ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 43, Number 6, 2013

RESULTS ON VALUES OF BARNES POLYNOMIALS

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ABSTRACT. In this paper, we investigate rationality of the Barnes numbers, and we find explicit good bounds for their denominators. In addition, we give Fourier expansion of Barnes polynomials and, from this study, we connect generalized Barnes numbers to values of Dirichlet L-function at non-negative integers.

1. Introduction and statement of results. Throughout this paper we use the following notation: $\mathbf{N} = \{0, 1, \ldots\}$ set of naturals numbers. It's well-known that Bernoulli numbers and polynomials are connected to many areas in mathematics and physics. Their most important properties are that Bernoulli polynomials have an easy Fourier series with the coefficients expressed in closed form, and Bernoulli numbers are rational numbers, and their denominators are controlled by Von Staudt-Clausen theorem and Kummer congruences and are values of Riemann zeta function at positive integers, and the values of L-Dirichlet functions at negative integers give us the generalized Bernoulli numbers. In this paper, we prove the extension of these properties to Barnes numbers and polynomials. Let us recall the definitions of Bernoulli and Barnes polynomials and numbers. The Bernoulli polynomials $B_n(x)$ are defined by

$$\frac{te^{tx}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}, \quad |t| < 2\pi. \tag{1}$$

$B_n := B_n(0)$ is the nth Bernoulli number. Let $\{x\}$ be the fractional part of the real number x. Then the Bernoulli functions $\overline{B}_n(x)$ are

2010 AMS *Mathematics subject classification.* Primary 11B68, 11B83, 11B99, 11M32.
Keywords and phrases. Bernoulli numbers and polynomials, Barnes numbers and polynomials, Von Staudt theorem, Dirichlet L-function.
The present research has been conducted by a Research Grant of Kwangwoon University in 2011.
Received by the editors on February 13, 2011, and in revised form on June 22, 2011.
DOI:10.1216/RMJ-2013-43-6-1857

defined by

$$\overline{B}_n(x) = \begin{cases} 0 & \text{if } n = 1,\, x \in \mathbf{Z}, \\ B_n(\{x\}) & \text{otherwise.} \end{cases} \tag{2}$$

For more information about properties of Bernoulli numbers and their generalization, see [**1, 4, 8, 9, 10, 12, 15**]. Let N positive integer and $a_1, \dots, a_N$ complex with positive real part. The Barnes polynomials and numbers are given by

$$\frac{t^N}{\prod_{j=1}^{N}(e^{a_j t} - 1)} e^{xt} = \sum_{n=0}^{\infty} B_n(x \mid a_1, \dots, a_N) \frac{t^n}{n!}, \qquad |t| < \min\left(\frac{2\pi}{|a_1|}, \dots, \frac{2\pi}{|a_N|}\right) \tag{3}$$

(see [**2, 3, 5, 7, 13, 14, 16**]). Writing

$$e(x) = e^{2\pi i x}, \quad x \in \mathbf{C}.$$

The essential series is

$$2\pi i \frac{e(uv)}{e(u) - 1} = \sum_{n \in \mathbf{Z}} \frac{e(nv)}{u - n}. \tag{4}$$

As Kronecker emphasized [**11, 18**], this identity is the foundation of the theory of classical Bernoulli functions and their relation to special values of the Riemann zeta function and Dirichlet L-functions. More precisely, let $f \geq 2$ be integer and χ be a Dirichlet character modulo f (see [**6**, page 253]). The *generalized Bernoulli numbers* $B_{m,\chi} \in \mathbf{Q}(\chi(1), \chi(2), \dots,)$ associated to χ ($m = 0, 1, \dots$) are defined by the generating function

$$\sum_{a=1}^{f} \chi(a) \frac{t}{e^{ft} - 1} e^{at} = \sum_{n=0}^{\infty} B_{n,\chi} \frac{t^n}{n!}, \quad |t| < \frac{2\pi}{f}. \tag{5}$$

The main interest of these numbers is that they give the values at negative integers of Dirichlet L-series: if $L(s, \chi) = \sum_{n=1}^{\infty} (\chi(n))/(n^s)$ ($\mathrm{Re}\,(s) > 1$) is the L-series attached to χ, then we have the formula

$$L(-n, \chi) = -\frac{B_{n+1,\chi}}{n+1} \quad (n \geq 0) \tag{6}$$

(see [**17**, Theorem 4.2]). Indeed, in equation (4), expanding the left and the right side into a Laurent series in u yields at once the most important property Fourier expansion of the classical Bernoulli

$$\overline{B}_n(x) = \frac{-n!}{(2\pi i)^n} \sum_{k\in\mathbf{Z}}^{*} \frac{e(kx)}{k^n}, \quad n \geq 1, \tag{7}$$

where the sum $\sum_{k\in\mathbf{Z}}^{*}$ means that $k=0$ is to be omitted (and in the non-absolutely convergent case $n=1$ the sum is to be interpreted as a Cauchy principal value). From this, it is not difficult to relate Bernoulli numbers and polynomials to special values of zeta and partial zeta functions defined over the field of rational numbers. This paper can now be summarized as a generalization of these facts to the Barnes polynomials and numbers.

For $\lambda \in \mathbf{C}\backslash\{0\}$, note that

$$\frac{t^N}{\prod_{j=1}^{N}(e^{(\lambda a_j)t}-1)} e^{\lambda x t} = \lambda^{-N} \frac{(\lambda t)^N}{\prod_{j=1}^{N}\left(e^{a_j(\lambda t)}-1\right)} e^{x(\lambda t)}. \tag{8}$$

Then, we obtain

$$\sum_{n=0}^{\infty} B_n(\lambda x \mid \lambda a_1, \dots, \lambda a_N)\frac{t^n}{n!} = \lambda^{-N} \sum_{n=0}^{\infty} B_n(x \mid a_1, \dots, a_N)\frac{\lambda^n t^n}{n!}. \tag{9}$$

Therefore, by comparing the coefficients of both sides of equation (9), it's easy to see that

Proposition 1 (Homogeneity). *For any $a_1, \dots, a_N$ non nul positive real numbers and $\lambda \in \mathbf{C}\backslash\{0\}$, we have*

$$B_n(\lambda x \mid \lambda a_1, \dots, \lambda a_N) = \lambda^{n-N} B_n(x \mid a_1, \dots, a_N), \quad (n \geq 1). \tag{10}$$

Now we state our main results.

Theorem 2 (Explicit formula and rationality). *Let $a_1, \ldots, a_N$ be non nul positive real numbers. Then*

$$\frac{B_n(x \mid a_1, \ldots, a_N)}{n!} = \sum_{m_1+\cdots+m_N=n} a_1^{m_1-1} \cdots a_N^{m_N-1} \frac{B_{m_1}(X)}{m_1!} \cdots \frac{B_{m_N}(X)}{m_N!}, \quad (n \in \mathbf{N}) \tag{11}$$

where $X = x/A_N$ and $A_N = a_1 + \cdots + a_N$. In addition, if $a_1, \ldots, a_N$ are rational numbers, then $[B_n(x \mid a_1, \ldots, a_N)]/n!$ is a polynomial with rational coefficients.

Theorem 3 (Fourier expansion). *Let $a_1, \ldots, a_N$ be non nul positive real numbers, and set $A_N = a_1 + \cdots + a_N$ and $X = x/A_N$. Then, for any $n \geq 1$ and $|X| < 1$, we have*

$$B_n(x \mid a_1, \ldots, a_N) = \frac{(-1)^N n!}{(2\pi i)^n} \sum^{*}_{m_1+\cdots+m_N=n} a_1^{m_1-1} \cdots a_N^{m_N-1} \sum'_{k_1,\ldots,k_N \in \mathbf{Z}\setminus\{0\}} \frac{e\left((k_1+\cdots+k_N)X\right)}{k_1^{m_1} \cdots k_N^{m_N}}.$$

Here $\sum'_{k_1,\ldots,k_N \in \mathbf{Z}\setminus\{0\}}$ means that $k_1, \ldots, k_N \in \mathbf{Z}\setminus\{0\}$ and, in the non-absolutely convergent case $m_i = 1$, for any $1 \leq i \leq N$, the sum $\sum'_{k_i \in \mathbf{Z}\setminus\{0\}}$ is to be interpreted as a Cauchy principal value for each i, and $\sum^*_{m_1+\cdots+m_N=n}$ means that $m_1, \ldots, m_N \in \mathbf{N}$ with the usual convention the sum $\sum_{k_i \in \mathbf{Z}\setminus\{0\}} e(k_i X) = -1$.

Theorem 4 (Universal bounds). *Let $a_1, \ldots, a_N$ be non nul positive integers. Put*

$$\lambda(n) = \prod_{p \text{ prime } \leq n} p^{[n/(p-1)]}, \quad (n \geq 1).$$

Then the denominator of $[B_n(0 \mid a_1, \ldots, a_N)]/n!$ divides $\lambda(n)$, where $[x]$ denotes the integer part of the real number x.

Let q be an integer ≥ 2 and χ a Dirichlet character modulo q. We set

$$L(s,\chi)=\sum_{k=1}^{\infty}\frac{\chi(k)}{k^s},\qquad G_\chi=\sum_{t=1}^{q}\chi(t)e(t/q),$$

where s is a complex variable and $L(s,\chi)$ converges for $\Re(s)>0$ if χ is non principal and for the principal character it converges for $\Re(s)>1$. G_χ is the so-called Gauss sum attached to the character χ.

By homogeneity, Proposition 1, without loss of generality, we can assume for the following theorems that: $a_1+\cdots+a_N=1$.

Theorem 5 (Values of L-function at non-negative integers). *Let $q\geq 2$ be a natural number, $a_1,\ldots,a_N$ non nul positive real numbers with $a_1+\cdots+a_N=1$, and χ a non trivial Dirichlet character modulo $q\geq 2$. Then we have*

$$\sum_{t=1}^{q}\overline{\chi}(t)B_n\left(t/q\mid a_1,\ldots,a_N\right)$$
$$=\begin{cases}\frac{2^N(-1)^N(n!)}{(2\pi i)^n}G_{\overline{\chi}}\sum^{*}a_1^{m_1-1}\cdots a_N^{m_N-1}L(m_1,\chi)\cdots L(m_N,\chi), & \\ & \text{if } \chi(-1)=(-1)^n,\\ 0 & \text{otherwise,}\end{cases}$$

with the usual convention $L(0,\chi)=(-1/2)$, $\overline{\chi}$ denotes the conjugate character of χ, and where the summation $\sum^{*}$ runs on $m_1,\ldots,m_N\geq 0$ such that

$$m_1+\cdots+m_N=n,\qquad \chi(-1)=(-1)^{m_i},\quad i=1,\ldots,N\,.$$

From Theorem 5 above we immediately get the following corollaries.

Corollary 6 (Values of L-function for even character). *Let $q\geq 2$, n be natural numbers, $a_1,\ldots,a_N$ non nul positive real numbers with $a_1+\cdots+a_N=1$, χ an even non trivial Dirichlet character modulo*

$q \geq 2$. Then we have

$$\sum_{t=1}^{q} \overline{\chi}(t) B_{2n}\left(t/q \mid a_1, \ldots, a_N\right) = \frac{2^N(-1)^N(2n)!}{(2\pi i)^{2n}}\, G_{\overline{\chi}} \sum_{\substack{j_1+\cdots+j_N=n \\ j_1,\ldots,j_N\geq 0}} a_1^{2j_1-1}\cdots a_N^{2j_N-1} L(2j_1,\chi)\cdots L(2j_N,\chi),$$

with the usual convention $L(0,\chi) = -1/2$.

Corollary 7 (Values of L-function for odd character). *Let $q \geq 2$, n be natural numbers, $a_1, \ldots, a_N$ non nul positive real numbers with $a_1 + \cdots + a_N = 1$ and χ an odd non trivial Dirichlet character modulo $q \geq 2$. Then we have*

$$\sum_{t=1}^{q} \overline{\chi}(t) B_{2n+1}\left(t/q \mid a_1, \ldots, a_N\right) = \frac{2^N(-1)^N(2n+1)!}{(2\pi i)^{2n+1}}\, G_{\overline{\chi}} \sum_{\substack{m_1+\cdots+m_N=2n+1 \\ m_1,\ldots,m_N \text{ odd}}} a_1^{m_1-1}\cdots a_N^{m_N-1} L(m_1,\chi)\cdots L(m_N,\chi).$$

Corollary 8. *Let $a_1, \ldots, a_N$ be non nul positive real numbers with $a_1 + \cdots + a_N = 1$ and χ a non trivial Dirichlet character modulo $q \geq 2$. If $\chi(-1) \neq (-1)^n$, we have*

$$\sum_{t=1}^{q} \overline{\chi}(t) B_n(t/q \mid a_1, \ldots, a_N) = 0.$$

Remark 1. Our Theorem 5 gives another way to obtain formula (6). Precisely, we show how to use Theorem 5 to obtain

$$L(-n,\chi) = -\frac{B_{n+1,\chi}}{n+1} \quad (n \geq 0).$$

Set

$$\delta = \begin{cases} 0 & \text{if } \chi(-1) = 1, \\ 1 & \text{if } \chi(-1) = -1. \end{cases}$$

Using the following functional equations

$$G_{\overline{\chi}} = \chi(-1)q/G_{\chi}, \qquad \Gamma(s)\cos\left(\pi\frac{s-\delta}{2}\right) = \frac{G_{\chi}}{2i^{\delta}}\left(\frac{2\pi}{q}\right)^{n} L(1-s,\chi),$$

(see [**17**, Chapter 4, pages 29–37]), at $s=n$, we obtain

$$-\frac{2(n!)G_{\overline{\chi}}L(n,\chi)}{(2\pi i)^n} = -nq^{1-n}L(1-n,\chi).$$

Now we take $N=1$ and $a_1=1$ and, by using Theorem 5, we derive the following equality

$$\frac{2(n!)G_{\overline{\chi}}L(n,\chi)}{(2\pi i)^n} = -\sum_{t=1}^{q}\overline{\chi}(t)B_n\left(t/q\right).$$

Hence,

$$L(1-n,\chi) = -\frac{1}{n}q^{n-1}\sum_{t=1}^{q}\overline{\chi}(t)B_n\left(t/q\right),$$

and, from definition (5), it's easy to see that

$$q^{n-1}\sum_{t=1}^{q}\overline{\chi}(t)B_n\left(t/q\right) = B_{n,\chi}.$$

Therefore, we get the well-known equality (6).

2. Proofs of main results.

Proof of Theorem 2. Writing $X = x/A_N$, we have

$$\begin{aligned}\frac{t^N}{\prod_{j=1}^{N}(e^{a_j t}-1)}e^{xt} &= \frac{1}{a_1\cdots a_N}\prod_{i=1}^{N}\frac{a_i t e^{X(a_i t)}}{e^{a_i t}-1}\\ &= \sum_{n=0}^{\infty}\left(\sum_{m_1+\cdots+m_N=n} a_1^{m_1-1}\cdots a_N^{m_N-1}\frac{B_{m_1}(X)}{m_1!}\cdots\frac{B_{m_N}(X)}{m_N!}\right)t^n.\end{aligned}$$

On the other hand,

$$\frac{t^N}{\prod_{j=1}^N (e^{a_j t}-1)} e^{xt} = \sum_{n=0}^{\infty} \frac{B_n}{n!}(x|a_1,\ldots,a_N)t^n.$$

By comparing the right members of both equations above we obtain

$$\begin{aligned}&\frac{B_n(x\mid a_1,\ldots,a_N)}{n!}\\&= \sum_{m_1+\cdots+m_N=n} a_1^{m_1-1}\cdots a_N^{m_N-1}\frac{B_{m_1}(X)}{m_1!}\cdots\frac{B_{m_N}(X)}{m_N!}, \quad (n\in \mathbf{N}).\end{aligned}$$

Let $a_1,\ldots,a_N$ be rational numbers. By using the fact that the classical Bernoulli polynomials are in $\mathbf{Q}[x]$, we obtain that $[B_n(x\,|\,a_1,\ldots,a_N)]/n!$ is a polynomial with rational coefficients. This completes the proof of our theorem. □

Proof of Theorem 3. Using equation (7) and Theorem 2, we can write

$$\begin{aligned}\frac{B_n(x\mid a_1,\ldots,a_N)}{n!} &= \sideset{}{^*}\sum_{\substack{m_1+\cdots+m_N=n\\ m_1,\ldots,m_N\ge 0}} \frac{a_1^{m_1-1}\cdots a_N^{m_N-1}}{m_1!\cdots m_N!}\frac{(-1)^N m_1!\cdots m_N!}{(2\pi i)^{m_1+\cdots+m_N}}\\&\quad\times \sideset{}{'}\sum_{k_1,\ldots,k_N\in\mathbf{Z}\setminus\{0\}} \frac{e((k_1+\cdots+k_N)X)}{k_1^{m_1}\cdots k_N^{m_N}}\\&= \frac{(-1)^N}{(2\pi i)^n}\sideset{}{^*}\sum_{\substack{m_1+\cdots+m_N=n\\ m_1,\ldots,m_N\ge 0}} a_1^{m_1-1}\cdots a_N^{m_N-1}\\&\quad\times \sideset{}{'}\sum_{k_1,\ldots,k_N\in\mathbf{Z}\setminus\{0\}} \frac{e((k_1+\cdots+k_N)X)}{k_1^{m_1}\cdots k_N^{m_N}}.\end{aligned}$$

This yields the theorem. □

Proof of Theorem 4. Let n be non-negative integers, p a prime number, we denote by $v_p(n)$ the p-adic valuation of n. In order to prove our theorem we need the following two lemmas:

Lemma 9. *For any $n \in \mathbf{N}$, $n \geq 1$ and p a prime number, we have*

$$v_p(n!) \leq \begin{cases} [\frac{n}{p-1}] & \text{if } p-1 \nmid n, \\ [\frac{n}{p-1}] - 1 & \text{if } p-1 \mid n. \end{cases}$$

For this, we set $K = [\log(n)/\log(p)]$. It's easy to see that

$$v_p(n!) = \sum_{i=1}^{K} [n/p^i].$$

(1) If $p-1 \nmid n$, we have

$$v_p(n!) = \sum_{i=1}^{K} [n/p^i] \leq n \sum_{i=1}^{\infty} n/p^i = \frac{n}{p-1}.$$

(2) If $p-1 \mid n$, we write $n = m(p-1)$ with $m \geq 1$. Then we have

$$v_p(n!) \leq n \sum_{i=1}^{K} n/p^i = m(1-p^{-K}) < m.$$

This proves Lemma 9. □

Lemma 10. *For any $n \in \mathbf{N}$, $n \geq 1$ and p a prime number, let d_n denote the denominator of $B_n/n!$. We have*

$$v_p(d_n) \leq \left[\frac{n}{p-1}\right].$$

From the Von Staudt theorem [**6**, page 233], we know that

$$v_p\ (\text{denominator}\,(B_n)) = \begin{cases} 0 & \text{if } p-1 \nmid n, \\ 1 & \text{if } p-1 \mid n. \end{cases}$$

From this and Lemma 9, we obtain

$$v_p(d_n) = v_p(n!) + v_p(\text{denominator}\,(B_n)) \leq \left[\frac{n}{p-1}\right].$$

Then we get our Lemma 10. $\square$

Now we are ready to prove our theorem. Let D_n be the denominator of $[B_n(0 \mid a_1, \dots, a_N)]/n!$. By Theorem 2, we get

$$\begin{aligned}\frac{B_n(0 \mid a_1, \dots, a_N)}{n!} & \\ &= \sum_{m_1+\cdots+m_N=n} a_1^{m_1-1} \cdots a_N^{m_N-1} \frac{B_{m_1}}{m_1!} \cdots \frac{B_{m_N}}{m_N!} \quad (n \in \mathbf{N}).\end{aligned}$$

Therefore, we can write

$$v_p(D_n) \leq \max_{m_1+\cdots+m_N=n} v_p(d_{m_1} \cdots d_{m_N}).$$

From this relation and Lemma 10, we obtain

$$\begin{aligned} v_p(D_n) &\leq \max_{m_1+\cdots+m_N=n} \left(\left[\frac{m_1}{p-1}\right] + \cdots + \left[\frac{m_N}{p-1}\right] \right) \\ &\leq \max_{m_1+\cdots+m_N=n} \left[\frac{m_1+\cdots+m_N}{p-1}\right] = \left[\frac{n}{p-1}\right]. \end{aligned}$$

Hence, the denominator D_n of $[B_n(0 \mid a_1, \dots, a_N)]/n!$ divides $\lambda(n) = \prod_{p \text{ prime } \leq n} p^{[n/(p-1)]}$. This gives our theorem. $\square$

Proof of Theorem 5. Using Theorem 3, we have

$$\begin{aligned}\sum_{t=1}^{q} \overline{\chi}(t) B_n\left(t/q \mid a_1, \dots, a_N\right) & \\ &= \frac{(-1)^N n!}{(2\pi i)^n} \sideset{}{^*}\sum_{m_1+\cdots+m_N=n} a_1^{m_1-1} \cdots a_N^{m_N-1} \\ &\quad \sideset{}{'}\sum_{k_1,\dots,k_N \in \mathbf{Z}\setminus\{0\}} \frac{1}{k_1^{m_1} \cdots k_N^{m_N}} \sum_{t=1}^{q} \overline{\chi}(t) e\left((k_1+\cdots+k_N)t/q\right).\end{aligned}$$

Since

$$\sum_{t=1}^{q} \overline{\chi}(t) e\left(kt/q\right) = \chi(k) G_{\overline{\chi}},$$

then we obtain

$$\begin{aligned}
&\sum_{t=1}^{q} \overline{\chi}(t) B_n\left(t/q \mid a_1, \ldots, a_N\right) \\
&\qquad = \frac{(-1)^N n!}{(2\pi i)^n} G_{\overline{\chi}} \sum_{m_1+\cdots+m_N=n}^{*} a_1^{m_1-1} \cdots a_N^{m_N-1} \\
&\qquad\quad \sum_{k_1,\ldots,k_N \in \mathbf{Z}\setminus\{0\}}^{\prime} \frac{1}{k_1^{m_1} \cdots k_N^{m_N}} \chi(k_1 + \cdots + k_N) \\
&\qquad = \frac{(-1)^N n!}{(2\pi i)^n} G_{\overline{\chi}} \sum_{m_1+\cdots+m_N=n}^{*} a_1^{m_1-1} \cdots a_N^{m_N-1} \\
&\qquad\quad \sum_{k_1,\ldots,k_N \in \mathbf{Z}\setminus\{0\}}^{\prime} \frac{\chi(k_1)}{k_1^{m_1}} \cdots \frac{\chi(k_N)}{k_1^{m_N}},
\end{aligned}$$

while

$$\sum_{k_i \in \mathbf{Z}\setminus\{0\}}^{\prime} \frac{\chi(k_i)}{k_i^{m_i}} = \left(1 + \chi(-1)(-1)^{m_i}\right) L(m_i, \chi).$$

Therefore, we get the following equalities

$$\begin{aligned}
&\sum_{t=1}^{q} \overline{\chi}(t) B_n\left(t/q \mid a_1, \ldots, a_N\right) \\
&= \frac{(-1)^N (n!)}{(2\pi i)^n}\, G_{\overline{\chi}} \sum_{\substack{m_1+\cdots+m_N=n \\ m_1,\ldots,m_N \geq 0}} a_1^{m_1-1} \cdots a_N^{m_N-1} \\
&\quad \prod_{i=1}^{N} \left(1 + \chi(-1)(-1)^{m_i}\right) L(m_i, \chi) \\
&= \frac{2^N (-1)^N (n!)}{(2\pi i)^n}\, G_{\overline{\chi}} \\
&\quad \sum_{\substack{m_1,\ldots,m_N \geq 0, m_1+\cdots+m_N=n \\ \chi(-1)=(-1)^{m_i}}} a_1^{m_1-1} \cdots a_N^{m_N-1} L(m_1, \chi) \cdots L(m_N, \chi).
\end{aligned}$$

Since the summation is over $m_1, \ldots, m_N \geq 0$, $m_1 + \cdots + m_N = n$, $\chi(-1) = (-1)^{m_i}$. It's easy to see that, if $\chi(-1) \neq (-1)^n$, then the sum is zero. Hence, we obtain our desired theorem. $\square$

Acknowledgments. We are very grateful to the anonymous referee for many helpful comments and suggestions on a previous version of this work.

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