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THREE TRIADS OF INTEGERS WITH EQUAL SUMS OF SQUARES AND CUBES

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Dedicated to the anonymous referee for an extremely fast and insightful report

ABSTRACT. While infinitely many examples of two triads of integers with equal sums of squares and cubes are known, until now there is no published example of three triads of integers with this property. In this paper we obtain, in parametric terms, three triads of integers with equal sums of squares and cubes and show that infinitely many similar parametric solutions for such triads can be obtained.

This paper deals with the simultaneous diophantine chains given by

$$\begin{aligned} x_1^2+x_2^2+x_3^2 &= y_1^2+y_2^2+y_3^2 = z_1^2+z_2^2+z_3^2, \\ x_1^3+x_2^3+x_3^3 &= y_1^3+y_2^3+y_3^3 = z_1^3+z_2^3+z_3^3. \end{aligned} \tag{1}$$

No solutions of these diophantine chains have been published until now. A parametric solution for x_i, y_i, $i=1, 2, 3$, satisfying simultaneously the first part of each of the two chain equations may be obtained as described in [**1,** page 200], but this solution is complicated and cannot be extended easily to yield a solution of the diophantine chains (1).

A computer program was accordingly devised to obtain all numerical solutions of the diophantine equations (1) with $|x_i|$, $|y_i|$, $|z_i| \leq 20{,}000$. There are 27 nontrivial primitive solutions in this range, including 6 solutions bounded by 3,000 listed in Table I. Analysis of these numerical solutions showed that 17 of these solutions satisfied the auxiliary condition,

$$x_1+x_2=y_1+y_2=z_1+z_2. \tag{2}$$

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Table I. Solutions of the chain (1)

x_1	x_2	x_3	y_1	y_2	y_3	z_1	z_2	z_3	
463	-311	-72	457	-305	120	445	-293	180	*
726	-125	-374	643	438	-278	526	451	450	
1033	-545	-240	991	-503	432	955	-467	540	*
1375	-551	-216	1369	-545	264	907	-83	1188	*
2105	-513	-748	1907	-315	1232	1221	371	1904	*
2803	-635	-1080	2281	-113	2052	2053	115	2280	*

Solutions marked with an asterisk ($*$) satisfy condition (2)

We will accordingly solve equations (1) together with the additional equation (2). We first obtain a parametric solution for x_i, y_i, $i = 1, 2, 3$, satisfying simultaneously the first part of each of the three chain equations given by (1) and (2) by taking $y_2 = x_1 + x_2 - y_1$, when the condition $\sum_i x_i^2 = \sum_i y_i^2$ can be written as

$$2(x_1 - y_1)(x_2 - y_1) = (x_3 - y_3)(x_3 + y_3), \tag{3}$$

which is satisfied if and only if there exist a, b, p, q such that

$$\begin{aligned} 2(x_1 - y_1) &= pa, & x_2 - y_1 &= qb, \\ x_3 - y_3 &= pb, & x_3 + y_3 &= qa, \end{aligned} \tag{4}$$

which gives

$$\begin{aligned} x_2 &= x_1 - pa/2 + qb, & x_3 &= (qa + pb)/2, \\ y_1 &= x_1 - pa/2, & y_3 &= (qa - pb)/2. \end{aligned} \tag{5}$$

With the above values of x_2, x_3, y_1, y_3 and $y_2 = x_1 + x_2 - y_1$, as before, the condition $\sum_i x_i^3 = \sum_i y_i^3$ reduces to a linear equation in x_1 which is readily solved, and we thus obtain a parametric solution of the first part of each of the three chain equations given by (1) and (2). On clearing denominators, this solution may be written as $x_i = \alpha_i$,

$y_i = \beta_i$, $i = 1, 2, 3$, with α_i, β_i, $i = 1, 2, 3$, being defined as follows:

$$
\begin{aligned}
\alpha_1 &= 3(p+q)qa^2 - 6q^2ab + p^2b^2,\\
\alpha_2 &= 3(q-p)qa^2 + 6q^2ab + p^2b^2,\\
\alpha_3 &= 6q^2a^2 + 6pqab,\\
\beta_3 &= 6q^2a^2 - 6pqab,\\
\beta_1 &= 3(q-p)qa^2 - 6q^2ab + p^2b^2,\\
\beta_2 &= 3(p+q)qa^2 + 6q^2ab + p^2b^2,
\end{aligned} \tag{6}
$$

where a, b, p and q are arbitrary parameters.

We will solve the simultaneous equations (1) and (2) by obtaining three distinct solutions of the following three simultaneous equations,

$$
\begin{aligned}
X_1 + X_2 &= s_1,\\
X_1^2 + X_2^2 + X_3^2 &= s_2,\\
X_1^3 + X_2^3 + X_3^3 &= s_3,
\end{aligned} \tag{7}
$$

where $s_1 = \alpha_1 + \alpha_2$, $s_2 = \alpha_1^2 + \alpha_2^2 + \alpha_3^2$, $s_3 = \alpha_1^3 + \alpha_2^3 + \alpha_3^3$, with α_1, α_2, α_3 being defined by (6) so that equations (7) already have two known solutions given by

$$(X_1, X_2, X_3) = (\alpha_1, \alpha_2, \alpha_3)$$

and

$$(X_1, X_2, X_3) = (\beta_1, \beta_2, \beta_3).$$

To obtain a third solution of the simultaneous equations (7), we eliminate X_1 and X_2 from these three equations when we get the following cubic equation in X_3:

$$(X_3 - \alpha_3)(X_3 - \beta_3)(X_3 - \gamma) = 0, \tag{8}$$

where $\gamma = 3p^2b^2 - 3q^2a^2$. While the roots $X_3 = \alpha_3$ and $X_3 = \beta_3$ of equation (8) lead to the two known solutions of (7), the third root $X_3 = \gamma$ will yield a new solution of (7). Substituting $X_3 = \gamma$ in the first two equations of (7) and eliminating X_2 from these two equations, we get the equation

$$
\begin{aligned}
2X_1^2 - (12q^2a^2 + 4p^2b^2)X_1 - 9q^2(2p^2 + q^2)a^4&\\
-6q^2(7p^2 + 12q^2)a^2b^2 + 11p^4b^4 &= 0.
\end{aligned} \tag{9}
$$

This is a quadratic equation in X_1 and, for its roots to be rational, its discriminant must be a perfect square. This discriminant is a quartic function of a, b, and it is given by

$$36\{2q^2(2p^2+3q^2)a^4+4q^2(3p^2+4q^2)a^2b^2-2p^4b^4\}. \tag{10}$$

By writing $t=p/q$ and $x=a/b$, the condition that the above discriminant is a perfect square may be considered as defining a parametrized quartic curve given by

$$y^2=2(2t^2+3)x^4+4(3t^2+4)x^2-2t^4. \tag{11}$$

We will denote points on the curve (11) by P_n and, given any such point $P_n=(x,\,y)$, we note that the point $P_n'=(-x,\,y)$ also lies on the curve (11).

We observe that there are two rational points on the curve (11) given by $P_0=(t,\,2t(t^2+2))$ and $P_1=(t/3,\,2t(t^2-6)/9)$, and thus (11) is, in fact, an elliptic curve. Neither the point P_0 nor the point P_1 leads to a nontrivial solution of our diophantine chains. However, by following a well-known procedure described by Dickson [**2,** page 639], and using either of these two known points on the curve (11), we can easily find another point on (11) and hence we can get a pair of values of a and b that make the discriminant (10) a perfect square. While the point P_0 yields the point P_1' which does not lead to a nontrivial solution of our diophantine chains, the point P_1 yields the point P_4' (this nomenclature will become clear at the end of the paper) on (11), where

$$\begin{aligned}P_4=(&t(t^6+246t^4-180t^2+5832)\{9(t^6-42t^4-564t^2+72)\}^{-1},\\ &(t^{12}-1020t^{10}-25236t^8-1201824t^6-5900688t^4-9906624t^2-1259712)\\ &\times\{2t(t^2-6)\}\{9(t^6-42t^4-564t^2+72)\}^{-2}).\end{aligned}$$

This point readily yields values of a and b that make the discriminant (10) a perfect square so that equation (9) has two rational roots, and then, equations (7) have three rational solutions, and these can be effectively determined. This leads to a parametric solution of the diophantine chains (1) in terms of polynomials of degree 14.

The referee was quick to point out in his report that the point P_4 on the elliptic curve (11) is divisible by 2, and this provides a simpler point P_2 on the curve (11), namely,

$$P_2=(t(t^2+18)/(5t^2-6),2t(t^2+2)(t^4-84t^2-108)/(5t^2-6)^2). \tag{12}$$

While we verify below the observation of the referee, we give another way of finding the point P_2 on the quartic curve (11).

By applying a birational transformation, as described by Mordell [**3,** page 77], the quartic curve (11) is reduced to the following cubic form of an elliptic curve:

$$\begin{aligned} Y^2 = \quad & 4X^3 + \left(8t^6 - 32t^2 - \frac{64}{3}\right)X + 16t^8 \\ & + \frac{160}{3}t^6 + 64t^4 + \frac{128}{3}t^2 + \frac{512}{27}, \end{aligned} \tag{13}$$

with the birational transformation being given by

$$\begin{aligned} x &= -t(12X - 3Y - 12t^4 - 24t^2 - 16) \\ &\quad \times \{12(t^2+1)X + 3Y + 12t^4 + 8t^2 - 16\}^{-1}, \\ y &= 2t(t^2+2)\{216X^3 - 432X^2 - 27Y^2 - 216t^2(t^2+2)Y \\ &\quad - 432t^8 - 1728t^6 - 1728t^4 + 256\} \\ &\quad \times [3\{12(t^2+1)X + 3Y + 12t^4 + 8t^2 - 16\}^2]^{-1}, \end{aligned} \tag{14}$$

and

$$\begin{aligned} X = \quad & \{(6t^4 + 12t^2 + 4)x^2 + 4(3t^2+4)tx + 3(t^2+2)ty + 4t^2\} \\ & \times\{3(t-x)^2\}^{-1}, \\ Y = \quad & 4t(t^2+2)\{t(2t^2+3)x^3 + (3t^2+4)x^2 + t(3t^2+4)x \\ & + (t^2+1)xy + ty - t^4\}(t-x)^{-3}. \end{aligned} \tag{15}$$

Using this birational transformation, we find that the points P_2 and P_4 on the quartic curve (11) correspond respectively to the points Q_2 and Q_4 on the cubic curve given by (13) where

$$\begin{aligned} Q_2 &= (t^4/4 - t^2 - 5/3,\ (t^2+2)^2(t^2+6)/4), \\ Q_4 &= ((3t^8 - 264t^6 + 712t^4 + 5088t^2 + 6192)\{48(t^2+6)^2\}^{-1}, \\ &\qquad (t^2+2)^2(t^2-30)(t^6 + 150t^4 + 108t^2 + 648) \\ &\qquad \times\{32(t^2+6)^3\}^{-1}). \end{aligned} \tag{16}$$

It is readily verified on applying the group law that the point Q_4 on the elliptic curve (13) is indeed $2Q_2$, thus confirming the assertion of the referee. It can also be verified that the point Q_2 is $2Q_1$, where

Q_1 corresponds to P_1; hence, we can find the point P_2 without first determining the point P_4.

The point P_2 on the curve (11) readily yields the following values of a and b that make the discriminant (10) a perfect square:

$$a = p(p^2 + 18q^2), \qquad b = q(5p^2 - 6q^2), \tag{17}$$

Thus, with these values of a and b, equation (9) has two rational roots. It follows that when a and b are given by (17), equations (7) have three rational solutions and this leads to the following parametric solution of the diophantine chains (1) in terms of polynomials of degree 6:

$$\begin{aligned}
x_1 &= 3p^6 + 28p^5q + 78p^4q^2 + 48p^3q^3 + 468p^2q^4\\
&\quad + 1008pq^5 + 648q^6,\\
x_2 &= -3p^6 + 28p^5q - 78p^4q^2 + 48p^3q^3 - 468p^2q^4\\
&\quad + 1008pq^5 - 648q^6,\\
x_3 &= 36pq(p^2 + 2q^2)(p^2 + 18q^2),\\
y_1 &= -3p^6 + 28p^5q - 138p^4q^2 + 48p^3q^3 - 1476p^2q^4\\
&\quad + 1008pq^5 + 648q^6,\\
y_2 &= 3p^6 + 28p^5q + 138p^4q^2 + 48p^3q^3 + 1476p^2q^4\\
&\quad + 1008pq^5 - 648q^6,\\
y_3 &= -24pq(p^2 - 6q^2)(p^2 + 18q^2),\\
z_1 &= -3p^6 + 28p^5q + 246p^4q^2 + 48p^3q^3 + 828p^2q^4\\
&\quad + 1008pq^5 + 648q^6,\\
z_2 &= 3p^6 + 28p^5q - 246p^4q^2 + 48p^3q^3 - 828p^2q^4\\
&\quad + 1008pq^5 - 648q^6,\\
z_3 &= 72pq(p^2 - 6q^2)(p^2 + 2q^2).
\end{aligned}$$

Using the group law on the elliptic curve (13), we can find infinitely many rational points $Q_n = nQ_1$ on (13), and hence also on the quartic curve (11), and thus obtain infinitely many pairs of values of a and b that make the discriminant of equation (9) a perfect square. We can thus find infinitely many parametric solutions of the simultaneous diophantine chains (1). For example, for $n = 3, 5, 6$, we get polynomials of degrees 10, 22, 30, respectively. These parametric solutions, however, do not give the complete solution of the simultaneous chains (1) since all of them necessarily satisfy the auxiliary condition (2), whereas we have already noted the existence of solutions of the chains (1) that do not satisfy the condition (2).

Finally, we note that the values of x_3, y_3, z_3 given by our solution of degree 6 satisfy the following identity:

$$x_3y_3 + y_3z_3 + z_3x_3 = 0. \tag{18}$$

In fact, it readily follows from equation (8) that all nontrivial solutions of the diophantine chains (1) subject to the constraints (2) will satisfy the identity (18). This identity was pointed out to us by the referee in his report.

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