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HOMOCLINIC ORBITS OF NONLINEAR FUNCTIONAL DIFFERENCE EQUATIONS WITH JACOBI OPERATORS

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ABSTRACT. By using the critical point theory, the existence of a nontrivial homoclinic orbit which decays exponentially at infinity for difference equations containing both advance and retardation is obtained. The proof is based on the mountain pass lemma in combination with periodic approximations. Our results extend the results of 2007.

1. Introduction. As usual, **N**, **Z** and **R** denote the sets of all natural numbers, integers and real numbers, respectively. For a, $b \in \mathbf{Z}$, define $\mathbf{Z}(a) = \{a, a+1, \dots\}$, $\mathbf{Z}(a,b) = \{a, a+1, \dots, b\}$ when $a \leq b$. l^2 denotes the space of all real functions whose second powers are summable on **Z**. $*$ is the transpose sign for a vector. I denotes the identity operator.

Time symmetric electrodynamics contradicts experience because we do not observe advanced interactions. A lot of papers are known about differential equations with retarded arguments [**15, 18, 22**], including the retarded equations of electrodynamics [**27**]. For advanced interactions, many of the theorems do not hold. Wheeler and Feynman [**32**] proposed that the universe acts to absorb advanced interactions, leaving a residual radiation reaction. In time symmetric electrodynamics Schulman [**28**] has studied the following equations

$$\frac{d^2u(s)}{ds^2} + \omega^2 u(s) = \frac{1}{2}\alpha u(s-\tau) + \frac{1}{2}\beta u(s+\sigma) + \psi(s), \quad s \in \mathbf{R} \tag{1.1}$$

and

$$\frac{d^2u(s)}{ds^2} + \omega^2 u(s) = \frac{1}{2}\alpha u(s-\tau) + \frac{1}{2}\alpha u(s+\tau) + \psi(s), \quad s \in \mathbf{R}, \tag{1.2}$$

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where α, β, ω, τ and σ are given constants, $\tau, \sigma > 0$ and $\psi(s)$ is a given function. An explicit solution was given to a boundary value problem for equations (1.1) and (1.2), respectively. The approach used in [**28**] was simply to give boundary values rather than initial values for the solution. The solution was well behaved, even in the presence of advanced interactions.

Equation (1.2) with $\psi = 0$ can be derived by variation of the action [**9, 17**]

$$J = \frac{1}{2} \int \left[\left(\frac{du(s)}{ds} \right)^2 - \omega^2 \left(u(s)\right)^2 + \alpha u\left(s + \frac{\tau}{2}\right) u\left(s - \frac{\tau}{2}\right) \right] ds. \tag{1.3}$$

Guo and Xu in [**1**] have given some criteria for the existence of periodic solutions to a class of second-order neutral differential difference equations as the following type

$$u''(s-\tau) - u(s-\tau) + f(s, u(s), u(s-\tau), u(s-2\tau)) = 0, \quad s \in \mathbf{R},$$

with a boundary value problem by means of critical point theory. In [**29**], using a variant of the mountain pass theorem, Smets and Willem have proved the existence of solitary waves with prescribed speed on infinite lattices of particles with nearest neighbor interaction for the following forward and backward differential difference equation

$$c^2 u''(s) = V'(u(s+1) - u(s)) - V'(u(s) - u(s-1)), \quad s \in \mathbf{R}.$$

Certainly, an immediate generalization of equation (1.2) is the following equation

$$Su(s) - \omega u(s) = f(s, u(s+T), u(s), u(s-T)), \quad s \in \mathbf{R}. \tag{1.4}$$

Here S is the Sturm-Liouville differential expression and $\omega \in \mathbf{R}$, T is a given nonnegative integer, $f \in C(\mathbf{R}^4, \mathbf{R})$.

In this paper, we consider the existence of a nontrivial homoclinic orbit which decays exponentially at infinity for the following equation

$$Lu(t) - \omega u(t) = f(t, u(t+T), u(t), u(t-T)), \quad t \in \mathbf{Z}, \tag{1.5}$$

containing both advance and retardation. We may think of equation (1.5) as being a discrete analogue of equation (1.4) which includes equation (1.2). Here the operator L is the Jacobi operator

$$Lu(t) = a(t)u(t+1) + a(t-1)u(t-1) + b(t)u(t),$$

where $a(t)$ and $b(t)$ are real valued for each $t \in \mathbf{Z}$, $\omega \in \mathbf{R}$, $f \in C(\mathbf{R}^4, \mathbf{R})$, $a(t)$, $b(t)$ and $f(t, v_1, v_2, v_3)$ are all M-periodic in t for a given positive integer M.

Jacobi operators appear in a variety of applications, see for example, [**31**]. They can be viewed as the discrete analogue of Sturm-Liouville operators, and their investigation has many similarities with Sturm-Liouville theory. Whereas numerous books about Sturm-Liouville operators have been written, only a few exist on Jacobi operators. In particular, there is currently less research available which covers some basic topics (like positive solutions, periodic operators, boundary value problems, etc.) typically found in textbooks on Sturm-Liouville operators.

For the case $T = 1$, Chen and Fang [**5**] have obtained a sufficient condition for the existence of periodic and subharmonic solutions of the second-order p-Laplacian difference equation

$$\Delta(\varphi_p(\Delta u(t-1))) + f(t, u(t+1), u(t), u(t-1)) = 0, \quad t \in \mathbf{Z},$$

using critical point theory.

In recent years, there has been much progress on the qualitative properties of difference equations, which included results on stability and attractivity [**20, 34**] and results on oscillation and other topics [**1, 7, 24**]. For the general background of difference equations, one can refer to the books [**1, 7, 16**]. However, the results on the existence and multiplicity of periodic solutions, subharmonic solutions, homoclinic and heteroclinic solutions of difference equations are relatively rare, see for example [**8**]. In the theory of differential equations, a trajectory which is asymptotic to a constant state as $|s| \to \infty$ (s denotes the time variable) is called a homoclinic orbit. Such orbits have been found in various models of continuous dynamical systems and frequently have tremendous effects on the dynamics of such nonlinear systems. So the homoclinic orbits have been extensively studied since the time of

Poincaré, see [**6, 14, 19, 21, 23, 25, 30**] and the references therein. In recent research we have found that the trajectory which is asymptotic to a constant state as $|t| \to \infty$ also exists in discrete dynamical systems. We still call it a homoclinic orbit.

Some special cases of equation (1.5) have been studied by many researchers via variational methods, see [**2, 3, 11–13, 24, 33**]. However, to the best of our knowledge, no similar results have been obtained in the literature for equation (1.5). Since f in (1.5) depends on $u(t+T)$ and $u(t-T)$, the traditional ways of establishing the functional in [**2, 3, 11–13, 24, 33**] are inapplicable to our case. The main purpose of this paper is to give some sufficient conditions for the existence of a nontrivial homoclinic orbit which decays exponentially at infinity. The main approach used in our paper is variational technique and the notable mountain pass lemma introduced by Ambrosetti and Rabinowitz [**4, 26**] in combination with periodic approximations. One of our results is more general than the result in the literature [**19**]. In fact, one can see the following Remark 1.3 for more detail.

Now we state the main results of this paper.

Theorem 1.1. *Assume that the following hypotheses are satisfied*:

(L) $a(t) \neq 0$, $b(t) - |a(t-1)| - |a(t)| > \omega$, *for all* $t \in \mathbf{Z}$;

(F_1) there exists a functional $F(t, v_1, v_2) \in C^1(\mathbf{Z} \times \mathbf{R}^2, \mathbf{R})$ *with* $F(t, v_1, v_2) \geq 0$, *and it satisfies*

$$\begin{gathered} F(t+M, v_1, v_2) = F(t, v_1, v_2), \\ F'_{v_1}(t-T, v_2, v_3) + F'_{v_2}(t, v_1, v_2) = f(t, v_1, v_2, v_3), \\ \lim_{\rho \to 0} \frac{F(t, v_1, v_2)}{\rho^2} = 0, \quad \rho = \sqrt{v_1^2 + v_2^2}, \\ \lim_{r \to 0} \frac{f(t, v_1, v_2, v_3)}{v_2} = 0, \quad r = \sqrt{v_1^2 + v_2^2 + v_3^2}; \end{gathered}$$

(F_2) there exists a constant $\beta > 2$ *such that*

$$0 < \beta F(t, v_1, v_2) \leq F'_{v_1}(t, v_1, v_2) v_1 + F'_{v_2}(t, v_1, v_2) v_2,$$

for all $(t, v_1, v_2) \in \mathbf{Z} \times \mathbf{R}^2 \setminus \{(0,0)\}$.

Then (1.5) *has a nontrivial homoclinic orbit.*

Remark 1.1. The above hypotheses imply that $u(t) \equiv 0$ is a trivial solution of (1.5).

Remark 1.2. Assumption (F_2) implies that there exist constants $a_1 > 0$ and $a_2 > 0$ such that (F_2') $F(t, v_1, v_2) \geq a_1(\sqrt{v_1^2 + v_2^2})^\beta - a_2$, for all $(t, v_1, v_2) \in \mathbf{Z} \times \mathbf{R}^2$.

Remark 1.3. Theorem 1.1 extends Theorem 1.1 in [**19**], which is the special case of our Theorem 1.1, by letting $T = 0$ and $a(t) < 0$.

Theorem 1.2. *The homoclinic orbit u of* (1.5) *obtained in Theorem* 1.1 *decays exponentially at infinity*:

$$|u(t)| \leq Ce^{-\gamma|t|}, \quad t \in \mathbf{Z},$$

with some constants $C > 0$ and $\gamma > 0$.

2. Variational structure and some lemmas. In order to apply critical point theory, we shall establish the corresponding variational framework for (1.5) and give some lemmas which will be of fundamental importance in proving our main results. Firstly, we state some basic notations.

Let S be the vector space of all real sequences of the form

$$u = \{u(t)\}_{t \in \mathbf{Z}} = (\ldots, u(-t), \ldots, u(-1), u(0), u(1), \ldots, u(t), \ldots),$$

namely

$$S = \{\{u(t)\} | u(t) \in \mathbf{R},\ t \in \mathbf{Z}\}.$$

For any given positive integers M and m, E_m is defined by

$$E_m = \{u \in S | u(t + 2mM) = u(t), \quad \text{for all } t \in \mathbf{Z}\}.$$

Then E_m is a subspace of S and isomorphic to $\mathbf{R}^{2mM}$. E_m can be equipped with

$$(2.1) \qquad \langle u, v \rangle_m = \sum_{t=-mM}^{mM-1} [(L - \omega I)u(t) \cdot v(t)], \quad \text{for all } u, v \in E_m.$$

Equation (2.1) allows us to deduce that the norm is

$$\|u\|_m = \sqrt{\sum_{t=-mM}^{mM-1} [(L-\omega I)u(t)\cdot u(t)]}, \quad \text{for all } u \in E_m. \tag{2.2}$$

It is obvious that $(E_m, \langle\cdot,\cdot\rangle_m)$ is a finite-dimensional Hilbert space and linearly homeomorphic to $\mathbf{R}^{2mM}$.

In what follows, l_m^2 defines the different norm on space E_m equipped with

$$\|u\|_{l_m^2} = \left(\sum_{t=-mM}^{mM-1} |u(t)|^2\right)^{1/2}, \quad \text{for all } u \in l_m^2.$$

Clearly, $\|u\|_{l_m^2}$ is just the usual inner product norm $\|\cdot\| = \sqrt{(\cdot,\cdot)}$, where $(\cdot,\cdot)$ is the usual inner product in $\mathbf{R}^{2mM}$, $|\cdot|$ denotes the standard absolute value in $\mathbf{R}$. Moreover, l_m^∞ denotes the different norm on space E_m endowed with

$$\|u\|_{l_m^\infty} = \max_{t\in\mathbf{Z}(-mM,mM-1)} |u(t)|, \quad \text{for all } u \in l_m^\infty.$$

For all $u \in E_m$, define the functional J_m on E_m as follows:

$$\begin{aligned} J_m(u) &:= \sum_{t=-mM}^{mM-1}\left[\frac{1}{2}(L-\omega I)u(t)\cdot u(t) - F(t, u(t+T), u(t))\right] \\ &= \frac{1}{2}\|u\|_m^2 - \sum_{t=-mM}^{mM-1} F(t, u(t+T), u(t)), \end{aligned} \tag{2.3}$$

where

$$F'_{v_1}(t-T, v_2, v_3) + F'_{v_2}(t, v_1, v_2) = f(t, v_1, v_2, v_3). \tag{2.4}$$

Clearly, $J_m \in C^1(E_m, \mathbf{R})$ and, for any $u = \{u(t)\}_{t\in\mathbf{Z}} \in E_m$, by the periodicity of $\{u(t)\}$, we can compute the partial derivative as

$$\frac{\partial J_m(u)}{\partial u(t)} = Lu(t) - \omega u(t) - f(t, u(t+T), u(t), u(t-T)),$$
$$\text{for all } t \in \mathbf{Z}(-mM, mM-1).$$

Thus, u is a critical point of J_m on E_m if and only if

$$Lu(t) - \omega u(t) = f(t, u(t+T), u(t), u(t-T)), \quad \text{for all } t \in \mathbf{Z}(-mM, mM-1). \tag{2.5}$$

Due to the periodicity of $u = \{u(t)\}_{t\in\mathbf{Z}} \in E_m$ and $f(t, v_1, v_2, v_3)$ in the first variable t, we reduce the existence of periodic solutions of (1.5) to the existence of critical points of J_m on E_m. That is, the functional J_m is just the variational framework of (1.5).

For convenience, we identify $u \in E_m$ with $u = u((-mM), u(-mM+1), \dots, u(mM-1))^*$.

Three lemmas should be stated here which will be used in the proof of our main results. First, let us recall the Palais-Smale condition.

Let E be a real Banach space, $J \in C^1(E, \mathbf{R})$, i.e., J is a continuously Fréchet-differentiable functional defined on E. J is said to be satisfying the Palais-Smale condition (P.S. condition for short) if any sequence $\{u(t)\} \subset E$ for which $\{J(u(t))\}$ is bounded and $J'(u(t)) \to 0 (t \to \infty)$ possesses a convergent subsequence in E.

Let B_ρ denote the open ball in E with radius ρ and centered at 0, and let ∂B_ρ denote its boundary.

Lemma 2.1 (mountain pass lemma [**4, 26**]). *Let E be a real Banach space and $J \in C^1(E, \mathbf{R})$ satisfies the* P.S. *condition. If $J(0) = 0$ and*

(J_1) there exist constants ρ, $\alpha > 0$ such that $J|_{\partial B_\rho} \geq \alpha$, and

(J_2) there exists an $e \in E \setminus B_\rho$ such that $J(e) \leq 0$, then J possesses a critical value $c \geq \alpha$, given by

$$c = \inf_{g\in\Gamma} \max_{s\in[0,1]} J(g(s)), \tag{2.6}$$

where

$$\Gamma = \{g \in C([0,1], E) | g(0) = 0, g(1) = e\}. \tag{2.7}$$

Remark 2.1. In fact, the P.S. condition was developed by Palais and Smale in order to deal with the existence of critical points of functionals defined on the Hilbert space with infinite dimensions. Of course, it can also be used to cope with problems in finite-dimensional spaces. In our setting, E_m is just the Hilbert space, which is homeomorphic to $\mathbf{R}^{2mM}$. So Lemma 2.1 can also be used to investigate the existence of critical points of J_m.

Lemma 2.2. *Assume that* (L) *holds. Then there exist constants* $\underline{\lambda}$ *and* $\overline{\lambda}$, *independent of* m, *such that the following inequalities are true*:

$$\underline{\lambda}\|u\|^2 \le \|u\|_m^2 \le \overline{\lambda}\|u\|^2, \tag{2.8}$$

$$\underline{\lambda}\|u\|_{l_m^\infty}^2 \le \|u\|_m^2. \tag{2.9}$$

Proof. Let

$$\sum_{t=-mM}^{mM-1}[(L-\omega I)u(t)\cdot u(t)] = (P_m u, u),$$

where

$$u = (u(-mM), \dots, u(-1), u(0), u(1), \dots, u(mM-1))^*,$$

$$P_m = \begin{bmatrix} b(-mM)-\omega & a(-mM) & 0 & \cdots & 0 & a(mM-1) \\ a(-mM) & b(-mM+1)-\omega & a(-mM+1) & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & b(mM-2)-\omega & a(mM-2) \\ a(mM-1) & 0 & 0 & \cdots & a(mM-2) & b(mM-1)-\omega \end{bmatrix},$$

where P_m is a $2mM \times 2mM$ matrix.

Define

$$\underline{\lambda} = \min_{t\in\mathbf{Z}(1,M)} (b(t)-\omega-|a(t-1)|-|a(t)|) > 0,$$

$$\overline{\lambda} = \max_{t\in\mathbf{Z}(1,M)} (b(t)-\omega+|a(t-1)|+|a(t)|).$$

By matrix theory and simple computation, we know that all eigenvalues of the matrix P_m satisfy $\underline{\lambda} \le \lambda_i \le \overline{\lambda}$, $i \in \mathbf{Z}(-mM, mM-1)$ and (2.8) is obviously true. By the definition of the norm $\|\cdot\|_{l_m^\infty}$ and $\|\cdot\|$, together with (2.8), (2.9) follows immediately. $\square$

Lemma 2.3. *Suppose that* (L), (F_1) *and* (F_2) *are satisfied. Then* J_m *satisfies the* P.S. *condition.*

Proof. Let $\{u_n\} \subset E_m$ be such that $\{J_m(u_n)\}$ is bounded and $J_m'(u_n) \to 0$ as $n \to \infty$. Then there exists a positive constant K such that $|J_m(u_n)| \le K$. Thus, by (2.3), (F_2) and the periodicity of $\{u(t)\}$ and $F(t, v_1, v_2)$ in the first variable t, for n large enough, we have

$$\begin{aligned}
\beta K + \|u_n\|_m &\ge \beta J_m(u_n) - \langle J_m'(u_n), u_n\rangle_m \\
&= \frac{\beta-2}{2}\|u_n\|_m^2 + \sum_{t=-mM}^{mM-1} [F_{v_1}'(t-T, u_n(t), u_n(t-T))u_n(t) \\
&\quad + F_{v_2}'(t, u_n(t+T), u_n(t))u_n(t) - \beta F(t, u_n(t+T), u_n(t))] \\
&= \frac{\beta-2}{2}\|u_n\|_m^2 + \sum_{t=-mM}^{mM-1} [F_{v_1}'(t, u_n(t+T), u_n(t))u_n(t+T) \\
&\quad + F_{v_2}'(t, u_n(t+T), u_n(t))u_n(t) - \beta F(t, u_n(t+T), u_n(t))] \\
&\ge \frac{\beta-2}{2}\|u_n\|_m^2.
\end{aligned}$$

Since $\beta > 2$, it is not difficult to know that $\{u_n\}$ is a bounded sequence in E_m. As a consequence, $\{u_n\}$ possesses a convergent subsequence in E_m, and thus, the P.S. condition is verified. $\square$

3. Proof of the main results. In this section, we firstly prove the existence of a nontrivial $2mM$-periodic solution of equation (1.5), known as a subharmonic solution [**12**]. Actually, we show that the functional J_m possesses a nontrivial critical point. Secondly, we obtain uniform estimates of u_m in E_m, l_m^2 and l_m^∞ which are independent of $m \in \mathbf{N}$. Thirdly, we get $\{u_m\}_{m\in\mathbf{N}}$ to pass to a nontrivial homoclinic orbit and complete the proof of Theorem 1.1. Finally, we show that the homoclinic orbit of (1.5) obtained in Theorem 1.1 decays exponentially fast at infinity and completes the proof of Theorem 1.2.

3.1. Existence of a subharmonic solution. For our setting, clearly $J_m(0)=0$. We have known that J_m satisfies the P.S. condition. By (F_1), there exists a $\delta>0$ such that $|F(t,v_1,v_2)|\leq 1/8\underline{\lambda}(v_1^2+v_2^2)$ for $t\in\mathbf{Z}$ and $v_1^2+v_2^2\leq\delta^2$. Letting $\rho=\sqrt{\underline{\lambda}}/\sqrt{2}\delta$, for any $u\in E_m$ and $\|u\|_m\leq\rho$, we have $|u(t)|\leq\|u\|\leq 1/\sqrt{\underline{\lambda}}\|u\|_m\leq 1/\sqrt{\underline{\lambda}}\rho=1/\sqrt{2}\delta$, for all $t\in\mathbf{Z}(-mM,mM-1)$. Thus, we have $(u(t+T))^2+(u(t))^2\leq\delta^2$, for all $t\in\mathbf{Z}(-mM,mM-1)$, which leads to

$$\begin{aligned}J_m(u)&\geq\frac{1}{2}\|u\|_m^2-\frac{1}{8}\underline{\lambda}\sum_{t=-mM}^{mM-1}\left[(u(t))^2+(u(t+T))^2\right]\\&\geq\frac{1}{2}\|u\|_m^2-2\times\frac{1}{8}\underline{\lambda}\cdot\frac{1}{\underline{\lambda}}\|u\|_m^2\\&=\frac{1}{4}\|u\|_m^2.\end{aligned}$$

Taking $\alpha=(1/4)\rho^2>0$, we obtain

$$J_m(u)|_{\partial B_\rho}\geq\alpha>0,$$

which implies that J_m satisfies the condition (J_1) of the mountain pass lemma.

Next, we shall verify the condition (J_2).

By (F_2'), for all $\tau\in\mathbf{R}$ and any given $w\in E_m\setminus\{0\}$, $\|w\|_m=1$, we obtain

$$\begin{aligned}J_m(\tau w)&=\frac{1}{2}\tau^2-\sum_{t=-mM}^{mM-1}F(t,\tau w(t+T),\tau w(t))\\&\leq\frac{1}{2}\tau^2-a_1|\tau|^\beta\sum_{t=-mM}^{mM-1}\left[\sqrt{(w(t+T))^2+(w(t))^2}\right]^\beta\\&\quad+2mMa_2.\end{aligned}\tag{3.1}$$

Since $\beta>2$, we can choose τ large enough to ensure that $J_m(\tau w)\leq 0$.

All the assumptions of the mountain pass lemma have been verified. Consequently, J_m possesses a critical value c_m given by (2.6) and (2.7) with $E=E_m$ and $\Gamma=\Gamma_m$, where $\Gamma_m=\{g_m\in C([0,1],E_m)|g_m(0)=0,g_m(1)=e_m,e_m\in E_m\backslash B_\rho\}$. We denote u_m as the corresponding critical point of J_m on E_m. It is obvious that $\|u_m\|_m\neq 0$ since $c_m>0$.

3.2. Uniform estimates for $\|u_m\|_m$ and $\|u_m\|_{l^\infty_m}$. The next step in the proof is to obtain estimates independent of m for c_m and u_m.

Lemma 3.2.1. *Suppose that (L), (F_1) and (F_2) are satisfied. Then there exists a constant d independent of m such that*

$$\|u_m\|_m \le d,$$

for any $m \in \mathbf{N}$.

Proof. Let $e \in E_1 \setminus \{0\}$ be such that $e(-M) = e(M) = 0$ and $J_1(e) \le 0$. Define $e_m = \{e_m(t)\}$, satisfying

$$e_m(t) = \begin{cases} e(t) & t \in \mathbf{Z}(-M, M), \\ 0 & M < |t| \le mM. \end{cases}$$

Then $e_m \in E_m \setminus \{0\}$ and $J_m(e_m) = J_1(e) \le 0$. By the mountain pass lemma, we know that $g_m(s) = se_m \in \Gamma_m$, for any $m \in \mathbf{N}$ and $J_m(g_m(s)) = J_1(g_1(s)) = J_1(se)$. Therefore, by (2.6),

$$\begin{aligned} 0 < \alpha \le c_m &\le \max_{0\le s\le 1} J_m(g_m(s)) \\ &= \max_{0\le s\le 1} J_1(g_1(s)) = \max_{0\le s\le 1} J_1(se) = d_1, \end{aligned} \tag{3.2}$$

where d_1 is a constant independent of m.

Estimate (3.2) leads to an a priori bound for u_m. Since $J_m'(u_m) = 0$, by (F_2),

$$\begin{aligned} c_m &= J_m(u_m) - \frac{1}{2}\langle J_m'(u_m), u_m\rangle_m \\ &= \sum_{t=-mM}^{mM-1} \Big[-F(t, u_m(t+T), u_m(t)) \\ &\qquad\qquad + \frac{1}{2}(F_{v_1}'(t-T, u_m(t), u_m(t-T))u_m(t) \\ &\qquad\qquad\qquad + F_{v_2}'(t, u_m(t+T), u_m(t))u_m(t))\Big] \\ &= \sum_{t=-mM}^{mM-1} \Big[-F(t, u_m(t+T), u_m(t)) \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2}(F'_{v_1}(t,u_m(t+T),u_m(t))u_m(t+T)\\
&\qquad\qquad + F'_{v_2}(t,u_m(t+T),u_m(t))u_m(t))\Big]\\
\geq \left(\frac{\beta}{2}-1\right)&\sum_{t=-mM}^{mM-1}F(t,u_m(t+T),u_m(t)).
\end{aligned}
$$

Combining with (2.3), we obtain a bound independent of m for $\|u_m\|_m$ as follows:

$$
\begin{aligned}
\|u_m\|_m &= \left[2c_m + 2\sum_{t=-mM}^{mM-1}F(t,u_m(t+T),u_m(t))\right]^{1/2}\\
&\leq \left(2c_m+\frac{4}{\beta-2}c_m\right)^{1/2} \leq \left[\left(2+\frac{4}{\beta-2}\right)d_1\right]^{1/2}\\
&\equiv d. \qquad \square
\end{aligned}
\tag{3.3}
$$

Lemma 3.2.2. *Assume that* (L), (F_1) *and* (F_2) *are satisfied. Then there exist positive constants* ξ *and* η *independent of* m *such that*

$$
\xi \leq \|u_m\|_{l_m^\infty} \leq \eta, \tag{3.4}
$$

where

$$
\|u_m\|_{l_m^\infty} = \max_{t\in \mathbf{Z}(-mM,mM-1)} |u_m(t)|\,.
$$

Proof. By (F_1), we can find a $\delta>0$ such that

$$
|f(t,v_1,v_2,v_3)| \leq \frac{1}{2}\underline{\lambda}|v_2|,
$$

for $t\in\mathbf{Z}$ and $v_1^2+v_2^2+v_3^2\leq\delta^2$. Suppose that $\|u_m\|_{l_m^\infty}\leq 1/\sqrt{3}\delta$. Then, $(u_m(t+T))^2+(u_m(t))^2+(u_m(t-T))^2\leq\delta^2$, $t\in\mathbf{Z}(-mM,mM-1)$, and we have

$$
|f(t,u_m(t+T),u_m(t),u_m(t-T))| \leq \frac{1}{2}\underline{\lambda}|u_m(t)|. \tag{3.5}
$$

From the definition of J_m, we have

$$\begin{aligned}&\langle J_m'(u_m), u_m\rangle_m\\ &\quad = \|u_m\|_m^2 - \sum_{t=-mM}^{mM-1} f(t, u_m(t+T), u_m(t), u_m(t-T))u_m(t) = 0.\end{aligned}$$

Therefore, we get

$$\begin{aligned}\|u_m\|_m^2 &= \sum_{t=-mM}^{mM-1} f(t, u_m(t+T), u_m(t), u_m(t-T))u_m(t)\\ &\leq \frac{1}{2}\underline{\lambda} \sum_{t=-mM}^{mM-1} (u_m(t))^2 = \frac{1}{2}\underline{\lambda}\|u_m\|^2 \leq \frac{1}{2}\|u_m\|_m^2.\end{aligned}$$

Thus, we have $u_m = 0$. But this contradicts $\|u_m\|_m \neq 0$, which shows that there exists a constant $\xi > 0$ independent of m such that $\|u_m\|_{l_m^\infty} \geq \xi$ for any $m \in \mathbf{N}$.

On the other hand, by (2.9) and Lemma 3.2.1, we have

$$\|u_m\|_{l_m^\infty} \leq \frac{1}{\sqrt{\underline{\lambda}}}\|u_m\|_m \leq \frac{d}{\sqrt{\underline{\lambda}}}.$$

Taking $\eta = d/\sqrt{\underline{\lambda}}$ independent of m, we obtain the desired result. □

3.3. Limit process for $\{u_m\}_{m\in\mathbf{N}}$. In the following, we shall give the existence of a nontrivial homoclinic orbit.

Consider the sequence $\{u_m = \{u_m(t)\}\}$ of $2mM$-periodic solutions found in subsection 3.1. Firstly, by (3.4), for any $m \in \mathbf{N}$, there exists a constant $t_m \in \mathbf{Z}$ independent of m such that

$$|u_m(t_m)| \geq \xi. \tag{3.6}$$

Indeed, since $a(t)$, $b(t)$ and $f(t, v_1, v_2, v_3)$ are all M-periodic in t, $\{u_m(t + jM)\}$ (for all $j \in \mathbf{N}$) is also $2mM$-periodic solution of (1.5). Hence, making such shifts, we can assume that $t_m \in \mathbf{Z}(1, M)$ in (3.6). Moreover, passing to a subsequence of m's, we can even assume that $t_m = t_0$ is independent of m.

Next, we extract a subsequence, still denoted by u_m, such that

$$u_m(t) \longrightarrow u(t), \qquad m \longrightarrow \infty, \quad \text{for all } t \in \mathbf{Z}.$$

Inequality (3.6) implies that $|u(t_0)| \geq \xi$ and, hence, $u = \{u(t)\}$ is a nonzero sequence. Moreover,

$$\begin{aligned} &Lu(t) - \omega u(t) - f(t, u(t+T), u(t), u(t-T)) \\ &\quad = \lim_{m\to\infty} [Lu_m(t) - \omega u_m(t) - f(t, u_m(t+T), u_m(t), u_m(t-T))] \\ &\quad = \lim_{m\to\infty} 0 = 0. \end{aligned}$$

So $u = \{u(t)\}$ is a solution of (1.5).

Finally, for any fixed $D \in \mathbf{Z}$ and m large enough, we have that

$$\sum_{t=-D}^{D} |u_m(t)|^2 \leq \underline{\lambda}^{-1} \|u_m\|_m^2 \leq \underline{\lambda}^{-1} d^2.$$

Since $\underline{\lambda}^{-1} d^2$ is a constant independent of m, passing to the limit, we have that

$$\sum_{t=-D}^{D} |u(t)|^2 \leq \underline{\lambda}^{-1} d^2.$$

Due to the arbitrariness of D, $u \in l^2$. Therefore, u satisfies $u(t) \to 0$ as $|t| \to \infty$. The existence of a nontrivial homoclinic orbit is obtained. $\square$

3.4. Proof of Theorem 1.2.

Proof. Take

$$\begin{aligned} \widetilde{\underline{\lambda}} &= \min_{t\in\mathbf{Z}(1,M)} (b(t) - |a(t-1)| - |a(t)|), \\ \widetilde{\overline{\overline{\lambda}}} &= \max_{t\in\mathbf{Z}(1,M)} (b(t) + |a(t-1)| + |a(t)|). \end{aligned}$$

Operator L is a bounded and self-adjoint operator in l^2. By the Jacobi operators theory [**31**], it is easy to know that the spectrum $\sigma(L) \subset [\widetilde{\underline{\lambda}}, \widetilde{\overline{\overline{\lambda}}}]$ and the frequency ω belong to a spectral gap $(-\infty, \widetilde{\underline{\lambda}})$.

It is obvious that the conclusion holds if $u(t) = 0$; in what follows, we consider the case $u(t) \neq 0$.

Let

$$v(t) = -\frac{f(t, u(t+T), u(t), u(t-T))}{u(t)}, \quad t \in \mathbf{Z}.$$

Then

$$\widetilde{L}u(t) - \omega u(t) = 0, \tag{3.7}$$

where

$$\widetilde{L}u(t) = Lu(t) + v(t)u(t).$$

By (F_1) and $u \in l^2$, we know that $\lim_{|t|\to\infty} v(t) = 0$, the multiplication by $v(t)$ is a compact operator in l^2. Hence,

$$\sigma_{ess}(\widetilde{L}) = \sigma_{ess}(L),$$

where σ_{ess} stands for the essential spectrum. Now (3.7) means that $u = \{u(t)\}$ is an eigenfunction that corresponds to the eigenvalue of finite multiplicity $\omega \notin \sigma_{ess}(\widetilde{L})$ of the operator $\widetilde{L}$. Therefore, the result follows from the standard theorem on exponential decay for such eigenfunctions, see, for example [**31**]. The proof of Theorem 1.2 is complete. $\square$

4. Example. As an application of Theorems 1.1 and 1.2, finally, we give an example to illustrate our results.

Example 4.1. For all $t \in \mathbf{Z}$, assume that

$$\begin{aligned} u(t+1) + u(t-1) - (2+\omega)u(t) &= \beta u(t)\Big[\varphi(t)\left((u(t+T))^2 + (u(t))^2\right)^{(\beta/2)-1} \\ &\quad + \varphi(t-T)\left((u(t))^2 + (u(t-T))^2\right)^{(\beta/2)-1}\Big], \end{aligned} \tag{4.1}$$

where $\omega < -4$, $\beta > 2$ and φ are continuously differentiable and $\varphi(t) > 0$, T a given nonnegative integer, M a given positive integer, $\varphi(t+M) = \varphi(t)$.

We have

$$a(t) = a(t-1) \equiv 1, \ b(t) \equiv -2,$$

$$f(t, v_1, v_2, v_3) = \beta v_2 \Big[\varphi(t)(v_1^2 + v_2^2)^{(\beta/2)-1} + \varphi(t-T)(v_2^2 + v_3^2)^{(\beta/2)-1}\Big]$$

and

$$F(t, v_1, v_2) = \varphi(t)(v_1^2 + v_2^2)^{\beta/2}.$$

Then

$$\begin{aligned} &F'_{v_1}(t-T, v_2, v_3) + F'_{v_2}(t, v_1, v_2) \\ &\qquad = \beta v_2 \Big[\varphi(t)(v_1^2 + v_2^2)^{(\beta/2)-1} + \varphi(t-T)(v_2^2 + v_3^2)^{(\beta/2)-1}\Big]. \end{aligned}$$

It is easy to verify that all the assumptions of Theorems 1.1 and 1.2 have been satisfied. Consequently, (4.1) has a nontrivial homoclinic orbit which decays exponentially at infinity.

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