[0,1] \rtimes_l G\big) \otimes_{B[0,1] \rtimes_l G} \big(B \rtimes_l G\big) \;&\cong\; \mathcal{F}_t \otimes_B \big(B \rtimes_l G\big)
\end{aligned}$$

for $0 \leq t \leq 1$, where the first isomorphism is by Lemma 13.3 and the second isomorphism follows from Lemma 7.7.

To prove (a), let $x_1 = (\mathcal{E}_1, T_1)$, $x_2 = (\mathcal{E}_2, T_2)$, $\mathcal{E}_{12} = \mathcal{E}_1 \otimes_{B_1} \mathcal{E}_2$, and let $(\mathcal{E}_{12}, T_{12})$ be a Kasparov product of x_1 and x_2. We have to check that $j^G(\mathcal{E}_{12}, T_{12}) = (\mathcal{E}_{12} \rtimes_l G, T_{12} \otimes 1)$ is a Kasparov product of $j^G(x_1) = (\mathcal{E}_1 \rtimes_l G, T_1 \otimes 1)$ and $j^G(x_2) = (\mathcal{E}_2 \rtimes_l G, T_2 \otimes 1)$. For the definition of a Kasparov product $(\mathcal{E}_{12}, T_{12})$ of $(\mathcal{E}_1, T_1)$ and $(\mathcal{E}_2, T_2)$ we shall use [**5,** Definition 19] (cf., [**18**]). It states that $\mathcal{E}_{12} = \mathcal{E}_1 \otimes_{B_1} \mathcal{E}_2$, $T_1 \otimes 1$ is a T_2-connection on $\mathcal{E}_{12}$, and $a[T_1 \otimes 1, T_{12}]a^* \geq 0$ in the quotient $\mathcal{L}(\mathcal{E}_{12})/\mathcal{K}(\mathcal{E}_{12})$ for all $a \in A$. For the definition of a T_2-connection on $\mathcal{E}_{12}$ see [**18**], [**9,** Definition 2.6], or [**5,** Definition 18].

We use the isomorphism given in Lemma 13.1. For the H-equivariant $*$-homomorphism

$$f : B_2 \longrightarrow B_2 \rtimes_l G, \quad f(b) = b1_G, \tag{13.2}$$

$j^G(\mathcal{E}_{12}, T_{12}) = f_*((\mathcal{E}_{12}, T_{12}))$ is a cycle in $\mathbf{E}^H(A \rtimes_l G, B \rtimes_l G)$ by [**5,** Definition 24].

The G-action on $\mathcal{E}_{12}$ will be denoted by U. The inclusion

$$\mathcal{K}(\mathcal{E}_2, \mathcal{E}_1 \otimes_{B_1} \mathcal{E}_2) \otimes 1_{B_2 \rtimes_l G} \subseteq \mathcal{K}\big(\mathcal{E}_2 \otimes_{B_2} (B_2 \rtimes_l G),\, \mathcal{E}_1 \otimes_{B_1} \mathcal{E}_2 \otimes_{B_2} (B_2 \rtimes_l G)\big),$$

where B_2 acts by f, is similarly proved as [**5,** Lemma 15].

We use it to check

$$\theta_\eta(T_2^t \otimes 1) - (-1)^{\partial\eta \cdot \partial T_2}(T_{12}^t \otimes 1)\theta_\eta \in \mathcal{K}(\mathcal{E}_2 \rtimes_l G, \mathcal{E}_{12} \rtimes_l G)$$

for $\eta \in \mathcal{E}_1, t \in \{1, *\}$ and

$$\theta_\eta(\xi \otimes z) = \eta \otimes \xi \otimes z$$

for $\xi \in \mathcal{E}_2, z \in B_2 \rtimes_l G$. This shows that $T_{12} \otimes 1$ is a $T_2 \otimes 1$-connection on $\mathcal{E}_{12} \rtimes_l G$.

By [**5,** Lemma 15] and the homomorphism f, we have

$$\mathcal{K}(\mathcal{E}_{12}) \otimes 1 \quad \subseteq \quad \mathcal{K}(\mathcal{E}_{12} \rtimes_l G). \tag{13.3}$$

By Lemma 13.2, we have $R + k \geq 0$ for $R = [T_1 \otimes 1, T_{12}]$ and some $k \in I_A(\mathcal{E}_{12})$. Let $a \in A$ (actually $\pi(A) \otimes 1!$), $g \in G'$, and note that $aU_g = U_g U_g^* a U_g = U_g g^*(a)$ for $a \in A$ and $g \in G'$. Using inclusion (13.3), Lemma 13.2 and the fact that $U_g \otimes V_g$ is in $\mathcal{L}(\mathcal{E}_{12} \rtimes_l G)$, we have the next computation in $\mathcal{E}_{12} \rtimes_l G = \mathcal{E}_{12} \otimes_{B_2} (B \rtimes_l^{\mathrm{Mod}} G)$ modulo $\mathcal{K}(\mathcal{E}_{12} \rtimes_l G)$ for $g \in G'$.

$$\begin{aligned} a(U_g \otimes V_g)(R \otimes 1) &= U_g g^*(a) U_g^* U_g R \otimes V_g \\ &\equiv a U_g R U_g^* U_g \otimes V_g \\ &\equiv a R U_g \otimes V_g = a(R \otimes 1)(U_g \otimes V_g). \end{aligned}$$

By induction on the length of a word in G^*, we see that this identity holds true also for all $g \in G^*$.

Let $a = \sum_g a_g g \in C_c(G, A)$. Let $\phi_l = \phi_{A,G,l}$. By the last computation, we have the following computation in the quotient $\mathcal{L}(\mathcal{E}_{12} \rtimes_l G)/\mathcal{K}(\mathcal{E}_{12} \rtimes_l G)$, where $\underline{R} := R + k \geq 0$.

$$\begin{aligned} &(\Theta \otimes 1)(\phi_l(a))\,(R \otimes 1)\,(\Theta \otimes 1)(\phi_l(a))^* \\ &\quad = \left[\Theta \otimes 1 \left(\phi_l\Big(\sum_{g \in G^*} a_g g\Big)\right)\right] \\ &\qquad (R \otimes 1)\left[\Theta \otimes 1 \left(\phi_l\Big(\sum_{h \in G^*} a_h h\Big)\right)\right]^* \\ &\quad = \sum_{g,h \in G^*} a_g U_g R U_h^* a_h^* \otimes V_g V_h^* \\ &\quad = \sum_{g,h \in G^*} U_g g^*(a_g) \underline{R} U_h^* a_h^* \otimes V_g V_h^* \\ &\quad = \sum_{g,h \in G^*} a_g \underline{R}^{1/2} U_g U_h^* \underline{R}^{1/2} a_h^* \otimes V_g V_h^* \geq 0. \end{aligned}$$

Note that

$$R \otimes 1 = [T_1 \otimes 1 \otimes 1, T_{12} \otimes 1].$$

This shows that $(\mathcal{E}_{12} \rtimes_l G, T_{12} \otimes 1)$ is a Kasparov product. We have thus checked point (a).

Point (b) follows from $j_l^G(A, 0) = (A \otimes_A (A \rtimes_l G), 0) = (A \rtimes_l G, 0)$ by using a map as in (13.2). $\square$

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Doppler Institute for mathematical physics, Trojanova 13, 12000 Prague, Czech Republic

Email address: **bernhardburgstaller@yahoo.de**