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TRACE FORMS AND IDEALS ON COMMUTATIVE ALGEBRAS SATISFYING AN IDENTITY OF DEGREE FOUR

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ABSTRACT. This paper deals with the variety of commutative algebras satisfying the identity $((xy)z)t - ((xy)t)z + ((yt)x)z - ((yt)z)x + ((yz)t)x - ((yz)x)t = 0$. These algebras appeared in the classification of the degree four identities in Carini, Hentzel and Piacentini-Cattaneo [**2**]. We prove the existence of a trace form. Moreover, if we assume the existence of a non degenerate trace form, then A satisfies the identity $((yx)x)x = y((xx)x)$, a generalization of right-alternativity. Finally we prove that $\mathrm{Ass}\,[A]$ and $N(A)$ are ideals in these algebras.

1. Introduction. In [**2**], Carini, Hentzel and Piacentini-Cattaneo extended Osborn's results [**7**] by classifying all degree four identities not implied by commutativity.

This classification is stated in the following important result [**2**].

Theorem 1. *Let A be a commutative algebra over F, $\mathrm{char}\,(F) \neq 2, 3$, and let A satisfy an identity of degree four not implied by the commutative law. Then A satisfies an identity from at least one of the families of identities*:

$$\alpha(x^2x^2) + \beta x^4 = 0 \tag{1}$$

(2)
$$2\beta \left\{(xy)^2 - x^2y^2\right\} + \gamma\left\{((xy)x)y + ((xy)y)x - (y^2x)x - (x^2y)y\right\} = 0$$

$$\beta\left\{(x^2y)x - ((xy)x)x\right\} + \gamma\left\{x^3y - ((xy)x)x\right\} = 0 \tag{3}$$

(4)
$$((xy)z)t - ((xy)t)z + ((yt)x)z - ((yt)z)x + ((yz)t)x - ((yz)x)t = 0.$$

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If A is an algebra over a field F and B is a bilinear symmetric map over A, then B is a trace form on A if and only if:

$$B(xy, z) = B(x, yz) \quad \text{for all } x, y, z \in A.$$

The study of existence of trace forms is motivated by the following result [**9**, page 24].

Theorem 2. *Let A be a finite dimensional algebra over a field F satisfying*:

(i) *There is a non degenerate trace form $B(x, y)$ defined on A.*

(ii) *$J^2 \neq 0$ for every ideal $J \neq 0$ of A.*

Then A is unique expressible as a direct sum $A = S_1 \oplus \cdots \oplus S_t$ of simple ideals S_i.

Algebras satisfying (1) were studied by Osborn in [**7**]. Results about trace forms in algebras satisfying identity (3) can be found in Arenas and Labra [**1**]. They prove that there always exists a trace form in terms of the trace of some right multiplication operators, specifically, for all $\beta, \gamma \in F$, the form $T : A \times A \to F$ defined by $T(x, y) = (3\beta + \gamma)\mathrm{tr}\,(R_{xy}) + (\beta + 3\gamma)\mathrm{tr}\,(R_x R_y)$ for all $x, y \in A$, is a trace form on A. Moreover, they prove that, if generalized almost-Jordan algebras with $\beta, \gamma \in F$ admit a non degenerate trace form, then they are Jordan algebras. This result needs the condition $\beta \neq 0$.

Results about trace forms in algebras satisfying identity (2) can be found in Labra and Rojas-Bruna [**6**]. They prove that, for all $\beta, \gamma \in F$, the form τ defined by $\tau(x, y) = \gamma\mathrm{tr}\,(R_{xy}) + (\gamma + 4\beta)\mathrm{tr}\,(R_x R_y)$ for all $x, y \in A$, is a trace form on A. They prove that, if A has a non degenerate trace form, then A satisfies the identity $((yx)x)x = y((xx)x)$ a generalization of right-alternativity.

In this paper we deal with trace form and ideals on algebras satisfying identity (4). In the case of identity (1), we give examples where we can find a non zero trace form given by a linear combination of the operators R_{xy}, R_x, R_y, but to find a general form for a trace form for algebras satisfying identity (1) is still an open problem.

Example 1. Examples of algebras satisfying the identity (4) can be constructed using the Albert program by Jacobs, Muddanna and Offutt [**4**].

1. Let A be the commutative algebra in two generators with basis $\{a, b, ab, a^2, (ab)a, a^2b\}$. This algebra is constructed using Albert and satisfies identity (4); clearly, this is not associative and also it is not alternative.

2. In the same way, we can construct a commutative algebra A satisfying (4) with basis $\{a, b, a^2, (ab)a, a^2b, a^3, a^2(ba), ((ba)a)a, (a^2b)a, a^3b\}$; then, A is not a Jordan algebra, that is, $x^2 \cdot yx - x^2y \cdot x = 0$ is not an identity in this algebra.

2. Trace forms. Let A be an algebra over a field F, and let R_x denote the right multiplication operator defined by $aR_x = ax = xa$ for any $a \in A$. If $\mathrm{Tr}\,(L)$ denotes the trace of a linear operator L, then we have the following proposition:

Proposition 3. *Let A be a commutative algebra over a field F,* $\mathrm{char}\,(F) \neq 2, 3$, *satisfying identity* (1). *Then*

$$\tau(x, y) = Tr(B(x, y)) \quad \textit{where } B(x, y) = R_{xy} - R_x R_y,$$

is a trace form on A.

Proof. It is clear from the definition of τ that τ is a symmetric bilinear map on A. Then we only need to prove that:

$$\tau(xz, y) = \tau(x, yz) \quad \text{for all } x, y, z \in A.$$

Using the right multiplication operators, identity (4) reduces to

$$R_{(xy)z} - R_{xy}R_z + R_y R_x R_z - R_y R_z R_x + R_{yz} R_x - R_{(yz)x} = 0. \tag{5}$$

Interchanging y by z in (5), we have

$$R_{(xz)y} - R_{xz}R_y + R_z R_x R_y - R_z R_y R_x + R_{yz} R_x - R_{(yz)x} = 0, \tag{6}$$

and interchanging x by z in (6), we obtain

$$R_{(xz)y} - R_{xz}R_y + R_x R_z R_y - R_x R_y R_z + R_{yx} R_z - R_{(yx)z} = 0. \tag{7}$$

Subtracting (7) from (5) gives

$$\begin{aligned} &R_{(xy)z} - R_{xy}R_z + R_yR_xR_z - R_yR_zR_x + R_{yz}R_x - R_{(yz)x} \\ &- \left\{R_{(xz)y} - R_{xz}R_y + R_xR_zR_y - R_xR_yR_z + R_{yx}R_z - R_{(yx)z}\right\} = 0. \end{aligned}$$

Reordering terms, and since $\mathrm{char}\,(F) \neq 2, 3$, we have

$$\begin{aligned} 2\left\{R_{(xy)z} - R_{xy}R_z\right\} = \left\{R_{(yz)x} - R_{yz}R_x\right\} + \left\{R_{(xz)y} - R_{xz}R_y\right\} \\ - \{R_yR_xR_z - R_xR_zR_y + R_xR_yR_z - R_yR_zR_x\}, \end{aligned}$$

or

$$\begin{aligned} 2\left\{R_{(xy)z} - R_{xy}R_z\right\} = \left\{R_{(yz)x} - R_{yz}R_x\right\} + \left\{R_{(xz)y} - R_{xz}R_y\right\} \\ - [R_y, R_xR_z] - [R_x, R_yR_z]\,. \end{aligned} \tag{8}$$

Now, if we apply trace in both sides of (8) and, since the trace of a commutator is always zero, we get

$$2\tau(xy, z) = \tau(yz, x) + \tau(xz, y). \tag{9}$$

Interchanging y by z in (9),

$$2\tau(xz, y) = \tau(yz, x) + \tau(xy, z), \tag{10}$$

subtracting (9) from (10), we obtain

$$3\tau(xy, z) = 3\tau(xz, y), \tag{11}$$

and, since $\mathrm{char}\,(F) \neq 3$, we get the equation

$$\tau(xy, z) = \tau(xz, y).$$

Finally, interchanging x by z in the last equation and, since τ is symmetric, we prove that

$$\tau(xy, z) = \tau(yz, x) = \tau(x, yz).$$

And then, $\tau(xz, y) = \tau(x, yz)$ for all $x, y, z \in A$ and τ is a trace form on A. □

For every algebra A with trace form τ, if I is an ideal of A, then the set $I^{\perp} = \{a \in A \mid \tau(a, y) = 0 \text{ for all } y \in I\}$ is an ideal of A. Then, if A is a simple commutative algebra and the trace form $\tau \neq 0$, then it is a non degenerated trace form.

The next example show that there exist non zero and not degenerated trace forms on algebras satisfying identity (4).

Example 2. Let be F a field of $\operatorname{char}(F) \neq 2$. Let $A = F \times F$ with scalar multiplication defined as usual, and multiplication of elements in A, defined by $(a, b)(c, d) = (ac + \alpha bd, -ad - bc)$. We can show by straightforward calculation that A satisfies identity (4).

This algebra was proved to be simple in Carini, Hentzel and Piacentini-Cattaneo [**2**]. So, if τ is the trace form defined in Proposition 3, then the ideal $\{x \in A : \tau(x, a) = 0 \text{ for all } a \in A\}$ must be equal to zero or A. But, since $\tau((1, 0), (1, 0)) = -2$, the trace form τ is not null, and hence this ideal must be zero and τ is a nondegenerated trace form on A.

We say that an algebra A is *alternative* if it satisfies the identities $x^2y = x(xy)$ and $xy^2 = (xy)y$ for every $x, y \in A$, respectively known as *the left and the right alternatives laws* (see Schafer [**9**] and Zhevlakov et al. [**10**]). If we assume the existence of a non degenerate trace form on A, we have the following result.

Proposition 4. *Let A be a commutative algebra satisfying identity* (4). *Assume that A has a nondegenerate trace form ρ. Then A satisfies identity* (2) *for $\beta = -1/2$, $\gamma = 1$.*

Proof. Setting $x = y$ in (4), we get the identity

$$(x^2z)t - (x^2t)z + ((xt)x)z - ((xt)z)x + ((xz)t)x - ((xz)x)t = 0.$$

Then, for any $w \in A$.

$$\rho((x^2z)t - (x^2t)z + ((xt)x)z - ((xt)z)x + ((xz)t)x - ((xz)x)t, w) = 0,$$

and since ρ is a trace form,

$$\rho(z, (wt)x^2 - (x^2t)w + ((xt)x)w - (xw)(xt) + ((xw)t)x - ((wt)x)x) = 0. \quad (12)$$

Moreover, since x, y, t, w, z are arbitrary and ρ is nondegenerate, equation (12) implies

$$(13)\quad (wt)x^2 - (x^2t)w + ((xt)x)w - (xw)(xt) + ((xw)t)x - ((wt)x)x = 0;$$

hence, reordering and setting $w = t$ in (13),

$$-\left\{(xt)(xt) - t^2x^2\right\} + \left\{((xt)t)x + ((xt)x)t - (x^2t)t - (t^2x)x\right\} = 0,$$

and then A satisfies identity (2), with $\beta = -1/2$, $\gamma = 1$. □

Algebras satisfying identity (2) have been studied by Labra and Rojas-Bruna in [**6**], and they prove that, if A has a non degenerate trace form and satisfies identity (2) for $2\beta + \gamma \neq 0$, then A satisfies the identity $((yx)x)x = y((xx)x)$, which is a generalization of right-alternativity. But, since $2\beta + \gamma = 0$ for $\beta = -1/2$, $\gamma = 1$, this result cannot be applied in this case.

3. Ideals. Let A be a commutative algebra, and let (a, b, c) denote the associator of a, b, c, defined as $(a, b, c) = (ab)c - a(bc)$ where $a, b, c \in A$.

Recall the identity

$$x(y, z, w) + (x, y, z)w = (xy, z, w) - (x, yz, w) + (x, y, zw),$$

which is called the *Teichmüller identity* and is valid on any nonassociative algebra [**9**].

If A is a simple commutative algebra and ρ is a nontrivial trace form, then ρ must be a non degenerate trace form. This fact, along with the natural relation between simple algebras and ideals and the results given by Carini, Hentzel and Piacentini-Cattaneo in [**2**], and Osborn in [**7**], about simple algebras satisfying (3) and algebras satisfying (2) where $\beta = 1/2$ and $\gamma = -2$, motivated us to study the existence of certain ideals in algebras satisfying identity (4).

If A is a commutative algebra, we define the set $\mathrm{Ass}\,[A]$ to be the linear subspace of A spanned by all the elements of the form (a, b, c) with $a, b, c \in A$.

To improve the calculations and the readability of this paper, we will use the notation $x \equiv y$ if and only if $x - y \in \mathrm{Ass}\,[A]$.

Proposition 5. *Let A be a commutative algebra over a field F,* $\mathrm{char}\,(F) \neq 2$. *If A has a non degenerated trace form and A satisfies identity* (4)*, then* $\mathrm{Ass}\,[A]$ *is an ideal of A.*

Proof. We will prove that $(y, x, w)z \equiv 0$ for any $y, w, x, z \in A$.

Using Proposition 4, we have that A satisfies identity (2) for $\beta = -1/2$, $\gamma = 1$. That is, A satisfies the identity

(14)
$$-\left\{(xy)^2 - x^2y^2\right\} + \left\{((xy)x)y + ((xy)y)x - (y^2x)x - (x^2y)y\right\} = 0.$$

The complete linearization of (14) is given by

(15)
$$-\{2\,(xy)\,(wz) + 2\,(wy)\,(xz) - 4(xw)(yz)\}$$
$$+\left\{\begin{array}{c} ((wz)x)y + ((xz)w)y + ((wy)x)z + ((xy)w)z + ((wz)y)x \\ +((xz)y)w + ((wy)z)x + ((xy)z)w \\ -2((yz)w)x - 2((yz)x)w - 2((xw)z)y - 2((xw)y)z \end{array}\right\} = 0.$$

Reordering terms, and writing this equation in terms of associators, we obtain

$$4(xw)(yz) - 2(xy)(wz) - 2(yw)(xz) + \left\{\begin{array}{c} (z, w, x)y + (z, x, w)y + (y, w, x)z + (y, x, w)z \\ +(w, z, y)x + (w, y, z)x + (x, z, y)w + (x, y, z)w \end{array}\right\} = 0.$$

Now, by the Teichmüller identity and, since $(a, b, c) = -(c, b, a)$, we get the following equations

$$\begin{aligned} y(z, w, x) &\equiv -(y, z, w)x \equiv (w, z, y)x, \\ y(z, x, w) &\equiv -(y, z, x)w \equiv (x, z, y)w, \\ z(y, w, x) &\equiv -(z, y, w)x \equiv (w, y, z)x, \\ w(x, y, z) &\equiv -(w, x, y)z \equiv (y, x, w)z. \end{aligned}$$

Then, if we replace the above equations in (15) and, since $\operatorname{char}(F) \neq 2$,

$$\begin{aligned}(16)\quad & 2(xw)(yz) - (xy)(wz) - (yw)(xz) + \\ & \{(z,w,x)y + (z,x,w)y + (y,w,x)z + (y,x,w)z\} = 0.\end{aligned}$$

or

$$\begin{aligned}(17)\quad & (y,w,x)z + (y,x,w)z \\ & + \left\{ \begin{array}{c} 2(xw)(yz) - (xy)(wz) - (yw)(xz) \\ +((zw)x)y - 2(z(wx))y + ((zx)w)y \end{array} \right\} = 0.\end{aligned}$$

Now, the right summand of (17) can be rewritten as

$$\begin{aligned}& 2(xw)(yz) - (xy)(wz) - (yw)(xz) + ((zw)x)y - 2(z(wx))y + ((zx)w)y \\ & = 2(y,z,xw) + (zw,x,y) + (xz,w,y) \equiv 0.\end{aligned}$$

Thus,

$$(y,w,x)z + (y,x,w)z \equiv 0,$$

and interchanging y by w, we get

$$(w,y,x)z + (w,x,y)z \equiv 0.$$

Finally, subtracting the last two equations, we obtain

$$3(y,x,w)z \equiv 0,$$

and, since $\operatorname{char}(F) \neq 3$,

$$(y,x,w)z \equiv 0,$$

which implies that $\operatorname{Ass}[A]A \subseteq \operatorname{Ass}[A]$ and $\operatorname{Ass}[A]$ is an ideal of A. □

Corollary 6. *Let $A = B_0 + B_1$ a simple commutative algebra over a field F of characteristic zero, satisfying identity* (4) *and having an idempotent e. If A has a non trivial trace form and $\operatorname{Tr}(R_{xz}R_y) = \operatorname{Tr}(R_{yz}R_x)$ for any $x, y, z \in A$, then A is associative.*

Proof. Proposition 4 implies that A must satisfy identity (2) with $\gamma = 1$ and $\beta = -1/2$. Moreover, Labra and Rojas-Bruna in [**6**] showed that an algebra satisfying identity (2) satisfies the following relation

$$\mathrm{Tr}\,(R_{(y,x,z)}) = \mathrm{Tr}\,(R_{yz}R_x) - \mathrm{Tr}\,(R_{xz}R_y).$$

This relation, and our initial assumption that $\mathrm{Tr}\,(R_{xz}R_y) = \mathrm{Tr}\,(R_{yz}R_x)$ implies

$$\mathrm{Tr}\,(R_{(y,x,z)}) = 0.$$

Since A is simple and $\mathrm{Ass}\,[A]$ is an ideal, we must have $\mathrm{Ass}\,[A] = 0$ or $\mathrm{Ass}\,[A] = A$.

If $\mathrm{Ass}\,[A] = A$, we can write up e as a linear combination of associators $e = \sum \alpha_i(x_i, y_i, z_i)$ and $\mathrm{Tr}\,(R_e) = 0$. On the other hand, since $A = B_0 + B_1$ and $\mathrm{char}\,(F) = 0$, we can build up a basis such that $\mathrm{Tr}\,(R_e) \neq 0$, which is a contradiction. Therefore, $\mathrm{Ass}\,[A] = 0$ and A is an associative algebra. ◻

Additionally, we can prove the existence of an ideal which is related to the associativity of the algebra.

We define the set $N(A)$ as

$$N(A) = \{x \in A : (x, A, A) = 0\}.$$

Clearly, if A satisfies identity (4), then $N(A)$ is a subalgebra of A. Moreover, we have the following result:

Proposition 7. *If A is a commutative algebra satisfying identity* (4)*, then $N(A)$ is an ideal of A.*

Proof. We will prove that $xy \in N(A)$ for any $y \in A, x \in N(A)$.

Recall identity (4):

$$((xy)z)t - ((xy)t)z + ((yt)x)z - ((yt)z)x + ((yz)t)x - ((yz)x)t = 0.$$

If $x \in N(A)$, then the Teichmüller identity reduces to:

$$x(y,z,t) = (xy,z,t), \quad \text{for any } y,z,t \in A. \tag{18}$$

On the other hand, we can rewrite identity (4) as

$$(x, y, z)t - (x, y, t)z + (z, y, t)x = 0, \tag{19}$$

so if $x \in N(A)$, then (19) becomes

$$(z, y, t)x = 0, \quad \text{for any } z, y, t \in A.$$

Then, by (18), we have

$$(xy, z, t) = 0, \quad \text{for any } y, z, t \in A,$$

and therefore $xy \in N(A)$ for any $y \in A$, and $N(A)$ is an ideal of A. □

Remark 1. Clearly, if $N(A) = A$, then A must be associative.

Remark 2. Recall the trace form $\tau(x, y) = \mathrm{Tr}\,(B(x, y))$ where $B(x, y) = R_{xy} - R_x R_y$. If $x \in N(A)$, then $(x, z, y) = 0$ and $(x, y, z) = 0$ imply that $\tau(x, y) = 0$ for any $y \in A$. Thus, if τ is non degenerate, then $N(A)$ must be equal to zero.

Remark 3. Related to identity (1), the following example shows that, in some algebras satisfying identity (1), we can find a non zero trace form given by a linear combination of the operators R_{xy}, R_x, R_y, but to find a general form for a trace form for algebras satisfying identity (1) is still an open problem.

Example 3. Let $A =$ polynomials in x of degree ≤ 1 over a field F.

We define a product $*$ in A by

$$g * h = \frac{d}{dx}(gh), \quad g, h \in A.$$

Straightforward calculation shows that A satisfies identity (1), and also using AXIOM [**5**] software we can check that the algebra A is neither associative, alternative, nor a Jordan algebra.

Moreover, if $g = px + r$, $h = sx + t \in A$.

$$R_{g*h} = \begin{bmatrix} 2ps & 4ps \\ 0 & pt + sr \end{bmatrix}.$$

Then, we can define a trace form B on A by $B(g,h) = \mathrm{Tr}\,(R_{g*h})$, and since

$$B(x+1,x+1) = \mathrm{Tr}\,(R_{x+1*x+1}) = \mathrm{Tr}\begin{bmatrix} 2 & 4 \\ 0 & 2 \end{bmatrix} = 4 \neq 0,$$

the bilinear form B is non trivial. Moreover, since $B(g,h) = 0$ for any $g = px + r \in A$ implies that $h = sx + t = 0$, then B is not degenerated. In fact, $B(g,h) = 0$ for any $g = px + r \in A$ implies that $\mathrm{Tr}\,(R_{g*h}) = 0$ for any $g = px + r \in A$. So, for any $g = px + r \in A$, $2ps + pt + sr = 0$. Taking $p = 0$, $r = 1$, we obtain that $s = 0$ and $p = 1$, $r = -2$ imply that $t = 0$. Therefore, $h = 0$ and this trace is not degenerated.

Example 4. Another example of an algebra satisfying (1) can be constructed in a similar way, taking A as the vector space of all polynomials of degree less than or equal to 2 over a field F, and defining the product $*$ by

$$u * v = k(u,v) = \left(\frac{d}{dx}u\right)\left(\frac{d}{dx}v\right).$$

This algebra is neither associative, alternative nor a Jordan algebra.

Remark 4. Algebras shown in the above examples belong to a variety of algebras, called *Novikov-Jordan algebras* [**3**].

REFERENCES

1. M. Arenas and A. Labra, *On nilpotency of generalized almost-Jordan algebras*, Algebra Colloq. **15** (2008), 69–82.

2. L. Carini, I.R. Hentzel and Piacentini-Cattaneo, *Degree four identities not implied by commutativity*, Comm. Alg. **16** (1988), 339–356.

3. A. Dzhumadil'daev, *Novikov-Jordan algebras*, Comm. Alg. **30** (2002), 5207-5240.

4. P.D. Jacobs, S.V. Muddanna and A.J. Offutt, *A computer algebra system for nonassociative identities*, Hadronic mechanics and nonpotential interactions, Part 1 (Cedar Falls, IA, 1990), Nova Scientific Publishers, Commack, NY, 1992.

5. R.D. Jenks and R. Sutor, *Axiom: The scientific computation system*, Springer-Verlag, Inc., New York, 1992.

6. A. Labra and C. Rojas-Bruna, *On trace forms on a class of commutative algebras satisfying an identity of degree four*, Alg. Colloq. **17** (2010), 875–880.

7. J.M. Osborn, *Commutative non-associative algebras and identities of degree four*, Canad. J. Math. **20** (1968), 769–794.

8. ———, *Varieties of algebras*, Adv. Math. **8** (1972), 163–329.

9. R.D. Schafer, *An introduction to nonassociative algebras*, Academic Press Inc., New York, 1966.

10. K.A. Zhevlakov, A.M. Slin'ko, I.P. Shestakov and A.I. Shirshov, *Rings that are nearly associative*, Pure Appl. Math. **104** (1982), 123–135.

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