

## CHARACTERIZATIONS OF HILBERT SPACE AND THE VIDAV-PALMER THEOREM

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ABSTRACT. New results in numerical range theory provide a much simplified proof of the Vidav-Palmer Theorem, leading to new proofs of Berkson's characterizations of Hilbert space and of Ptak's characterization of C\*-algebras.

An early question about Banach spaces, namely, what additional properties force them to be Hilbert spaces, has led to an extensive literature: see, for instance, [1, 15].

A norm answer (the parallelogram law) was given by Jordan and von Neumann [16]: that

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

should hold for all pairs  $x, y$  of vectors in the space; the problem is in some sense two-dimensional.

Later, we find characterizations in terms of projections onto subspaces. The norm is determined by an inner-product if every two-dimensional subspace (in a space of dimension other than two) is the range of a projection of norm 1 (Kakutani [17], Phillips [25] and Bohnenblust [7]). Later still, we find that if every subspace is the range of a projection, with no a priori bound on the norms of these projections, then the space is linearly homeomorphic to a Hilbert space (Lindenstrauss and Tzafriri [20]). With the study of Banach algebras came the realization that the presence of an involution on the ring of operators on a space is an indication of Hilbert space structure (e.g., Kawada [19], Kakutani and Mackey [18]).

The impetus towards these new numerical range results, specifically Theorems 2.4, 2.5 and 5.3, came from the search for appreciably simpler proofs of two results in numerical range vein originally proved by Berkson: if every one-dimensional subspace of a Banach space  $X$  is

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the range of a hermitian projection ([4, 6]), or if  $L(X)$  is the linear span of its hermitian elements ([5]), then  $X$  is a Hilbert space.

The Vidav-Palmer Theorem (which might more properly be called the Berkson-Glickfeld-Vidav-Palmer Theorem) is an essential ingredient, and for this, too, I have developed a streamlined proof, finessing previously published treatments (see, for example, [8, 11, 24]). In particular, I do not need to show that hermitian algebras are symmetric (the Shirali-Ford Theorem, based on Ford's Square Root Lemma). The key new result, Theorem 2.5, is a variation on Ptak's fundamental inequality.

Further by-products are an elementary proof that a Banach  $*$ -algebra is  $C^*$ -equivalent if its set of exponential unitaries is bounded (see Theorem 4.1) and a version of the Kawada-Kakutani-Mackey Theorem (Theorem 6.1).

**1. Banach algebras, Banach  $*$ -algebras and  $C^*$ -algebras.** I start with a résumé of essential results from the general theory, as can be found in [8, 11, 23, 24], for instance. For readability, to make the paper more self-contained, I have included a number of proofs of results already known.

Throughout,  $(\mathcal{A}, \|\cdot\|)$  will be a complex unital Banach algebra. The unit element will be assumed to have norm 1. The *spectral radius*, defined by  $|a|_\sigma = \sup \{|z| : z \in \sigma(a)\}$ , satisfies  $|ab|_\sigma = |ba|_\sigma$  (any  $a, b \in \mathcal{A}$ ), and I will exploit this frequently without explicit mention.

It is the work of the moment to show that, for  $a \in \mathcal{A}$ , we have  $\sigma(a) \subseteq \mathbf{R}$  if and only if  $\sigma(e^{ia}) \subseteq \mathbf{T}$  if and only if  $|e^{ira}|_\sigma = 1$  for all  $r \in \mathbf{R}$ .

There is an interplay between renormings of an underlying space and of (semigroups of) the algebra of operators acting on it. As discussed in [23, Proposition 1.1.9], for instance:

**Lemma 1.1.** *If  $\mathcal{G}$  is a bounded semigroup in a Banach subalgebra  $\mathcal{A}$  of some  $L(Y)$ , and if  $1_{L(Y)} \in \mathcal{G}$ , one can define a norm  $|\cdot|_{\mathcal{G}}$  on  $Y$ , equivalent to the original norm  $\|\cdot\|$  of  $Y$ , by*

$$|y|_{\mathcal{G}} = \sup \{\|sy\| : s \in \mathcal{G}\}, \quad (y \in Y).$$

*In turn, this determines an algebra norm, the operator norm  $|\cdot|_{\mathcal{G}}$  on*

$L(Y)$  (hence on  $\mathcal{A}$ ), for which  $|s|_{\mathcal{G}} \leq 1$  ( $s \in \mathcal{G}$ ). If  $\mathcal{G}$  is a group then  $|s|_{\mathcal{G}} = 1$  ( $s \in \mathcal{G}$ ).

**Banach  $*$ -algebras.** The involution  $*$  of a complex unital Banach  $*$ -algebra  $\mathcal{A}$  need not be an isometry, nor even continuous, but  $\sigma(a^*) = \overline{\sigma(a)}$  for any  $a \in \mathcal{A}$ ; so  $|a^*|_{\sigma} = |a|_{\sigma}$ .

The set of *selfadjoint* elements in a complex unital Banach  $*$ -algebra  $\mathcal{A}$  will be denoted by  $\mathcal{S}$  (the *hermitian* elements, denoted by  $\mathcal{H}$ , are defined in terms of numerical range—see below).

Write  $\mathcal{U}$  for the set of unitaries in a complex unital Banach  $*$ -algebra  $\mathcal{A}$  and  $\mathcal{E}$  for the set  $\{e^{ih} : h \in \mathcal{S}\}$ . When the involution is continuous one has  $\mathcal{E} \subseteq \mathcal{U}$ , and it is natural to refer to the elements of  $\mathcal{E}$  as *exponential unitaries*.

The Ptak function will feature in the sequel, but I will not need to rely on Ptak's extensive development [27].

**Definition 1.2.** The Ptak function  $|\cdot|_{\Sigma}$  is defined on any unital Banach  $*$ -algebra by

$$|a|_{\Sigma} = |a^*a|_{\sigma}^{1/2}, \quad (a \in \mathcal{A}).$$

It has the C\*-property:  $|a^*a|_{\Sigma} = |a|_{\Sigma}^2$ , and  $|a^*|_{\Sigma} = |a|_{\Sigma}$  for all  $a \in \mathcal{A}$ .

**Hermitian Banach  $*$ -algebras.** Note that  $\sigma(h) = \overline{\sigma(h)}$  for  $h$  selfadjoint in a complex unital Banach  $*$ -algebra  $\mathcal{A}$ , but this does not restrict  $\sigma(h)$  to lie in  $\mathbf{R}$ .

**Definition 1.3.** A complex unital Banach  $*$ -algebra  $\mathcal{A}$  is *hermitian* if  $\sigma(h) \subseteq \mathbf{R}$  for every selfadjoint  $h \in \mathcal{S}$ .

**Lemma 1.4.** Suppose that  $\mathcal{A}$  is a hermitian Banach  $*$ -algebra. Then  $\sigma(u) \subseteq \mathbf{T}$  for  $u \in \mathcal{U}$ .

If the involution is continuous and if  $u \in \mathcal{U}$  with  $\sigma(u) \neq \mathbf{T}$  then there exists a selfadjoint  $h \in \mathcal{S}$  such that  $u = e^{ih}$ .

*Proof.* If  $w \in \sigma(u)$ , then  $w^{-1} \in \sigma(u^{-1}) = \sigma(u^*)$  and therefore, by commutative spectral theory,  $w + w^{-1} \in \sigma(u + u^*) \subseteq \mathbf{R}$  while  $w - w^{-1} \in \sigma(u - u^*) \subseteq i\mathbf{R}$ : that is,  $|w| = 1$ .

If  $\sigma(u) \neq \mathbf{T}$  we can choose an analytic branch of the logarithm, say  $L$ , on a neighborhood of  $\sigma(u)$  and so can define  $h = -iL(u)$ . Since the involution is continuous we have  $h^* = iL(u^*) = iL(u^{-1}) = -iL(u) = h$ , as required.  $\square$

**Theorem 1.5.** *If  $\mathcal{A}$  is a hermitian Banach \*-algebra with continuous involution, then*

$$\mathcal{U} \subseteq \overline{\text{co}} \mathcal{E}.$$

Hence,  $\overline{\text{co}} \mathcal{U} = \overline{\text{co}} \mathcal{E}$ .

*Proof.* Consider a unitary  $u \in \mathcal{U}$ . Choose  $t \in (0, 1)$  and put  $a = tu$ . Then  $|a|_\sigma = t < 1$ .

The *Potapov generalized Möbius transformation* [26] was first exploited by Harris [14] to prove the Russo-Dye Theorem. Here we need only the simplified form for *normal*  $a$ : that

$$F_a(\lambda) = (\lambda + a)(1 + \lambda a^*)^{-1}$$

is defined and analytic in the region  $|\lambda| < t^{-1}$ . Moreover,  $F_a(\lambda)$  is *unitary* for  $|\lambda| = 1$ ; and  $F_a(0) = a$ .

Since, by Lemma 1.4,  $|u|_\sigma = 1$ , it follows that  $2 + \lambda a^* + \bar{\lambda}a$  is invertible for any unimodular complex number  $\lambda$ . Hence

$$\lambda + F_a(\lambda) = \lambda(2 + \lambda a^* + \bar{\lambda}a)(1 + \lambda a^*)^{-1}$$

is invertible for  $|\lambda| = 1$ . Since  $-\lambda \notin \sigma(F_a(\lambda))$  we can use Lemma 1.4 to produce a selfadjoint  $h_\lambda$  such that  $F_a(\lambda) = e^{ih_\lambda} \in \mathcal{E}$ . Now

$$a = F_a(0) = \frac{1}{2\pi i} \int_{\mathbf{T}} F_a(\lambda) \lambda^{-1} d\lambda \in \overline{\text{co}} \mathcal{E}$$

so  $u = \lim_{t \rightarrow 1^-} tu \in \overline{\text{co}} \mathcal{E}$ . And  $\mathcal{E} \subseteq \mathcal{U}$ .  $\square$

**When  $\mathcal{E}$  is bounded.** Theorem 1.5 holds whether or not the set  $\mathcal{E}$  is bounded. We shall see below, Theorem 4.1, that algebras with  $\mathcal{E}$  bounded are in fact  $C^*$ -equivalent. In the meantime, we can show the following.

**Theorem 1.6.** *Let  $\mathcal{A}$  be a complex unital Banach  $*$ -algebra in which the set of exponential unitaries  $\mathcal{E}$  is bounded. Then  $\mathcal{A}$  is hermitian. Further,  $\mathcal{S}$  is closed in  $\mathcal{A}$ , and the involution is continuous. Moreover,  $\mathcal{U}$  is also bounded. Indeed,*

$$\sup \{\|u\| : u \in \mathcal{U}\} = \sup \{\|e^{ih}\| : h \in \mathcal{S}\}.$$

*Proof.* Let  $K = \sup \{\|e^{ih}\| : h \in \mathcal{H}\}$ . Then, first,  $|e^{irh}|_\sigma = \lim \|e^{inrh}\|^{1/n} \leq \lim K^{1/n} = 1$  for any real  $r$ . So  $\sigma(h) \subseteq \mathbf{R}$  for any  $h \in \mathcal{S}$ : that is,  $\mathcal{A}$  is hermitian.

Next, as remarked by Wichmann [32], the set

$$\mathcal{S}_{(t)} = \{h \in \mathcal{S} : \|e^{irh}\| \leq t \quad (\forall r \in \mathbf{R})\}$$

is closed in  $\mathcal{A}$  for any positive  $t$ . This follows from the absolute convergence of the power series for the exponential function. Hence  $\mathcal{S}$  ( $= \mathcal{S}_{(K)}$ ) is closed and the Closed Graph Theorem shows that the involution is continuous. So Theorem 1.5 applies.  $\square$

**C\*-algebras and the Russo-Dye Theorem.** An elegant proof of the Russo-Dye Theorem for C\*-algebras was provided by Gardner [12], based on the following recipe.

**Lemma 1.7.** *Suppose that  $a$  is an element of a C\*-algebra and that  $\|a\| < 1$ . Then  $\text{co } \mathcal{U}$  is invariant under the mapping*

$$A : u \longmapsto \frac{a + u}{2}.$$

*Proof.* It suffices to show that  $A(\mathcal{U}) \subseteq \text{co } \mathcal{U}$ . Given  $u \in \mathcal{U}$  define

$$b = A u = \frac{au^{-1} + 1}{2} u.$$

Then  $\|au^{-1}\| \leq \|a\| < 1$  so  $\|b\| < 1$ . Moreover,  $\|bu^{-1} - 1\| = \|(au^{-1} - 1)/2\| < 1$ , so  $bu^{-1}$  is invertible and, hence,  $b$  too is invertible. Thus  $b = v|b|$ , where  $|b| = (b^*b)^{1/2}$  is invertible and  $v$  is unitary; and,

since  $|b| < 1$ , we can define a unitary  $w = |b| + i(1 - |b|^2)^{1/2}$ . Hence  $b = v(w + w^*)/2 \in \text{co } \mathcal{U}$ .  $\square$

**Theorem 1.8** (Russo-Dye [29]). *The closed unit ball of a C\*-algebra is the closed convex hull of its (exponential) unitaries.*

*Proof.* It is enough to show that the *open* ball lies inside  $\overline{\text{co }} \mathcal{U}$ . Given  $a \in \mathcal{A}$  with  $\|a\| < 1$  choose any unitary  $u$ . Put  $u_0 = u$  and, inductively, let  $u_{n+1} = A u_n$ , with  $A$  as in Lemma 1.7. Then  $\|u_{n+1} - a\| = \|u_n - a\|/2$  so  $u_n \rightarrow a$ . But  $u_n \in \text{co } \mathcal{U}$ , by the lemma: so  $a \in \overline{\text{co }} \mathcal{U}$ . And,  $\overline{\text{co }} \mathcal{U} = \overline{\text{co }} \mathcal{E}$  by Theorem 1.5.  $\square$

**2. Numerical range in a Banach algebra.** In this section  $\mathcal{A}$  will denote a complex unital Banach algebra, not initially assumed to be endowed with an involution. The background on numerical range and hermitian operators can be found in [8], for instance. Again, I recapitulate enough to exhibit the essential strands of the argument.

The pairing of a Banach space  $X$  with its dual  $X'$  will be denoted by angle brackets:  $\langle x, \theta \rangle$  is the value of the functional  $\theta \in X'$  at  $x \in X$ .

**Hermitian elements of Banach algebras.** The *state space* of  $\mathcal{A}$  is

$$S(\mathcal{A}) = \{\phi \in \mathcal{A}' : \langle 1, \phi \rangle = 1 = \|\phi\|\}$$

and the *algebra numerical range* of an element  $a$  is

$$V(a) = \{\langle a, \phi \rangle : \phi \in S(\mathcal{A})\}.$$

The *numerical radius*  $|a|_v$  of an element  $a$  is defined as  $\sup \{|z| : z \in V(a)\}$ .

Lumer's Fundamental Theorem [21] sets bounds for the numerical range.

**Theorem 2.1** (Lumer). *For any element  $a$  of a complex unital Banach algebra  $\mathcal{A}$  one has*

$$\begin{aligned} \max\{\text{Re } \lambda : \lambda \in V(a)\} &= \sup \{r^{-1} \log \|\exp(ra)\| : r > 0\} \\ &= \lim_{r \rightarrow 0^+} r^{-1} \log \|\exp(ra)\| \\ &= \lim_{r \rightarrow 0^+} r^{-1} \{\|1 + ra\| - 1\}. \end{aligned}$$

The numerical radius is a norm on  $\mathcal{A}$  equivalent to the given norm:  $e^{-1}\|a\| \leq |a|_v \leq \|a\|$  for any  $a \in \mathcal{A}$ .

Since  $\sigma(a) \subseteq V(a)$ , we see that  $|a|_\sigma \leq |a|_v$  for any  $a \in \mathcal{A}$ .

**Definition 2.2.** An element  $h$  of  $\mathcal{A}$  is *hermitian* if  $V(h) \subseteq \mathbf{R}$ . Equivalently,  $h$  is hermitian if and only if  $\|1 + irh\| = 1 + o(r)$  ( $\mathbf{R} \ni r \rightarrow 0$ ), and, equivalently, if and only if

$$\|e^{irh}\| = 1 \quad (\forall r \in \mathbf{R}).$$

If so,  $V(h) = \text{co } \sigma(h)$ , the *convex hull* of  $\sigma(h)$ , and  $\|h\| = |h|_v$ : furthermore,  $|h|_\sigma = \|h\|$  (Sinclair's Theorem).

Write  $\mathcal{H}$  for the real linear subspace of hermitian elements of  $\mathcal{A}$ . Given  $h, k \in \mathcal{H}$  the *Lie product*  $hk - kh$  automatically lies in  $i\mathcal{H}$ ; in contrast, the *Jordan product*  $hk + kh$  need not be hermitian.

Although the square of a hermitian  $h$  need not be hermitian we do know that  $V(h^2)$  lies in the *right half plane* ([9]): for, given  $t \in \mathbf{R}$  and  $\phi \in S(\mathcal{A})$ , we have  $1 \geq \text{Re } \langle e^{ith}, \phi \rangle = 1 - t^2 \text{Re } \langle h^2, \phi \rangle / 2 \dots$ . This justifies the following lemma (on which Theorem 2.5 below depends).

**Lemma 2.3.** *Given  $h \in \mathcal{H}$  and  $\phi \in S(\mathcal{A})$  we have*

$$\langle h, \phi \rangle^2 \leq \text{Re } \langle h^2, \phi \rangle.$$

*Proof.* Since  $h - \langle h, \phi \rangle 1 \in \mathcal{H}$  we have  $0 \leq \text{Re } \langle (h - \langle h, \phi \rangle 1)^2, \phi \rangle$ . □

**Hermitian projections.** Hermitian projections enjoy some, if not all, properties of selfadjoint projections in a C\*-algebra. We will need these in Section 5.

For a *hermitian projection*  $p$  ( $\neq 0$ )

$$\sigma(p) = \{0, 1\}, \quad V(p) = [0, 1], \quad 1 = \|e^{i\pi p}\| = \|1 - 2p\|.$$

Thus,  $u = 1 - 2p$  is an *isometric symmetry*, and  $a \mapsto \langle uau, \phi \rangle \in S(\mathcal{A})$  if  $\phi \in S(\mathcal{A})$ ; so

$$V(uau) = V(a) \quad (a \in \mathcal{A})$$

and therefore

$h$  is hermitian if and only if  $uhu$  is hermitian.

Now  $uau = a - 2(pa + ap) + 4pap$  for any  $a$ . This establishes the following result, which seems not to have been presented explicitly before.

**Theorem 2.4.** *If  $h$  is hermitian and  $p$  is a hermitian projection in  $\mathcal{A}$ , then*

$ph + hp$  is hermitian if and only if  $php$  is hermitian.

**The Ptak function and the involution on  $\mathcal{H} + i\mathcal{H}$ .** Consider  $\mathcal{J} = \mathcal{H} + i\mathcal{H}$ , the complex linear span of  $\mathcal{H}$ , which need not be a subalgebra of  $\mathcal{A}$ . Any element of  $\mathcal{J}$  has a unique expression in the form  $h + ik$  ( $h, k \in \mathcal{H}$ ) for, if  $h + ik = h' + ik'$ , then  $h - h' = i(k' - k) \in \mathcal{H} \cap i\mathcal{H} = \{0\}$ . The *natural sesquilinear involution*  $*$  :  $\mathcal{J} \rightarrow \mathcal{J}$  :  $b = h + ik \mapsto b^* = h - ik$  is therefore well defined; clearly  $(\lambda b)^* = \bar{\lambda}b^*$  for  $\lambda \in \mathbf{C}$ ; and  $*$  is continuous: indeed,  $\|h\| = |h|_v \leq |h + ik|_v \leq \|h + ik\|$ , so  $\|b^*\| \leq 2\|b\|$ .

Even though  $\mathcal{J}$  need not be an algebra we can still define a Ptak function

$$|b|_\Sigma := |b^*b|_\sigma^{1/2} \quad (b \in \mathcal{J})$$

on  $\mathcal{J}$ , as in Definition 1.2. Then  $|b|_\Sigma = |bb^*|_\sigma^{1/2} = |b^*|_\Sigma$  for any  $b \in \mathcal{J}$ , and (Sinclair's Theorem)  $|h|_\Sigma = |h|_\sigma = |h|_v = \|h\|$  for any hermitian  $h \in \mathcal{H}$ .

This next result is critical. *Where it applies* it is formally stronger than Ptak's inequality ( $|a|_\sigma \leq |a|_\Sigma$  for elements of a hermitian Banach  $*$ -algebra).

**Theorem 2.5.** *If  $b \in \mathcal{H} + i\mathcal{H}$ , where  $\mathcal{H}$  is the real linear subspace of hermitian elements of a complex unital Banach algebra  $\mathcal{A}$ , and if  $b^*b$  is hermitian, then*

$$|b|_v \leq |b|_\Sigma.$$

*Proof.* First, with  $b = h + ik$  ( $h, k \in \mathcal{H}$ ), note that  $b^*b$  is hermitian if and only if  $h^2 + k^2$  is hermitian (if and only if  $bb^*$  is hermitian): for  $i(hk - kh)$  is automatically hermitian. If so,  $|b^*b|_v = |b^*b|_\sigma = |b|_\Sigma^2 = |bb^*|_\sigma = |bb^*|_v$ .

Next, for any  $\phi$  in  $S(\mathcal{A})$ :

$$\begin{aligned} |\langle b, \phi \rangle|^2 &= \langle h, \phi \rangle^2 + \langle k, \phi \rangle^2 \\ &\leq \operatorname{Re} \langle h^2, \phi \rangle + \operatorname{Re} \langle k^2, \phi \rangle \quad (\text{using Lemma 2.3}) \\ &= \operatorname{Re} \langle h^2 + k^2, \phi \rangle = \langle h^2 + k^2, \phi \rangle \quad (h^2 + k^2 \text{ is hermitian}) \\ &= \frac{1}{2} \langle b^*b + bb^*, \phi \rangle \leq \frac{1}{2} (|b^*b|_v + |bb^*|_v) = |b|_\Sigma^2. \quad \square \end{aligned}$$

*Remark 2.6.* Strict inequality is possible: when  $b = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$  we have  $|b|_\sigma = 0$ ,  $|b|_v = 1/2$  and  $|b|_\Sigma = 1$ .

### 3. $V$ -algebras and the Vidav-Palmer Theorem.

**Definition 3.1.** A complex unital Banach algebra  $\mathcal{A}$  is a  $V$ -algebra if  $\mathcal{A} = \mathcal{H} + i\mathcal{H}$ , where  $\mathcal{H}$  is the set of hermitian elements of  $\mathcal{A}$ : that is, if  $\mathcal{J} = \mathcal{A}$ .

Suppose that  $\mathcal{A}$  is a  $V$ -algebra. Then  $\mathcal{H}$  is closed under squaring and forming Jordan products. For, if  $h^2 = k + il$  with  $k, l \in \mathcal{H}$ , then  $h(k + il) = (k + il)h$  so  $hk - kh = i(lh - hl) \in \mathcal{H} \cap i\mathcal{H} = \{0\}$ . Spectral theory in the commutative unital algebra generated by  $h$  and  $k$  shows that  $\sigma(h^2 - k) = \sigma(il) \subseteq \mathbf{R} \cap i\mathbf{R} = \{0\}$  and therefore  $h^2 = k \in \mathcal{H}$ . Then  $hk + kh = (h + k)^2 - h^2 - k^2$ .

This done, we can show that the natural involution on  $\mathcal{A}$  is an algebra involution. Consider  $a = h + ik$ ,  $b = m + in \in \mathcal{H} + i\mathcal{H}$ . Then

$$\begin{aligned} ab + b^*a^* &= (hm + mh) - (kn + nk) + i(hn - nh) + i(km - mk) \in \mathcal{H} \\ ab - b^*a^* &= (hm - mh) - (kn - nk) + i(hn + nh) + i(km + mk) \in i\mathcal{H}, \end{aligned}$$

and therefore  $(ab + b^*a^* + (ab - b^*a^*))^* = ab + b^*a^* - (ab - b^*a^*)$ : that is,  $(ab)^* = b^*a^*$ .

Thus  $\mathcal{A}$  is a Banach \*-algebra with continuous involution in which the selfadjoints are the hermitians. In particular, all elements  $a^*a$  are hermitian: hence

$$\|a^*a\| = |a^*a|_\Sigma = |a^*a|_\sigma = |a|_\Sigma^2 \quad (a \in \mathcal{A}).$$

**Theorem 3.2** (Vidav-Palmer). *Let  $\mathcal{A}$  be a  $V$ -algebra. Then, with the natural involution and given norm,  $\mathcal{A}$  is a  $C^*$ -algebra.*

*Proof.* The only possible  $C^*$ -norm on  $\mathcal{A}$  is  $|\cdot|_\Sigma$ : so let us check that  $|\cdot|_\Sigma$  is a norm on  $\mathcal{A}$ . In the first place, if  $|a|_\Sigma = 0$  then  $|a|_v = 0$  by Theorem 2.5, forcing  $a = 0$ .

Next,  $|\cdot|_\Sigma$  is submultiplicative (as in any hermitian Banach \*-algebra): for, given any  $a, b \in \mathcal{A}$ , we have

$$|ab|_\Sigma^2 = |b^*a^*ab|_\sigma = |a^*abb^*|_\sigma \leq \|a^*abb^*\| \leq \|a^*a\|\|bb^*\| = |a|_\Sigma^2|b|_\Sigma^2.$$

And  $|\cdot|_\Sigma$  is subadditive: given  $a, b \in \mathcal{A}$ , we have

$$\begin{aligned} \|a^*b + b^*a\| &= |a^*b + b^*a|_v \\ &\leq |a^*b|_v + |b^*a|_v \\ &\leq |a^*b|_\Sigma + |b^*a|_\Sigma \quad (\text{using Theorem 2.5}) \\ &\leq |a^*|_\Sigma|b|_\Sigma + |b^*|_\Sigma|a|_\Sigma \quad (|\cdot|_\Sigma \text{ is submultiplicative}) \\ &= 2|a|_\Sigma|b|_\Sigma, \end{aligned}$$

and therefore

$$\begin{aligned} |a + b|_\Sigma^2 &= \|a^*a + b^*b + a^*b + b^*a\| \\ &\leq |a|_\Sigma^2 + |b|_\Sigma^2 + 2|a|_\Sigma|b|_\Sigma \\ &= (|a|_\Sigma + |b|_\Sigma)^2. \end{aligned}$$

This establishes that  $|\cdot|_\Sigma$  is a  $C^*$ -norm on  $\mathcal{A}$ , and it is equivalent to the original norm: for  $e^{-1}\|a\| \leq |a|_v \leq |a|_\Sigma \leq (\|a^*\|\|a\|)^{1/2} \leq 2^{1/2}\|a\|$ .

To conclude, apply the Russo-Dye Theorem (Theorem 1.8 above) to the  $C^*$ -algebra  $(\mathcal{A}, |\cdot|_\Sigma)$ : if  $|a|_\Sigma \leq 1$ , then  $a \in \overline{\text{co}} \mathcal{E}$ . Now, each  $e^{ih}$  has (original) norm 1, so  $|a|_\Sigma \leq 1$  implies  $\|a\| \leq 1$ . Thus,  $|a|_\Sigma \geq \|a\|$  for any  $a \in \mathcal{A}$ . Hence

$$\|a^*a\| = |a^*a|_\Sigma = |a|_\Sigma^2 \geq \|a\|^2,$$

from which  $\|a^*\|\|a\| \geq \|a^*a\| \geq \|a\|^2$ . Thus  $\|a^*\| \geq \|a\|$ , and, finally,  $\|a^*a\| = \|a\|^2$ .  $\square$

**4. C\*-equivalence.** Suppose that  $\mathcal{A}$  is a unital Banach  $*$ -algebra in which the set of exponential unitaries  $\mathcal{E}$  is bounded. Then its unitary group  $\mathcal{U}$  is also bounded and in consequence  $\mathcal{A}$  can be renormed to be a C\*-algebra. This result is perhaps the most elusive of the equivalences in [27, Theorem 8.4] but now follows quite directly.

**Theorem 4.1.** *If  $\mathcal{A}$  is a unital Banach  $*$ -algebra and its set of exponential unitaries  $\mathcal{E}$  is bounded, then  $\mathcal{A}$  is C\*-equivalent.*

*Proof.* Let  $K = \sup \{\|e^{ih}\| : h \in \mathcal{H}\}$ . By Theorem 1.6, the algebra  $\mathcal{A}$  is hermitian with continuous involution and  $\|u\| \leq K$  for each  $u \in \mathcal{U}$ . Thus,  $\mathcal{U}$  is a *bounded group* in  $\mathcal{A}$  and therefore by Lemma 1.1 with  $Y = \mathcal{A}$ , there is an equivalent algebra norm, say  $|\cdot|_{\mathcal{U}}$ , on  $\mathcal{A}$  for which each  $u \in \mathcal{U}$  has norm 1.

Now  $|e^{irh}|_{\mathcal{U}} = 1$  for every  $h \in \mathcal{S}$  and  $r \in \mathbf{R}$ : that is, every selfadjoint in  $\mathcal{A}$  is  $|\cdot|_{\mathcal{U}}$ -hermitian. Thus  $(\mathcal{A}, |\cdot|_{\mathcal{U}})$  is a  $V$ -algebra; hence  $(\mathcal{A}, |\cdot|_{\Sigma})$  is a C\*-algebra.  $\square$

Note that we do not need the full strength (the isometric part, clinched by the Russo-Dye Theorem) of the Vidav-Palmer Theorem—only that  $(\mathcal{A}, |\cdot|_{\Sigma})$  is a C\*-algebra.

**Local C\*-equivalence.** Barnes [2] introduced the concept of a *locally C\*-equivalent* Banach  $*$ -algebra, one in which the closed sub-algebra generated by any selfadjoint element is C\*-equivalent (so  $\mathcal{E}$  is bounded precisely when  $\mathcal{A}$  is uniformly locally C\*-equivalent), and proved that locally C\*-equivalent algebras are indeed C\*-equivalent if they are *commutative*.

Wichmann [32] provided a simpler proof: with notation as in Section 1 above, in such an algebra  $\mathcal{S} = \bigcup_{t \in \mathbf{N}} \mathcal{S}_{(t)}$  and, therefore, by Baire's Category Theorem, some  $\mathcal{S}_{(t)}$  has interior. Then  $\mathcal{S} = \mathcal{S}_{(t^2)}$  (a trivial calculation, given commutativity) so Theorem 4.1 applies.

Cuntz [10] later showed, by an intricate argument, that the commutativity hypothesis may be dropped. There is a full treatment of this topic in [11, Chapter 9].

**5. Characterizations of Hilbert space.** With the Vidav-Palmer Theorem now at our disposal we can pass on to Berkson's results.

Consider a Banach space  $X$  with dual  $X'$ . Then

$$\Pi_X = \{(x, \theta) \in X \times X' : \langle x, \theta \rangle = \|x\| = \|\theta\| = 1\}$$

is the *supporting set* of  $X$ , and an operator  $h$  on  $X$  is (*spatially hermitian*) if its (spatial) numerical range

$$V(h) = \{\langle hx, \theta \rangle : (x, \theta) \in \Pi_X\}$$

is real. An operator on a Banach space  $X$  is (spatially) hermitian if and only if it is hermitian as an element of the Banach algebra  $L(X)$ .

Consider a nontrivial operator  $f$  of rank 1 defined on the Banach space  $X$ . Its range is a one-dimensional subspace (a *ray*)  $Cy$  for some unit vector  $y$  in  $X$  and

$$fx = \langle x, \phi \rangle y = (y \otimes \phi)x$$

for some  $\phi \in X'$  with  $\|f\| = \|\phi\|$ . Now  $f^2 = \langle y, \phi \rangle f$  so  $f$  is a projection if and only if  $\langle y, \phi \rangle = 1$ . If so,  $\|f\| = 1$  if and only if  $(y, \phi) \in \Pi_X$ .

**Lemma 5.1.** *Suppose that  $f = y \otimes \phi$  is a nontrivial rank 1 projection on  $X$ . Then  $f$  is hermitian if and only if both  $(y, \phi) \in \Pi_X$  and*

$$\kappa_{(y, \phi), (z, \psi)} := \langle y, \psi \rangle \langle z, \phi \rangle \geq 0 \quad \text{for all } (z, \psi) \in \Pi_X.$$

*Proof.* First, any nontrivial hermitian projection has positive numerical range and norm 1. Second,  $V(f) = \{\langle fz, \psi \rangle : (z, \psi) \in \Pi_X\} = \{\langle y, \psi \rangle \langle z, \phi \rangle : (z, \psi) \in \Pi_X\}$ .  $\square$

**Lemma 5.2.** *Let  $f = y \otimes \phi$  and  $g = z \otimes \psi$  be rank 1 projections. Then  $\|fg\| = |\langle z, \phi \rangle| \|g\|$  and  $\|gf\| = |\langle y, \psi \rangle| \|f\|$ . Also,  $fgf = \kappa f$  and  $gfg = \kappa g$ , where*

$$\kappa = \kappa_{(y, \phi), (z, \psi)} = \langle y, \psi \rangle \langle z, \phi \rangle.$$

*Proof.*  $fgx = f\langle x, \psi \rangle z = \langle x, \psi \rangle \langle z, \phi \rangle y$  for  $x \in X$ , so  $\|fg\| = |\langle z, \phi \rangle| \|\psi\| = |\langle z, \phi \rangle| \|g\|$ . Further,  $gfgx = \langle x, \psi \rangle \langle z, \phi \rangle \langle y, \psi \rangle z = \kappa \langle x, \psi \rangle z = \kappa gx$ .  $\square$

**Theorem 5.3.** *Let  $f = y \otimes \phi$  and  $g = z \otimes \psi$  be rank 1 hermitian projections. Then  $fg + gf$  is hermitian.*

*Proof.* This is a direct corollary of Theorem 2.4, for  $fgf$  is hermitian, being equal to  $\kappa f$  with  $\kappa$  real (Lemmas 5.1 and 5.2).  $\square$

**Theorem 5.4.** *Let  $f = y \otimes \phi$  and  $g = z \otimes \psi$  be rank 1 hermitian projections. Then*

$$\langle y, \psi \rangle = \overline{\langle z, \phi \rangle}.$$

*Proof.* Each string in  $\{1, f, g\}$  reduces to a multiple of one of  $\{1, f, g, fg, gf\}$ . So the algebra generated by  $\{1, f, g\}$  is the complex linear span of  $\{1, f, g, fg + gf, i(fg - gf)\}$ . All of these are hermitian (Theorem 5.3 is critical here) so  $\text{alg}\{1, f, g\}$  is a  $V$ -algebra, and hence, by the Vidav-Palmer Theorem, a  $C^*$ -algebra. Hence,  $\|fg\| = \|(fg)^*\| = \|gf\|$ , from which  $|\langle z, \phi \rangle| = |\langle y, \psi \rangle|$  (Lemma 5.2); and  $\langle y, \psi \rangle \langle z, \phi \rangle \geq 0$  (Lemma 5.1).  $\square$

*Remark.* It is natural to wonder whether one might be able to show that  $\|fg\| = \|gf\|$  directly (without appealing to the Vidav-Palmer Theorem).

**Theorem 5.5** (Berkson). *Suppose that  $X$  is a Banach space in which every one-dimensional subspace is the range of a hermitian projection. Then  $X$  is a Hilbert space.*

*Proof.* Recall that a hermitian projection is determined uniquely by its range. Indeed, if  $eX = fX$  where  $e$  and  $f$  are hermitian projections, then  $ef = f$  and  $fe = e$  and therefore  $e - f = fe - ef$  is both hermitian and skew hermitian, hence zero.

Thus, given a unit vector  $y$ , there is a unique supporting functional  $Jy$  such that  $y \otimes Jy$  is the hermitian projection onto  $\mathbf{C}y$ .

We can extend  $J$  to all of  $X$  by putting  $Jy = \|y\|J(y/\|y\|)$  for  $y \neq 0$  and  $J0 = 0$ . Then, from Theorem 5.4,  $\langle y, Jz \rangle = \langle z, Jy \rangle$  for all  $y, z$  in  $X$ . Thus,

$$(y|z) = \langle y, Jz \rangle = \overline{\langle z, Jy \rangle}$$

is an inner product on  $X$  satisfying  $\|x\|^2 = (x|x)$  identically.  $\square$

Incidentally, Theorem 5.5 was originally derived, in part, from Theorem 5.6, here an immediate consequence of it.

**Theorem 5.6** (Berkson). *If  $X$  is a Banach space such that  $L(X)$  is the linear span of hermitian operators, that is, if  $L(X)$  is a  $V$ -algebra, then  $X$  is a Hilbert space.*

*Proof.* Given a unit vector  $y$ , choose  $\phi$  such that  $(y, \phi) \in \Pi_X$ . Then  $q = y \otimes \phi$  is a norm 1 operator with range  $\mathbf{C}y$ , and, as is easy to check,  $qq^*q = \langle q^*y, \phi \rangle q$ . Thus,  $qq^*qq^* = \langle q^*y, \phi \rangle qq^*$ , which shows that  $\langle q^*y, \phi \rangle \neq 0$  and that  $\langle q^*y, \phi \rangle^{-1}qq^*$  is the hermitian projection with range  $\mathbf{C}y$ .  $\square$

**6. The Kawada-Kakutani-Mackey Theorem.** As an immediate consequence of Theorems 4.1 and 5.6 we have

**Theorem 6.1** ( $\leq$  Kawada-Kakutani-Mackey). *If  $X$  is a Banach space and  $L(X)$  admits an algebra involution such that the set  $\mathcal{E}$  of all exponential unitaries in  $L(X)$  is bounded, then  $X$  is linearly homeomorphic to a Hilbert space.*

*Proof.* By Theorem 4.1,  $(L(X), |\cdot|_\Sigma)$  is a  $C^*$ -algebra and  $|\cdot|_\Sigma$  is equivalent to  $\|\cdot\|$  on  $L(X)$ . Now  $|\cdot|_{\mathcal{U}}$  on  $X$ , as in Lemma 1.1, is equivalent to the original norm; and  $|u|_{\mathcal{U}} = 1$  ( $u \in \mathcal{U}$ ). Thus  $(L(X), |\cdot|_{\mathcal{U}})$  is a  $V$ -algebra and, by Theorem 5.6,  $(X, |\cdot|_{\mathcal{U}})$  is a Hilbert space.  $\square$

The hypothesis of the original Kawada-Kakutani-Mackey Theorem is that the involution is *proper*: that  $T^*T = 0$  only for  $T = 0$ . The following lemma relates these two hypotheses.

**Lemma 6.2.** *If  $L(X)$  admits an algebra involution which is not proper one can find a projection  $P \neq 0$  for which  $P^*P = 0$ . The set of exponential unitaries is then not bounded.*

*Proof.* Given a  $T \neq 0$  with  $T^*T = 0$ , one can choose  $0 \neq z = Ty$ ; without loss of generality  $\|z\| = 1$ . Choose a supporting functional  $\phi$  for  $z$  and let  $Q = y \otimes \phi$ . Then  $P = TQ$  is a nonzero idempotent with  $P^*P = 0$ ,  $Pz = z$ , and  $P^*z = 0$ . Now, as noted by Vukman [31], the exponential unitary  $U_t = e^{t(P-P^*)}$  satisfies  $U_tz = e^t z$ : so  $\|U_t\| \geq e^t$  for real  $t$ .  $\square$

**Question.** What supplementary hypotheses will ensure that the unitaries form a bounded set in a Banach  $*$ -algebra with proper involution? Might it be possible to prove this directly when  $L(X)$  admits a proper involution?

It is interesting to compare the methods above with the mode of proof of the Kawada-Kakutani-Mackey Theorem that can be extracted from [28]; see also [24, subsection 11.6].

The essential steps are:

- Fix a nonzero  $\psi \in X'$ .
- Show that the minimal ideal  $\mathcal{L} = \{x \otimes \psi : x \in X\}$  is linearly homeomorphic to  $X$ .
- Show that there is a selfadjoint projection  $p$  in  $L(X)$  such that  $\mathcal{L} = L(X)p$ .
- Derive the existence of a bounded positive functional  $\Psi \in L(X)'$  such that  $pap = \Psi(a)p$  for each  $a \in L(X)$ .
- Put  $(bp | cp) = \Psi(c^*b)$  to define an inner product on  $\mathcal{L}$  whose associated norm is equivalent to the original norm on  $X$ .

This route produces the inner product directly on the space  $X$  rather than by first renorming  $L(X)$ . It seems to be essential to know that  $L(X)$  is strongly irreducible.

**7. Further remarks.** The term *parallelogram law* seems not to appear until quite late, after Jordan-von Neumann [16]. The earliest statement in classical geometry equivalent to the parallelogram law

that I can identify is a Theorem of Apollonius of Perga:  $AB^2 + AC^2 = 2(AD^2 + BD^2)$  when  $D$  is the midpoint of the side  $BC$  in the triangle  $ABC$ .

Ptak [27] provided a detailed analysis of hermitian Banach  $*$ -algebras based on his fundamental inequality  $|a|_\sigma \leq |a|_\Sigma$ . Consequences of this inequality include the Shirali-Ford Theorem, that any hermitian Banach  $*$ -algebra is *symmetric*: that is,  $\sigma(a^*a) \subseteq \mathbf{R}^+$  for every  $a$ .

Vidav [30] established that a  $V$ -algebra is homeomorphic and star isomorphic to a  $C^*$ -algebra, subject to an additional condition (that for every  $h \in \mathcal{H}$  one can express  $h^2$  in the form  $u + iv$  with  $u, v \in \mathcal{H}$  and  $uv = vu$ ). Berkson [3] and Glickfeld [13] independently established isometry. Palmer [22] showed that Vidav's extra hypothesis was not needed, and gave a simpler proof of isometry.

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