ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 43, Number 4, 2013

MULTIPLICITIES FOR ARBITRARY MODULES AND REDUCTION

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ABSTRACT. Let $(R, \mathfrak{m})$ be a d-dimensional Noetherian local ring and E a finitely generated R-submodule of the free module R^p. In this work, we introduce a multiplicity sequence $c_k(E)$, $k = 0, \dots, d+p-1$, for E that generalizes the Buchsbaum-Rim multiplicity defined when E has finite colength in R^p as well as the Achilles-Manaresi multiplicity sequence that applies when $E \subseteq R$ is an ideal. Our main results are that the new multiplicity sequence is an invariant of E up to reduction; we show that this multiplicity sequence behaves well with respect to sufficiently general hyperplane sections, and we also give a criterion for reduction of ideals involving the c_0-multiplicity in all localizations in prime ideals.

1. Introduction. Let $(R, \mathfrak{m})$ be a local Noetherian ring, N a finitely generated R-module of dimension d and $I \subseteq J$ two ideals in R. Recall that I is a reduction of (J, N) if $IJ^nN = J^{n+1}N$ for sufficiently large n. If $I \subseteq J$ are $\mathfrak{m}$-primary and I is a reduction of (J, N), then it is well known and easy to prove that the Hilbert-Samuel multiplicities $e(J, N)$ and $e(I, N)$ are equal. Rees proved his famous result, which nowadays has his name, that the converse also holds under an additional assumption:

Theorem 1.1 (Rees's theorem, [**15**]). *Let $(R, \mathfrak{m})$ be a quasi-unmixed local ring, N a finitely generated d-dimensional R-module and $I \subseteq J$ $\mathfrak{m}$-primary ideals of R. Then, the following conditions are equivalent*:

(i) *I is a reduction of (J, N);*

(ii) $e(J, N) = e(I, N)$.

Now assume that $I \subseteq J$ are arbitrary ideals with the same radicals. If I is a reduction of J, then we always have $e(J_\mathfrak{p}, R_\mathfrak{p}) = e(I_\mathfrak{p}, R_\mathfrak{p})$ for

2010 AMS *Mathematics subject classification.* Primary 13H15, Secondary 13B21, 32S15.
Keywords and phrases. Integral closure, multiplicity sequence.
Received by the editors on November 30, 2010.
DOI:10.1216/RMJ-2013-43-4-1077

all minimal primes of J. However, the converse is not true, in general. Under additional assumptions, Böger [**3**] was able to prove a converse as follows: *let* $I \subseteq J \subseteq \sqrt{I}$ *be ideals in a quasi-unmixed local ring* R *such that* $s(I) = \mathrm{ht}\,(I)$, *where* $s(I)$ *denotes the analytic spread of* I. *Then* I *is a reduction of* J *if and only if* $e(J_{\mathfrak{p}}, R_{\mathfrak{p}}) = e(I_{\mathfrak{p}}, R_{\mathfrak{p}})$ *for all minimal primes of* I.

Using the j-multiplicity defined by Achilles and Manaresi [**1**] (a generalization of the classical Hilbert-Samuel multiplicity), Flenner and Manaresi [**8**] gave a numerical characterization of reduction ideals which generalize Böger's theorem to arbitrary ideals: *let* $I \subseteq J$ *be ideals in a quasi-unmixed local ring* R, *and let* N *be a finitely generated* d-*dimensional* R-*module. Then* I *is a reduction of* (J, N) *if and only if* $j(J_{\mathfrak{p}}, N_{\mathfrak{p}}) = j(I_{\mathfrak{p}}, N_{\mathfrak{p}})$ *for all* $\mathfrak{p} \in \mathrm{Spec}\,(R)$.

There is another generalization of the classical Hilbert-Samuel multiplicity for arbitrary ideals due to Achilles and Manaresi [**2**]. They introduced, for each ideal I of a d-dimensional local ring $(R, \mathfrak{m})$ and N a finitely generated d-dimensional R-module, a sequence of multiplicities $c_0(I, N), \ldots, c_d(I, N)$ which generalize the Hilbert-Samuel multiplicity in the sense that, for $\mathfrak{m}$-primary ideals I, $c_0(I, N)$ is the Hilbert-Samuel multiplicity of I in N, and the remaining $c_k(I, N)$, $k = 1, \ldots, d$, are zero. In fact, their definition was given in the case that $N = R$ but their construction can be readily extended to this context.

Using the above Achilles-Manaresi multiplicity sequence, Ciupercă proved the following theorem.

Theorem 1.2. *Let* $(R, \mathfrak{m})$ *be a local ring and* N *a finitely generated* d-*dimensional* R-*module. Let* $I \subseteq J$ *be proper arbitrary ideals of* R. *If* I *is a reduction of* (J, N), *then* $c_k(I, N) = c_k(J, N)$ *for all* $k = 0, \ldots, d$.

On the other hand, the Buchsbaum-Rim multiplicity $e_{BR}(E)$ is a generalization of the Samuel multiplicity and is defined for submodules of free modules $E \subset R^p$ such that R^p/E has finite length. These were first described by Buchsbaum and Rim in [**4**]. The Buchsbaum-Rim multiplicity has been generalized, in the finite colength case, by Kirby [**12**], Kirby and Rees [**13**], Katz [**11**], Kleiman and Thorup [**14**] and Simis, Ulrich and Vasconcelos [**16**]. For an extensive history of Buchsbaum-Rim multiplicity, we refer to [**14**]. Using the Buchsbaum-

Rim multiplicity, Katz [**11**], Kleiman and Thorup [**14**] and Simis, Ulrich and Vasconcelos [**16**] proved the following generalization of Rees's theorem for modules:

Theorem 1.3. *Let $(R,\mathfrak{m})$ be a quasi-unmixed local ring, $E \subseteq F$ a finitely generated R-submodule of the free module R^p such that R^p/E has finite length. Then, the following conditions are equivalent*:

(i) *E is a reduction of F*;

(ii) $e_{BR}(E) = e_{BR}(F)$.

There have been some generalizations of the Buchsbaum-Rim multiplicity for arbitrary submodules E of the free module R^p which we now describe. Gaffney in [**9**] introduced a sequence of multiplicities $e_i(E)$, $0 \le i \le d = \dim R$ in the analytic context. This sequence satisfies a Rees type theorem: Suppose that $E \subset F \subset R^p$ are $R := \mathcal{O}_{X,x}$-modules where X^d is a complex analytic space which as a reduced space is equidimensional, and which is generically reduced. Suppose that $e_i(E,x) = e_i(F,x)$, $0 \le i \le d$. Then E is a reduction of F. Also, if E is of finite colength in R^p, then $e_d(E)$ is the standard Buchsbaum-Rim multiplicity of E, and the other e_i's are zero. Unfortunately, for ideals of non-finite colength, Gaffney's multiplicity sequence does not coincide with the Achilles-Manaresi multiplicity sequence and also the codimension condition of E in R^p is built into the definition of the multiplicity which uses a codimension filtration ascending from the integral closure of the module.

On the other hand, the authors in [**5**] extended the notion of the Buchsbaum-Rim multiplicity of a submodule of a free module to the case where the submodule no longer has finite colength. For a submodule E of R^p, they introduced a sequence $e^k_{BR}(E)$, $k = 0, \dots, d+p-1$, which in the ideal case coincides with the multiplicity sequence $c_0(I,R), \dots, c_d(I,R)$ defined for an arbitrary ideal I of R by Achilles and Manaresi [**2**]. They also proved that, if $E = I_1 \oplus \cdots \oplus I_p \subset R^p$ has finite colength, then $e^0_{BR}(E) = p\,!(e_{BR}(E))$ and $e^k_{BR}(E) = 0$ for $k = 1, \dots, d-1$. Nevertheless, no relation with reduction of modules and their multiplicity sequence was shown in their work.

There is also a generalization of the Flenner-Manaresi theorem for arbitrary submodules of a free module due to Ulrich and Validashti (see [**17**]). They introduced a multiplicity $j(E)$ for a submodule of the free module R^p that generalizes the Buchsbaum-Rim multiplicity defined when E has finite colength in R^p as well as the j-multiplicity of Achilles-Manaresi that applies when $E \subseteq R$ is an ideal. Their result is as follows:

Theorem 1.4. *Let $(R, \mathfrak{m})$ be a universally catenary ring, $E \subseteq F$ a finitely generated R-submodule of a free module R^p and N a finitely generated locally equidimensional Noetherian R-module. Assume that $E_{\mathfrak{p}} = F_{\mathfrak{p}}$ for every minimal prime $\mathfrak{p}$ of R. Then, the following are equivalent*:

(i) *E is a reduction of (F, N);*

(ii) *$j(E_{\mathfrak{q}}, N_{\mathfrak{q}}) = j(F_{\mathfrak{q}}, N_{\mathfrak{q}})$ for every $\mathfrak{q} \in \operatorname{Spec}(R)$.*

In this work we introduce a multiplicity sequence $c_k(E, N)$ with $k = 0, \ldots, d+p-1$ for the pair (E, N) that generalize the Buchsbaum-Rim multiplicity defined when E has finite colength in R^p as well as the Achilles-Manaresi multiplicity sequence that applies when $E \subseteq R$ is an ideal. One of our main results is that the new multiplicity sequence is an invariant of E with respect to N up to reduction:

Theorem 1.5. *Let $(R, \mathfrak{m})$ be a Noetherian local ring, let $E \subseteq F \subseteq R^p$ be R-modules and write $I := \mathcal{R}_1(E)A$ for the corresponding ideal of $A := \operatorname{Sym}(R^p)$. Let N be a d-dimensional finitely generated R-module, and set $M := A \otimes_R N$. If E is a reduction of (F, N), then $c_k(E, N) = c_k(F, N)$ for all $k = 0, \ldots, d+p-1$.*

We also show that this multiplicity sequence behaves well with respect to sufficiently general hyperplane sections:

Theorem 1.6. *Let $(R, \mathfrak{m})$ be a Noetherian local ring, $E \subseteq R^p$ an R-module, N a d-dimensional finitely generated R-module, and write $I := \mathcal{R}_1(E)A$ for the corresponding ideal of $A := \operatorname{Sym}(R^p)$. Let y be a superficial element for (E, N) and a non zero-divisor on N. Then*

$$c_k(E, N) = c_k(\overline{E}, \overline{N}), \quad k = 0, \ldots, d+p-2,$$

where $\overline{N} := N/yN$ and $\overline{E} := E \otimes_R \overline{R}$ with $\overline{R} := R/yR$.

We also give a criterion for reduction of ideals involving the c_0-multiplicity in all localizations in prime ideals.

Theorem 1.7. *Let* $(R, \mathfrak{m})$ *be a universally catenary local Noetherian ring,* $E \subseteq F$ *submodules of the free module* $L := R^p$ *and* N *a* d-*dimensional finitely generated* R-*module. Assume that the* R-*module* N *is locally equidimensional and that* $E_{\mathfrak{p}} = L_{\mathfrak{p}}$ *for every minimal prime* $\mathfrak{p}$ *in* $\mathrm{Supp}_R(N)$. *Then the following are equivalent*:

(i) E *is a reduction of* (F, N).

(ii) $c_0(E_{\mathfrak{p}}, N_{\mathfrak{p}}) = c_0(F_{\mathfrak{p}}, N_{\mathfrak{p}})$ *for all* $\mathfrak{p} \in \mathrm{Spec}\,(R)$.

(iii) $c_0(E_{\mathfrak{p}}, N_{\mathfrak{p}}) \leq c_0(F_{\mathfrak{p}}, N_{\mathfrak{p}})$ *for all* $\mathfrak{p} \in \mathrm{Spec}\,(R)$.

The paper is organized as follows. In Section 2, we recall the basic results of Hilbert functions of bigraded algebras, and we define the c^D-multiplicity sequence associated to a graded module. The important result of this section is the additivity formula for this multiplicity sequence. In Section 3, we define the first multiplicity sequence associated to ideals generated by linear forms, which we call the c^*-multiplicity sequence. The important results of this section are the additivity formula for this multiplicity sequence (Theorem 3.2) and Theorem 3.3 which shows that the c^*-multiplicity sequence is an invariant of M with respect to I up to reduction. In Section 4, we define two multiplicity sequences associated to ideals generated by linear forms, which we call $c^\sharp$-multiplicity and b-multiplicity sequences. These multiplicity sequences are related to the c^*-multiplicity sequence in Lemma 4.3. The b-multiplicity sequence has been introduced only with the purpose of proving that the $c^\sharp$-multiplicity sequence is an invariant of I up to reduction (see Theorem 4.6). In Section 5, we show that sufficiently general hyperplane sections behave well with respect to the c^*-multiplicity sequence (Proposition 5.4). In Section 6, we give a criterion for reduction of ideals involving the $c_0^\sharp$-multiplicity in all localizations in prime ideals (Theorem 6.3). In Section 7, we introduce a multiplicity sequence $c_k(E)$, $k = 0, \ldots, d + p - 1$, for an arbitrary submodule E of the free module R^p which generalizes the Buchsbaum-Rim multiplicity defined when E has finite colength in R^p as well as the Achilles-Manaresi multiplicity sequence that applies when $E \subseteq R$ is an ideal. The main result of this section, other than the definition of multiplicities for arbitrary modules, is Theorem 7.3 (an immediate consequence of Theorem 4.6) which says that the new multiplicity sequence is an invariant of E with respect to N up to reduction. We show that this multiplicity sequence behaves well with respect to sufficiently general hyperplane sections (Theorem 7.5) and we also give a criterion for reduction of ideals involving the c_0-multiplicity in all localizations in prime ideals (Theorem 7.6). Our approach is partly inspired by [**2, 17**].

2. Multiplicity sequence. In this section we recall some well-known facts on Hilbert functions and Hilbert polynomials of bigraded modules which will be essential for defining the multiplicity sequences associated to a pair (I, M).

Let $R = \oplus_{i,j=0}^{\infty} R_{i,j}$ be a bigraded ring, and let $T = \oplus_{i,j=0}^{\infty} T_{i,j}$ be a bigraded R-module. Assume that $R_{0,0}$ is an Artinian ring and that R is finitely generated as an $R_{0,0}$-algebra by elements of $R_{1,0}$ and $R_{0,1}$ (i.e., R is a standard bigraded algebra). The *Hilbert function* of T is defined to be

$$h_T(i,j) = \ell_{R_{0,0}}(T_{i,j}).$$

For i, j sufficiently large, the function $h_T(i,j)$ becomes a polynomial $P_T(i,j)$. If D denotes the dimension of the module T, we can write this polynomial in the form

$$P_T(i,j) = \sum_{\substack{k,l \geq 0 \\ k+l \leq D-2}} a_{k,l}(T) \binom{i+k}{k} \binom{j+l}{l}$$

with $a_{k,l}(T) \in \mathbf{Z}$ and $a_{k,l}(T) \geq 0$ if $k + l = D - 2$ [**18**, Theorem 7, page 757 and Theorem 11, page 759].

We also consider the sum transform of h_T with respect to the first variable defined by

$$h_T^{(1,0)}(i,j) = \sum_{u=0}^{i} h_T(u,j).$$

From this description, it is clear that, for i, j sufficiently large, $h_T^{(1,0)}$ becomes a polynomial with rational coefficients of degree at most $D-1$.

As usual, we can write this polynomial in terms of binomial coefficients

$$P_T^{(1,0)}(i,j) = \sum_{\substack{k,l\geq 0\\ k+l\leq D-1}} a_{k,l}^{(1,0)}(T)\binom{i+k}{k}\binom{j+l}{l}$$

with $a_{k,l}^{(1,0)}(T)$ integers and $a_{k,D-k-1}^{(1,0)}(T)\geq 0$.

Since

$$h_T(i,j) = h_T^{(1,0)}(i,j) - h_T^{(1,0)}(i-1,j),$$

we get $a_{k+1,l}^{(1,0)}(T) = a_{k,l}(T)$ for $k,l\geq 0$, $k+l\leq D-2$.

Definition 2.1. For the coefficients of the terms of highest degree in $P_T^{(1,0)}$, we introduce the symbols

$$c_k(T) := a_{k,D-k-1}^{(1,0)}(T), \quad k = 0,\dots,D-1,$$

which are called the *multiplicity sequence* of T.

We define next the c^D-multiplicity sequence associated to a module. Let $(R,\mathfrak{m})$ be a local ring, $S=\oplus_{j\in\mathbf{N}}S_j$ a standard graded R-algebra, $N=\oplus_{j\in\mathbf{N}}N_j$ a finitely generated graded S-module and

$$T := G_{\mathfrak{m}}(N) = \bigoplus_{i,j\in\mathbf{N}} \frac{\mathfrak{m}^i N_j}{\mathfrak{m}^{i+1}N_j}$$

the bigraded F-module with

$$F := G_{\mathfrak{m}}(S) = \bigoplus_{i,j\in\mathbf{N}} \frac{\mathfrak{m}^i S_j}{\mathfrak{m}^{i+1}S_j}.$$

Notice that $F_{0,0}=R/\mathfrak{m}$ is a field.

Definition 2.2. Consider an integer D such that $D\geq \dim N$. For all $k=0,\dots,D-1$, we set

$$c_k^D(N) = \begin{cases} 0 & \text{if } \dim N < D\\ c_k(T) & \text{if } \dim N = D,\end{cases}$$

which is called the c^D-*multiplicity sequence* of N. Moreover, we set $c_k(N) := c_k^{\dim N}(N)$.

First we show that this c^D-multiplicity sequence behaves well with respect to short exact sequences.

Proposition 2.3. *Let $(R, \mathfrak{m})$ be a local ring, $S = \oplus_{j\in\mathbf{N}} S_j$ a standard graded R-algebra, and $0 \to N_0 \to N_1 \to N_2 \to 0$ an exact sequence of finitely generated graded S-modules. Then, for $D \geq d := \dim N_1$,*

$$c_k^D(N_1) = c_k^D(N_0) + c_k^D(N_2)$$

for all $k = 0, \dots, D-1$.

Proof. Let $M_s := \mathbf{R}(\mathfrak{m}, N_s)^+ := \oplus_{i\in\mathbf{Z}} \oplus_{j\in\mathbf{N}} \mathfrak{m}^i (N_s)_j$ be the extended Rees module associated to N_s, $s = 0, 1, 2$. For any bigraded module T and for $i, j \gg 0$, we define the polynomial $h_T^D(i,j)$ of degree $D-2$ as the Hilbert polynomial of $h_T(i,j)$ adding coefficient zero to the terms of degree between $\dim(T) - 2$ and $D-2$.

Let u be an indeterminate, which we consider with degree one. Set $M_0' := \ker(M_1 \to M_2) = \oplus_{i\in\mathbf{Z},\, j\in\mathbf{N}} (N_0)_j \cap \mathfrak{m}^i (N_1)_j$. We consider the natural diagram

$$\begin{tikzcd}
0 \arrow[r] & M_0'(1,0) \arrow[r] \arrow[d, "u^{-1}"] & M_1(1,0) \arrow[r] \arrow[d, "u^{-1}"] & M_2(1,0) \arrow[r] \arrow[d, "u^{-1}"] & 0 \\
0 \arrow[r] & M_0' \arrow[r] & M_1 \arrow[r] & M_2 \arrow[r] & 0,
\end{tikzcd}$$

which gives an exact sequence of cokernels

$$0 \longrightarrow G' := \frac{M_0'}{u^{-1} M_0'} \longrightarrow G_{\mathfrak{m}}(N_1) \longrightarrow G_{\mathfrak{m}}(N_2) \longrightarrow 0. \tag{1}$$

Denote the cokernel of the natural injection $M_o \hookrightarrow M_0'$ by L. Using the diagram,

$$\begin{tikzcd}
0 \arrow[r] & M_0(1,0) \arrow[r] \arrow[d, "u^{-1}"] & M_0'(1,0) \arrow[r] \arrow[d, "u^{-1}"] & L(1,0) \arrow[r] \arrow[d, "u^{-1}"] & 0 \\
0 \arrow[r] & M_0 \arrow[r] & M_0' \arrow[r] & L \arrow[r] & 0,
\end{tikzcd}$$

the snake-lemma yields an exact sequence

$$0 \longrightarrow V \longrightarrow G_{\mathfrak{m}}(N_0) \longrightarrow G' \longrightarrow W \longrightarrow 0, \tag{2}$$

where V and W are the kernel and cokernel of $u^{-1} : L(1,0) \to L$, respectively, i.e., we have the exact sequence:

$$0 \longrightarrow V \longrightarrow L(1,0) \longrightarrow L \longrightarrow W \longrightarrow 0. \tag{3}$$

For $n \leq 1$, the coefficient modules of u^n in $\mathbf{R}(\mathfrak{m}, N_0)^+$ and in M_0' coincide; hence, the action of u^{-1} on L is nilpotent. Therefore, the dimension of L is at most that of G', which is bounded by D. Thus, all modules occurring in exact sequence (3) have dimension at most D.

Now (1), (2) and (3) are exact sequences of finitely generated modules of dimension at most D. We denote by $h_{T_s}(i,j)$ the Hilbert-Samuel function of $T_s := G_{\mathfrak{m}}(N_s)$.

From (1) and (2), we have

$$\begin{aligned} h_{T_0}^{D\,(1,0)}(i,j) + h_{T_2}^{D\,(1,0)}(i,j) - h_{T_1}^{D\,(1,0)}(i,j) \\ = h_V^{D\,(1,0)}(i,j) - h_W^{D\,(1,0)}(i,j). \end{aligned} \tag{4}$$

Because of (3), we have

$$\begin{aligned} h_V^{D\,(1,0)}(i,j) - h_W^{D\,(1,0)}(i,j) \\ = h_L^{D\,(1,0)}(i+1,j) - h_L^{D\,(1,0)}(i,j) = h_L^{D}(i,j). \end{aligned} \tag{5}$$

Hence by (4) and (5),

$$h_{T_0}^{D\,(1,0)}(i,j) + h_{T_2}^{D\,(1,0)}(i,j) - h_{T_1}^{D\,(1,0)}(i,j) = h_L^{D}(i,j)$$

is a polynomial of degree at most $D-2$, which concludes the proof. ◻

3. c^*-multiplicity sequence. We begin by recalling the notion of the Achilles-Manaresi multiplicity sequence for arbitrary ideals as introduced and developed in [**2**]. Let R be a d-dimensional Noetherian ring with a fixed maximal ideal $\mathfrak{m}$ and I an arbitrary ideal of R. Set $S := G_{\mathfrak{m}}(G_I(R)) =: \oplus_{i,j\in\mathbf{N}} S_{i,j}$ where

$$S_{i,j} = \frac{\mathfrak{m}^i I^j + I^{j+1}}{\mathfrak{m}^{i+1} I^j + I^{j+1}}.$$

Let $h(i,j) = \ell(S_{i,j})$ be the Hilbert-Samuel function of S and

$$h^{(1,1)}(i,j) = \sum_{v=0}^{j} \sum_{u=0}^{i} h(i,j)$$

its Hilbert sum. Notice that, for $i,j \gg 0$, $h^{(1,1)}(i,j)$ becomes a polynomial of degree d which can be written as

$$h^{(1,1)}(i,j) = \sum_{k=0}^{d} \frac{c_k(I)}{k!(d-k)!} i^k j^{d-k} + \cdots ,$$

where $\cdots$ means lower degree terms. The coefficients $c_k(I)$, $k = 0,\ldots,d$, are the Achilles-Manaresi multiplicity sequence for I. If I is an $\mathfrak{m}$-primary ideal of R, then $c_0(I)$ is the usual Hilbert-Samuel multiplicity of I, and the remaining $c_k(I)$, $k = 1,\ldots,d$, are zero. Hence, the Achilles-Manaresi multiplicity sequence is a generalization of the usual Hilbert-Samuel multiplicity.

We are now ready to introduce the c^*-multiplicity sequence. The main idea here is to consider a suitable grading on the extended Rees module as in the work of Ulrich and Validashti [**17**].

Let $(R,\mathfrak{m})$ be a Noetherian local ring, A a standard graded Noetherian R-algebra, I an ideal of A generated by elements of degree one and M a finitely generated graded A-module.

Let t be a variable. Consider the extended Rees ring of I

$$\mathbf{R}(I,A)^+ := \bigoplus_{i\in\mathbf{Z}} I^i t^i \subseteq A[t,t^{-1}],$$

and the extended Rees module

$$\mathbf{R}(I,M)^+ := \bigoplus_{i\in\mathbf{Z}} I^i M t^i \subseteq M \bigotimes_R R[t,t^{-1}],$$

where we set $I^i = R$ for $i \leq 0$. Notice that $\mathbf{R}(I,M)^+$ is a module over $\mathbf{R}(I,A)^+$ which gives rise to the *associated graded module* of M with respect to I,

$$G_I(M) := \frac{\mathbf{R}(I,M)^+}{t^{-1}\mathbf{R}(I,M)^+} = \bigoplus_{i\in\mathbf{N}} \frac{I^iM}{I^{i+1}M}\, t^i,$$

which is a module over the associated graded ring $G_I(A)$ of the same dimension as M. Note that here $\mathbf{N}$ contains 0.

Assigning degree zero to the variable t, the Laurent polynomial ring $A[t,t^{-1}]$ becomes a standard graded Noetherian $R[t,t^{-1}]$-algebra, and $M[t,t^{-1}] := M \otimes_R R[t,t^{-1}]$ a finitely generated graded module over this algebra. The extended Rees ring $\mathbf{R}(I,A)^+$ is a homogeneous $R[t^{-1}]$-subalgebra of $A[t,t^{-1}]$, and hence a standard graded Noetherian $R[t^{-1}]$-algebra. Furthermore $\mathbf{R}(I,M)^+$ is a homogeneous $\mathbf{R}(I,A)^+$-submodule of $M[t,t^{-1}]$, thus a finitely generated graded module over $\mathbf{R}(I,A)^+$. With respect to this grading, $G_I(A) := \mathbf{R}(I,A)^+/t^{-1}\mathbf{R}(I,A)^+$ becomes a standard graded Noetherian R-algebra and $G_I(M) := \mathbf{R}(I,M)^+/t^{-1}\mathbf{R}(I,M)^+$ a finitely generated graded module over this algebra. Notice that

$$[G_I(M)]_n = \bigoplus_{i\in\mathbf{N}} [I^iM/I^{i+1}M]_n.$$

The grading so defined on the extended Rees module and the associated graded module is called *internal grading*–for it is induced by the grading on the module M (see [**17**]).

Definition 3.1. Let D be any integer with $D \geq \dim M$. We define the *c^*-multiplicity sequence* of M with respect to I as

$$c^*_{k,D}(I,M) := c^D_k(G_I(M)), \quad k = 0,\dots,D-1,$$

where $G_I(M)$ is graded by the internal grading. In the case where $D = \dim M$, we simply write $c^*_k(I,M)$ instead of $c^*_{k,\,\dim M}(I,M)$, $k = 0,\dots,\dim M - 1$.

To be more explicit, consider the standard bigraded R-algebra $S^* := G_{\mathfrak{m}}(G_I(A)) = \oplus_{s,n=0}^{\infty} S^*_{s,n}$ with

$$S^*_{s,n} = \bigoplus_{i=0}^{\infty} \left[\frac{\mathfrak{m}^s I^i A + I^{i+1}A}{\mathfrak{m}^{s+1} I^i A + I^{i+1}A}\right]_n,$$

where $G_I(A)$ is graded by the internal grading, and the finitely generated bigraded module over this algebra

$$T^* = G_{\mathfrak{m}}(G_I(M)) = \bigoplus_{s,n=0}^{\infty} T^*_{s,n}$$

with

$$T^*_{s,n} = \bigoplus_{i=0}^{\infty} \left[\frac{\mathfrak{m}^s I^i M + I^{i+1}M}{\mathfrak{m}^{s+1} I^i M + I^{i+1}M}\right]_n,$$

where $G_I(M)$ is graded by the internal grading.

Observe that $S^*_{0,0} = R/\mathfrak{m}$ is a field and T^* has dimension $\dim M$. We denote the Hilbert-Samuel function $\ell_{S^*_{0,0}}(T^*_{s,n})$ of $T^* = G_{\mathfrak{m}}(G_I(M))$ by $h^*_{(I,M)}(s,n)$ and its first Hilbert sum by $h^{*\,(1,0)}_{(I,M)}(s,n)$. Thus,

$$h^*_{(I,M)}(s,n) = \sum_{i=0}^{\infty} \ell_R \left[\frac{\mathfrak{m}^s I^i M + I^{i+1}M}{\mathfrak{m}^{s+1} I^i M + I^{i+1}M}\right]_n$$

and

$$h^{*\,(1,0)}_{(I,M)}(s,n) = \sum_{i=0}^{\infty} \ell_R \left[\frac{I^i M}{\mathfrak{m}^{s+1} I^i M + I^{i+1}M}\right]_n.$$

For $s,n \gg 0$, we define the polynomial $h^{*\,D}_{(I,M)}(s,n)$ of degree $D-2$ as the Hilbert polynomial of $h^*_{(I,M)}(s,n)$, adding coefficient zero to the terms of degree between $\dim M - 2$ and $D-2$.

Thus, if $s,n \gg 0$, the sequence

$$\frac{c^*_{k,\,D}(I,M)}{k!(D-1-k)!}, \quad k = 0, \ldots, D-1,$$

is the coefficients of the leading form of the polynomial $h^{*\,D\,(1,0)}_{(I,M)}(s,n)$.

We will need the fact that the c^*-multiplicity sequence is additive on short exact sequences:

Theorem 3.2 (*Additivity*). *Let $(R,\mathfrak{m})$ be a local Noetherian ring, A a standard graded Noetherian R-algebra, and I an ideal of A generated* *by linear forms. If* $0 \longrightarrow M_0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow 0$ *is an exact sequence of finitely generated graded A-modules and D an integer with* $D \geq d := \dim M_1$. *Then,*

$$c^*_{k,\ D}(I, M_1) = c^*_{k,\ D}(I, M_0) + c^*_{k,\ D}(I, M_2)$$

for all $k = 0, \ldots, D-1$.

Proof. Let $N_j := \mathbf{R}(I, M_j)^+$ be the extended Rees module associated to M_j, graded by the internal grading. Set

$$N_0' := \ker(N_1 \to N_2) = \bigoplus_{n \in \mathbf{Z}} \bigoplus_{i=0}^{\infty} [M_0 \cap I^i M_1]_n.$$

We consider the natural diagram

$$\begin{tikzcd} 0 \arrow[r] & N_0' \arrow[r] \arrow[d, "t^{-1}"] & N_1 \arrow[r] \arrow[d, "t^{-1}"] & N_2 \arrow[r] \arrow[d, "t^{-1}"] & 0 \\ 0 \arrow[r] & N_0' \arrow[r] & N_1 \arrow[r] & N_2 \arrow[r] & 0, \end{tikzcd}$$

which gives an exact sequence of cokernels:

$$0 \longrightarrow G' := \frac{N_0'}{t^{-1}N_0'} \longrightarrow G_I(M_1) \longrightarrow G_I(M_2) \longrightarrow 0. \tag{6}$$

Next consider the following commutative diagram:

$$\begin{tikzcd} 0 \arrow[r] & N_0 \arrow[r] \arrow[d, "t^{-1}"] & N_0' \arrow[r] \arrow[d, "t^{-1}"] & L \arrow[r] \arrow[d, "t^{-1}"] & 0 \\ 0 \arrow[r] & N_0 \arrow[r] & N_0' \arrow[r] & L \arrow[r] & 0, \end{tikzcd}$$

where $L := \operatorname{coker}(N_0 \to N_0')$. The snake-lemma yields an exact sequence of finitely generated graded $\mathbf{R}(I, A)^+$-modules having dimension at most D,

$$0 \longrightarrow U \longrightarrow G_I(M_0) \longrightarrow G' \longrightarrow V \longrightarrow 0, \tag{7}$$

where $U := \ker(t^{-1} : L \to L)$ and $V := \operatorname{coker}(t^{-1} : L \to L)$. We also have the exact sequence:

$$0 \longrightarrow U \longrightarrow L \longrightarrow L \longrightarrow V \longrightarrow 0. \tag{8}$$

For $n \leq 1$, the coefficient modules of t^n in $\mathbf{R}_e(I, M_0)$ and in N_0' coincide; hence, the action of t^{-1} on L is nilpotent. Therefore, the dimension of L is at most that of G', which is bounded by D. Thus, all modules occurring in the exact sequence (8) have dimension at most D.

Now (6), (7) and (8) are exact sequences of graded $G_I(A)$-modules of dimension at most D. Hence, we may compute the c^D-multiplicity sequence of graded modules along these sequences. Using the additivity of this multiplicity sequence as stated in Proposition 2.3, we deduce that indeed

$$c^*_{k,\,D}(I, M_1) = c^*_{k,\,D}(I, M_0) + c^*_{k,\,D}(I, M_2)$$

for all $k = 0, \ldots, D-1$. $\square$

We will prove next that the c^*-multiplicity sequence is an invariant of M with respect to I up to reduction. For $I \subseteq J$ two A-ideals and M a Noetherian A-module, we say that I is a *reduction* of (J, M) if $IJ^iM = J^{i+1}M$ for some $i \leq 0$.

Theorem 3.3. *Let $(R, \mathfrak{m})$ be a local Noetherian ring, A a standard graded Noetherian R-algebra, M a finitely generated graded A-module and D an integer with $D \geq \dim M$. Let $I \subseteq J$ be ideals of A generated by linear forms. If I is a reduction of (J, M), then $c^*_{k,\,D}(I, M) = c^*_{k,\,D}(J, M)$ for all $k = 0, \ldots, D-1$.*

Proof. Since I is a reduction of (J, M), we have that $I(J^iM) = J^{i+1}M = J(J^iM))$ for some $i \geq 0$. Hence, we have that $G_I(J^iM) = G_J(J^iM)$, and thus $c^*_{k,\,D}(I, J^iM) = c^*_{k,\,D}(J, J^iM)$ for all $k = 0, \ldots, D-1$. On the other hand, set $M_j := J^jM/J^{j+1}M$ for $j \geq 0$. Notice that $JM_j = 0 = IM_j$; hence, $G_I(M_j) = M_j = G_J(M_j)$, and then $c^*_{k,\,D}(I, M_j) = c^*_{k,\,D}(J, M_j)$ for all $k = 0, \ldots, D-1$.

Using the additivity of the c^*-multiplicity sequence as proved in Theorem 3.2, we now conclude that

$$\begin{aligned} c^*_{k,\,D}(I,M) &= c^*_{k,\,D}(I,J^iM) + \sum_{j=0}^{i-1} c^*_{k,\,D}(I,M_j) \\ &= c^*_{k,\,D}(J,J^iM) + \sum_{j=0}^{i-1} c^*_{k,\,D}(J,M_j) \\ &= c^*_{k,\,D}(J,M). \qquad \square \end{aligned}$$

4. $c^\sharp$-multiplicity sequence. We introduce another multiplicity sequence, the $c^\sharp$-multiplicity sequence, that is more suited for generalizing the Achilles-Manaresi multiplicity sequence for arbitrary ideals as well as the Buchsbaum-Rim multiplicity for modules of finite colength in a free module. The definition is inspired by [**2, 17**].

In addition to the assumptions of the last section, suppose that M is generated in degree zero. Again, consider $G_I(M)$ as graded by the internal grading.

Definition 4.1. Let D be any integer with $D \geq \dim M$. We define the $c^\sharp$-*multiplicity sequence* of M with respect to I as

$$c^\sharp_{k,\,D}(I,M) := c^D_k(A_1 G_I(M)), \quad k = 0, \dots, D-1,$$

where $G_I(M)$ is graded by the internal grading. In the case where $D = \dim M$, we simply write $c^\sharp_k(I,M)$ instead of $c^\sharp_{k,\,\dim M}(I,M)$, $k = 0, \dots, \dim M - 1$.

To be more explicit, consider the standard bigraded R-algebra $S^\sharp := G_{\mathfrak{m}}(A_1 G_I(A)) = \oplus_{s,n=0}^{\infty} S^\sharp_{s,n}$, with

$$S^\sharp_{s,n} = \bigoplus_{i=0}^{\infty} \left[\frac{\mathfrak{m}^s I^i A_1 + I^{i+1}}{\mathfrak{m}^{s+1} I^i A_1 + I^{i+1}}\right]_n = \bigoplus_{i=0}^{n-1} \left[\frac{\mathfrak{m}^s I^i A_1 + I^{i+1}}{\mathfrak{m}^{s+1} I^i A_1 + I^{i+1}}\right]_n,$$

where $G_I(A)$ is graded by the internal grading, and the finitely generated bigraded module over this algebra

$$T^\sharp = G_{\mathfrak{m}}(A_1 G_I(M)) = \bigoplus_{s,n=0}^{\infty} T^\sharp_{s,n}$$

with

$$T^{\sharp}_{s,n} = \bigoplus_{i=0}^{\infty} \left[\frac{\mathfrak{m}^s I^i A_1 M + I^{i+1} M}{\mathfrak{m}^{s+1} I^i A_1 M + I^{i+1} M} \right]_n = \bigoplus_{i=0}^{n-1} \left[\frac{\mathfrak{m}^s I^i M + I^{i+1} M}{\mathfrak{m}^{s+1} I^i M + I^{i+1} M} \right]_n,$$

where $G_I(M)$ is graded by the internal grading.

We denote the Hilbert-Samuel function $\ell_R(T^{\sharp}_{s,n})$ of $T^{\sharp} = G_{\mathfrak{m}}(A_1 G_I(M))$ by $h^{\sharp}_{(I,M)}(s,n)$ and its first Hilbert sum by $h^{\sharp\,(1,0)}_{(I,M)}(s,n)$.

Thus,

$$h^{\sharp}_{(I,M)}(s,n) = \sum_{i=0}^{n-1} \ell_R \left[\frac{\mathfrak{m}^s I^i M + I^{i+1} M}{\mathfrak{m}^{s+1} I^i M + I^{i+1} M} \right]_n$$

and

$$h^{\sharp\,(1,0)}_{(I,M)}(s,n) = \sum_{i=0}^{n-1} \ell_R \left[\frac{I^i M}{\mathfrak{m}^{s+1} I^i M + I^{i+1} M} \right]_n.$$

For $s, n \gg 0$, we define the polynomial $h^{\sharp\,D}_{(I,M)}(s,n)$ of degree $D-2$ as the Hilbert polynomial of $h^{\sharp}_{(I,M)}(s,n)$, adding coefficient zero to the terms of degree between $\dim(A_1 G_I(M)) - 2$ and $D-2$.

Thus, if $s, n \gg 0$, the sequence

$$\frac{c^{\sharp}_{k,\,D}(I,M)}{k!(D-1-k)!}, \quad k = 0, \dots, D-1$$

is the coefficients of the leading form of the polynomial $h^{\sharp\,D\,(1,0)}_{(I,M)}(s,n)$.

Next we introduce another multiplicity sequence, called the b-multiplicity sequence, which also is an invariant of M with respect to I up to reduction. The motivation for introducing this new multiplicity sequence is to help prove one of our main results: that the $c^{\sharp}$-multiplicity sequence is an invariant of M with respect to I up to reduction (see Theorem 4.6).

Definition 4.2. Denote the graded $G_I(A)$-module $G_I(M)/A_1G_I(M)$ by $B(I,M)$. Let D be any integer with $D \geq \dim M$. We define the *b-multiplicity sequence* of M, with respect to I, as

$$b_{k,D}(I,M) := c_k^D(B(I,M)), \quad k = 0,\dots,D-1,$$

where $B(I,M)$ is graded by the internal grading. In the case where $D = \dim M$, we simply write $b_k(I,M)$ instead of $b_k^{\dim M}(I,M)$, $k = 0,\dots,\dim M - 1$.

It will be useful to clarify the relationship between the three multiplicity sequences c^*, b and $c^\sharp$.

Lemma 4.3. *We use the same notation of Definition* 4.1*. We have that*

$$c^*_{k,\,D}(I,M) = c^\sharp_{k,\,D}(I,M) + b_{k,D}(I,M).$$

Proof. Consider the exact sequence of $G_I(A)$-modules:

$$0 \longrightarrow A_1G_I(M) \longrightarrow G_I(M) \longrightarrow B(I,M) \longrightarrow 0.$$

By the additivity of the c^D-multiplicity sequence, Proposition 2.3, we have

$$c_k^D(G_I(M)) = c_k^D(A_1G_I(M)) + c_k^D(B(I,M)).$$

Recall that $c_k^D(G_I(M)) = c^*_{k,\,D}(I,M)$, $c_k^D(A_1G_I(M)) = c^\sharp_{k,\,D}(I,M)$ and $c_k^D(B(I,M)) = b_{k,D}(I,M)$. Hence, the result follows. $\square$

We will need the fact that the b-multiplicity sequence is additive on short exact sequences:

Proposition 4.4 (Additivity). *Let $(R,\mathfrak{m})$ be a local ring, A a standard graded Noetherian R-algebra, and I an ideal of A generated by linear forms. If $0 \to M_0 \to M_1 \to M_2 \to 0$ is an exact sequence of finitely generated graded A-modules and D an integer with $D \geq \dim M_1$, then*

$$b_{k,D}(I,M_1) = b_{k,D}(I,M_0) + b_{k,D}(I,M_2)$$

for all $k = 0,\dots,D-1$.

Proof. Notice that $B(I, M_j) = \oplus_{n\in\mathbf{N}}[I^n M_j]_n$, $j = 0, 1, 2$. Set

$$G' := \ker\left(B(I, M_1) \longrightarrow B(I, M_2)\right) = \bigoplus_{n\in\mathbf{N}}[M_0 \cap I^n M_1]_n.$$

We have the exact sequence

$$0 \longrightarrow G' \longrightarrow B(I, M_1) \longrightarrow B(I, M_2) \longrightarrow 0. \tag{9}$$

Set

$$L := \operatorname{coker}\left(B(I, M_0) \longrightarrow G'\right) = \bigoplus_{n\in\mathbf{N}}\left[\frac{M_0 \cap I^n M_1}{I^n M_0}\right]_n.$$

We have the exact sequence

$$0 \longrightarrow B(I, M_0) \longrightarrow G' \longrightarrow L \longrightarrow 0. \tag{10}$$

Now (9) and (10) are exact sequences of finitely generated graded modules of dimension at most D. Hence, we may compute the c^D-multiplicity sequence of graded modules along these sequences. Using the additivity of this multiplicity sequence as stated in Proposition 2.3, we deduce that

$$b_{k,D}(I, M_1) = b_{k,D}(I, M_0) + b_{k,D}(I, M_2) + c_k^D(L).$$

To obtain that $c_k^D(L) = 0$, we show that L has dimension less than D. In fact, by Artin-Rees we have that

$$\frac{M_0 \cap I^n M_1}{I^n M_0} = \frac{I^{n-c}(M_0 \cap I^c M_1)}{I^n M_0} \subseteq \frac{I^{n-c} M_0}{I^n M_0}.$$

Hence, $\dim(L) \le \dim(P)$ where $P := \oplus_{n\in\mathbf{N}}[I^{n-c}M_0/I^n M_0]_n$. Clearly, P has dimension less than the dimension of $G_I(M_0) = \oplus_{n\in\mathbf{N}} \oplus_{i=0}^{\infty} [I^i M_0/I^{i+1}M_0]_n$, which is at most D. $\square$

We will prove next that the b-multiplicity sequence is an invariant of M with respect to I up to reduction.

Proposition 4.5. *Let $(R, \mathfrak{m})$ be a local ring, A a standard graded Noetherian R-algebra, M a finitely generated graded A-module and D an integer with $D \geq \dim M$. Let $I \subseteq J$ be ideals of A generated by linear forms. If I is a reduction of (J, M), then $b_{k,D}(I, M) = b_{k,D}(J, M)$ for all $k = 0, \ldots, D-1$.*

Proof. Since I is a reduction of (J, M), we have that $I(J^iM) = J^{i+1}M = J(J^iM))$ for some $i \geq 0$. Hence, we have that $B(I, J^iM) = B(J, J^iM)$, and thus $b_{k,\,D}(I, J^iM) = b_{k,\,D}(J, J^iM)$ for all $k = 0, \ldots, D-1$. On the other hand, set $M_j := J^jM/J^{j+1}M$ for $j \geq 0$. Notice that $JM_j = 0 = IM_j$; hence, $B(I, M_j) = 0 = B(J, M_j)$ and then $b_{k,\,D}(I, M_j) = 0 = b_{k,\,D}(J, M_j)$ for all $k = 0, \ldots, D-1$.

Using the additivity of the b-multiplicity sequence as proved in Theorem 4.4, we now conclude that

$$\begin{aligned} b_{k,\,D}(I, M) &= b_{k,\,D}(I, J^iM) + \sum_{j=0}^{i-1} b_{k,\,D}(I, M_j) \\ &= b_{k,\,D}(J, J^iM) + \sum_{j=0}^{i-1} b_{k,\,D}(J, M_j) \\ &= b_{k,\,D}(J, M). \qquad \square \end{aligned}$$

Now we come to one of our main results: that the $c^\sharp$-multiplicity sequence is an invariant of M with respect to I up to reduction.

Theorem 4.6. *Let $(R, \mathfrak{m})$ be a Noetherian local ring, A a standard graded Noetherian R-algebra, M a graded A-module generated by finitely many homogeneous elements of degree zero and D an integer with $D \geq \dim M$. Let $I \subseteq J$ be ideals of A generated by linear forms. If I is a reduction of (J, M), then $c^\sharp_{k,D}(I, M) = c^\sharp_{k,D}(J, M)$ for all $k = 0, \ldots, D-1$.*

Proof. By Lemma 4.3. we have that

$$c^*_{k,\,D}(I, M) = c^\sharp_{k,\,D}(I, M) + b_{k,D}(I, M).$$

On the other hand, we have by Theorem 3.3 and Proposition 4.3 that $c^*_{k,D}(I,M) = c^*_{k,D}(J,M)$ and $b_{k,D}(I,M) = b_{k,D}(J,M)$ for all $k = 0,\dots,D-1$, respectively. Therefore, $c^\sharp_{k,D}(I,M) = c^\sharp_{k,D}(J,M)$ for all $k = 0,\dots,D-1$. $\square$

Remark 4.7. Let $(R,\mathfrak{m})$ be a Noetherian local ring. Given two standard finitely generated graded R-algebras G' and G, with G' a graded subalgebra of G and M a D-dimensional finitely generated graded G-module, we define a sequence of multiplicities $c_k(G',G;M)$, $k=0,\dots,D-1$, as follows: let $I := G'_1G$ and $A := G$. Set

$$c_k(G',G;M) := c^\sharp_k(I,M), \quad k=0,\dots,D-1.$$

Let G'' be another standard finitely generated graded R-algebras such that $G'' \subseteq G' \subseteq G$. Then Theorem 4.6 asserts that $c_k(G'',G;M) = c_k(G',G;M)$ for all $k = 0,\dots,D-1$ if G'' is a reduction of G' for M. This result is in the spirit of [**14**, Theorem 6.7 (a), item (iii) (a)]. It would be very important to prove the converse and also that G' is a reduction of G for M if, and only if, $c_k(G',G;M) = 0$ for all $k=0,\dots,D-1$, as in [**14**, Theorem 6.3 (a)].

5. Hyperplane sections. In this section we show that the multiplicity sequences $c^*, c^\sharp, b$ behave well with respect to sufficiently general hyperplane sections.

Given $g = \sum_{n\in\mathbf{N}} g_n \in M \setminus \{0\}$, let n be the smallest number such that $g_n \neq 0$. Let i be the largest number such that $g_n \in [I^iM]_n$, and define the initial form of g, denoted by g^*, by

$$g^* := g_n \text{ modulo}\, [I^{i+1}M]_n \in [I^iM/I^{i+1}M]_n.$$

If $g=0$, we define $g^*=0$. For a graded A-submodule N of M,

$$G_I(N,M) := \bigoplus_{n\in\mathbf{N}} \bigoplus_{i\in\mathbf{N}} \left[\frac{N\cap I^iM + I^{i+1}M}{I^{i+1}M}\right]_n$$

will denote the $G_I(A)$-submodule of $G_I(M)$ generated by the initial forms of all elements of N.

If x is an element of A, denote by x' the initial form of $x^* \in G_I(A)$ in $G_{\mathfrak{m}}(G_I(A))$. Similarly, if J is a graded ideal in A, let

$$J' := G_{\mathfrak{m}}(G_I(J, A), G_I(A)) \subseteq G_{\mathfrak{m}}(G_I(A))$$

be the ideal of A generated by all x' when $x \in J$, and, if N is a graded A submodule of M, we denote

$$N' = G_{\mathfrak{m}}(G_I(N, M), G_I(M)) \subseteq G_{\mathfrak{m}}(G_I(M)).$$

Definition 5.1. Let $S = G_{\mathfrak{m}}(G_I(A))$, and let $(0) = N_1 \cap N_2 \cap \cdots \cap N_r \cap N_{r+1} \cap \cdots \cap N_t$ be an irredundant primary decomposition of (0) in the S-module $T = G_{\mathfrak{m}}(G_I(M))$. Denote $P_i = \sqrt{(N_i :_S T)}$, $i = 1, \ldots, t$. Assume that

$$I' \subseteq P_{r+1}, \ldots, P_t$$

and

$$I' \nsubseteq P_1, \ldots, P_r.$$

We say that $x \in I$ is a *superficial element* for (I, M) if $x' \notin P_1, \ldots, P_r$.

Remark 5.2. Let $x \in I$ be a superficial element for (I, M). By definition, there exist k such that $(I')^k T \subseteq N_{r+1} \cap \cdots \cap N_t$. Then

$$(0' :_T x') = \bigcap_{i=1}^{t} (N_i :_T x') \subseteq N_1 \cap N_2 \cap \cdots \cap N_r;$$

hence,

$$(I')^k T \cap (0' :_T x') \subseteq N_1 \cap N_2 \cap \cdots \cap N_r \cap N_{r+1} \cap \cdots \cap N_t.$$

The following lemma, in its version for ungraded modules, is due to Ciupercă in [**6**, Lemma 2.10], whose proof can be easily adapted to the graded case we present here.

Lemma 5.3. *Let $(R, \mathfrak{m})$ be a local ring, A a standard graded Noetherian R-algebra and M a finitely generated graded A-module, and let $L \subseteq K$ be two graded submodules of M such that the length $\ell(K_n/L_n)$ is finite for all $n \in \mathbf{N}$. Then*

$$\ell\bigg(\frac{K_n}{L_n}\bigg) = \ell\bigg(\frac{[G_I(K, M)]_n}{[G_I(L, M)]_n}\bigg).$$

The following proposition shows that sufficiently general hyperplane sections behave well with respect to the c^*-multiplicity sequence. The proof is similar to that of Ciupercă [**6**, Proposition 2.11], but we include the proof in its graded context for completeness.

Proposition 5.4. *Let $(R, \mathfrak{m})$ be a local Noetherian ring, A a standard graded Noetherian R-algebra and M a D-dimensional finitely generated graded A-module. Suppose that $x \in I$ is a superficial element for (I, M) and a nonzero divisor on M with $x' \in S_{0,1}$. Denote $\overline{T} = G_{\overline{\mathfrak{m}}}(G_{\overline{I}}(\overline{M}))$, where $\overline{A} = A/xA$, $\overline{I} = I \otimes \overline{A}$ and $\overline{M} = M \otimes \overline{A}$. Then,*

$$c_k^*(I, M) = c_k^*(\overline{I}, \overline{M}), \quad \text{for } k = 0, \dots, D-2.$$

Proof. The proof relies on Lemma 5.3.

We have the following exact sequence:

$$0 \longrightarrow K \longrightarrow \left[\frac{I^i M}{\mathfrak{m}^{s+1} I^i M + I^{i+1} M}\right]_n \longrightarrow \left[\frac{I^i M + xM}{\mathfrak{m}^{s+1} I^i M + I^{i+1} M + xM}\right]_n \longrightarrow 0,$$

where

$$\begin{aligned} K &= \left[\frac{I^i M \cap (\mathfrak{m}^{s+1} I^i M + I^{i+1} M + xM)}{\mathfrak{m}^{s+1} I^i M + I^{i+1} M}\right]_n \\ &= \left[\frac{(\mathfrak{m}^{s+1} I^i M + I^{i+1} M) + I^i M \cap xM}{\mathfrak{m}^{s+1} I^i M + I^{i+1} M}\right]_n \\ &= \left[\frac{I^i M \cap xM}{(\mathfrak{m}^{s+1} I^i M + I^{i+1} M) \cap xM}\right]_n. \end{aligned}$$

From this exact sequence, we get

$$\begin{aligned} h_{\overline{T}}^{(1,0)}(s,n) &= \sum_{i=0}^{\infty} \ell\left(\left[\frac{I^iM + xM}{\mathfrak{m}^{s+1}I^iM + I^{i+1}M + xM}\right]_n\right) \\ &= \sum_{i=0}^{\infty} \ell\left(\left[\frac{I^iM}{\mathfrak{m}^{s+1}I^iM + I^{i+1}M}\right]_n\right) \\ &\quad - \sum_{i=0}^{\infty} \ell\left(\left[\frac{I^iM \cap xM}{(\mathfrak{m}^{s+1}I^iM + I^{i+1}M) \cap xM}\right]_n\right). \end{aligned}$$

Therefore, to conclude the proof, we need to show that for $s, n \gg 0$,

$$\sum_{i=0}^{\infty} \ell\left(\left[\frac{I^iM \cap xM}{(\mathfrak{m}^{s+1}I^iM + I^{i+1}M) \cap xM}\right]_n\right)$$

and

$$\sum_{i=0}^{\infty} \ell\left(\left[\frac{I^{i-1}M}{\mathfrak{m}^{s+1}I^{i-1}M + I^iM}\right]_{n-1}\right)$$

have the same leading forms.

We have that

$$\begin{aligned} \sum_{i=0}^{\infty} \ell\left(\left[\frac{I^iM \cap xM}{(\mathfrak{m}^{s+1}I^iM + I^{i+1}M) \cap xM}\right]_n\right) &= \sum_{i=0}^{\infty} \ell\left(\left[\frac{x(I^iM : x)}{x((\mathfrak{m}^{s+1}I^iM + I^{i+1}M) : x)}\right]_n\right) \\ &= \sum_{i=0}^{\infty} \ell\left(\left[\frac{(I^iM : x)}{(\mathfrak{m}^{s+1}I^iM + I^{i+1}M) : x}\right]_{n-1}\right) \\ &= \sum_{i=0}^{\infty} \ell\left(\left[\frac{(I^iM : x)'}{((\mathfrak{m}^{s+1}I^iM + I^{i+1}M) : x)'}\right]_{n-1}\right), \end{aligned}$$

where the last equality follows by a successive application of Lemma 5.3.

By Remark 5.2, there exist c such that $(I')^cT \cap (0' :_T x') = (0)$. We claim that, for $i > c$,

$$(I^iM : x)' \cap (I')^cT = (I^{i-1}M)' \tag{11}$$

and

$$((\mathfrak{m}^{s+1}I^iM + I^{i+1}M) : x)' \cap (I')^cT = (\mathfrak{m}^{s+1}I^iM + I^{i+1}M)'. \tag{12}$$

We first prove (11). Let $y \in (I^iM : x)$ be such that $0 \neq y' \in (I')^cT$. Since $(I')^cT \cap (0' :_T x') = (0)$, it follows that $y' \notin (0' :_T x')$; hence, $0 \neq (yx)' \in (I^iM)'$. But $(I^iM)'$ is

$$\begin{array}{ccccccccccccc}
0 & \oplus & 0 & \oplus & \cdots & \oplus & 0 & \oplus & T_{0,i} & \oplus & T_{0,i+1} & \oplus & \cdots \\
\oplus & & \oplus & & & & \oplus & & \oplus & & \oplus & & \\
0 & \oplus & 0 & \oplus & \cdots & \oplus & 0 & \oplus & T_{1,i} & \oplus & T_{1,i+1} & \oplus & \cdots \\
\oplus & & \oplus & & & & \oplus & & \oplus & & \oplus & & \\
0 & \oplus & 0 & \oplus & \cdots & \oplus & 0 & \oplus & T_{2,i} & \oplus & T_{2,i+1} & \oplus & \cdots \\
\oplus & & \oplus & & & & \oplus & & \oplus & & \oplus & & \\
\vdots & & \vdots & & & & \vdots & & \vdots & & \vdots & &
\end{array}$$

Since $x' \in S_{0,1}$, we must have $y' \in (I^{i-1}M)'$.

To see (12), consider $y \in ((\mathfrak{m}^{s+1}I^iM + I^{i+1}M) : x)$ such that $0 \neq y' \in (I')^cT$. By the choice of c, we have $y' \notin (0' : x')$; hence, $(yx)' \in (\mathfrak{m}^{s+1}I^iM + I^{i+1}M)'$ and $(yx)' \neq 0$. The homogeneous components of the bigraded submodule $(\mathfrak{m}^{s+1}I^iM + I^{i+1}M)' \subseteq T$ are represented below:

$$\begin{array}{ccccccccccccc}
0 & \oplus & \cdots & \oplus & 0 & \oplus & 0 & \oplus & T_{0,i+1} & \oplus & T_{0,i+2} & \oplus & \cdots \\
\oplus & & & & \oplus & & \oplus & & \oplus & & \oplus & & \\
0 & \oplus & \cdots & \oplus & 0 & \oplus & 0 & \oplus & T_{1,i+1} & \oplus & T_{1,i+2} & \oplus & \cdots \\
\oplus & & & & \oplus & & \oplus & & \oplus & & \oplus & & \\
\vdots & & & & \vdots & & \vdots & & \vdots & & \vdots & & \\
\oplus & & & & \oplus & & \oplus & & \oplus & & \oplus & & \\
0 & \oplus & \cdots & \oplus & 0 & \oplus & 0 & \oplus & T_{s,i+1} & \oplus & T_{s,i+2} & \oplus & \cdots \\
\oplus & & & & \oplus & & \oplus & & \oplus & & \oplus & & \\
0 & \oplus & \cdots & \oplus & 0 & \oplus & T_{s+1,i} & \oplus & T_{s+1,i+1} & \oplus & T_{s+1,i+2} & \oplus & \cdots \\
\oplus & & & & \oplus & & \oplus & & \oplus & & \oplus & & \\
0 & \oplus & \cdots & \oplus & 0 & \oplus & T_{s+2,i} & \oplus & T_{s+2,i+1} & \oplus & T_{s+2,i+2} & \oplus & \cdots \\
\oplus & & & & \oplus & & \oplus & & \oplus & & \oplus & & \\
\vdots & & & & \vdots & & \vdots & & \vdots & & \vdots & & .
\end{array}$$

Since $x' \in S_{0,1}$, we must have $y' \in (\mathfrak{m}^{s+1}I^iM+I^{i+1}M)'$. Then we have

$$\begin{aligned}
\sum_{i=0}^{\infty}\ell\biggl(\biggl[\frac{I^iM\cap xM}{(\mathfrak{m}^{s+1}I^iM+I^{i+1}M)\cap xM}\biggr]_n\biggr)&\\
&= \sum_{i=0}^{\infty}\ell\biggl(\biggl[\frac{x(I^iM:x)}{x((\mathfrak{m}^{s+1}I^iM+I^{i+1}M):x)}\biggr]_n\biggr)\\
&= \sum_{i=0}^{\infty}\ell\biggl(\biggl[\frac{(I^iM:x)}{(\mathfrak{m}^{s+1}I^iM+I^{i+1}M):x}\biggr]_{n-1}\biggr)\\
&= \sum_{i=0}^{\infty}\ell\biggl(\biggl[\frac{(I^iM:x)'}{((\mathfrak{m}^{s+1}I^iM+I^{i+1}M):x)'}\biggr]_{n-1}\biggr)\\
&= \sum_{i=0}^{\infty}\ell\biggl(\biggl[\frac{(I^iM:x)'\cap(I')^cT}{((\mathfrak{m}^{s+1}I^iM+I^{i+1}M):x)'\cap(I')^cT}\biggr]_{n-1}\biggr)\\
&\quad+\sum_{i=0}^{\infty}\ell\biggl(\biggl[\frac{(I^iM:x)'+(I')^cT}{((\mathfrak{m}^{s+1}I^iM+I^{i+1}M):x)'+(I')^cT}\biggr]_{n-1}\biggr).
\end{aligned}$$

Notice that, by (11) and (12) we have that the right-hand-side of the above equality and

$$\begin{aligned}
\sum_{i=0}^{\infty}\ell\biggl(\biggl[\frac{I^{i-1}M}{\mathfrak{m}^{s+1}I^iM+I^iM}\biggr]_{n-1}\biggr)&\\
&+\sum_{i=0}^{\infty}\ell\biggl(\biggl[\frac{(I^iM:x)'+(I')^cT}{((\mathfrak{m}^{s+1}I^iM+I^{i+1}M):x)'+(I')^cT}\biggr]_{n-1}\biggr)
\end{aligned}$$

have the same leading forms.

By the Artin-Rees lemma, there exists a p such that, for $i>p$,

$$I^iM\cap xM = I^{i-p}(I^pM\cap xM),$$

i.e.,

$$x(I^iM:_M x) = xI^{i-p}(I^pM:_M x),$$

or

$$(I^iM:_M x) = I^{i-p}(I^pM:_M x).$$

Then, for $i>p+c$, $(I^iM:_M x)'\subseteq (I')^cT$.

On the other hand, we also have

$$
\begin{aligned}
((\mathfrak{m}^{s+1}I^iM + I^{i+1}M) : x)' &\subseteq (I^iM :_M x)' \\
&\subseteq (I')^cT \quad \text{for } i > p + c \text{ and all } s.
\end{aligned}
$$

We can now conclude that

$$\sum_{i=0}^{\infty} \ell\bigg(\bigg[\frac{I^iM \cap xM}{(\mathfrak{m}^{s+1}I^iM + I^{i+1}M) \cap xM}\bigg]_n\bigg)$$

and

$$\sum_{i=0}^{\infty} \ell\bigg(\bigg[\frac{I^{i-1}M}{\mathfrak{m}^{s+1}I^{i-1}M + I^iM}\bigg]_{n-1}\bigg)$$

have the same leading forms, which finishes the proof. $\square$

Remark 5.5. Similarly, we can show that sufficiently general hyperplane sections behave well with respect to the b-multiplicity sequence. Since, by Lemma 4.3,

$$c^*_{k,\,D}(I, M) = c^\sharp_{k,\,D}(I, M) + b_{k,D}(I, M),$$

we have that the $c^\sharp$-multiplicity sequence also has this property.

6. Local criterion for reduction. In this section we give a criterion for reduction of ideals involving the $c^\sharp$-multiplicity sequence in all localizations in prime ideals.

In order to be able to prove the main result in a simple way, we need some notations. Let $(R, \mathfrak{m})$ be a local Noetherian ring and A a standard graded Noetherian R-algebra. Given a finitely generated graded A-module, $N^j = \oplus_{n\in\mathbf{N}}[N^j]_n v^n$, the extended Rees module

$$\mathbf{R}_e(\mathfrak{m}, N^j) := \bigoplus_{\substack{s\in\mathbf{Z}\\ n\in\mathbf{N}}} \mathfrak{m}^s[N^j]_n u^s v^n$$

will be denoted by P^j. It gives rise to the associated bigraded module

$$G^j := \frac{P^j}{u^{-1}P^j} = G_{\mathfrak{m}}(N^j) := \bigoplus_{s,n\in\mathbf{N}} [G^j]_{s,n},$$

where

$$[G^j]_{s,n} := \frac{\mathfrak{m}^s[N^j]_n}{\mathfrak{m}^{s+1}[N^j]_n}.$$

In general, given any bigraded module F, its generic s, nth piece will be denoted by $F_{s,n}$, i.e., $F = \oplus_{s,n\in\mathbf{N}} F_{s,n}$.

Lemma 6.1. *Let N^j, $j = 0, 1, 2$, be finitely generated A-modules and*

$$0 \longrightarrow N^0 \longrightarrow N^1 \longrightarrow N^2 \longrightarrow 0$$

an exact sequence. Let ${P'}^0 = \ker(P^1 \to P^2)$. Assume that P^0 is naturally injected in ${P'}^0$, and denote by L the cokernel of $P_0 \hookrightarrow {P'}^0$, Let U and V denote the kernel and cokernel of the map $u^{-1} : L(1,0) \to L$, respectively. Then we have that

(i) $\ell([G^1]_{s,n}) = \ell([G^0]_{s,n}) + \ell([G^2]_{s,n}) - [\ell(U_{s,n}) - \ell(V_{s,n})]$;

(ii) $0 \to U \to L(1,0) \to L \to V \to 0$ *is an exact sequence*; *and*

(iii) *all modules occurring in* (ii) *have dimension at most* $\dim(N^1)$.

Proof. From the exact sequence,

$$0 \longrightarrow N^0 \longrightarrow N^1 \longrightarrow N^2 \longrightarrow 0,$$

we get the natural diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & {P'}^0(1,0) & \longrightarrow & P^1(1,0) & \longrightarrow & P^2(1,0) & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle u^{-1}} & & \Big\downarrow{\scriptstyle u^{-1}} & & \Big\downarrow{\scriptstyle u^{-1}} & & \\ 0 & \longrightarrow & {P'}^0 & \longrightarrow & P^1 & \longrightarrow & P^2 & \longrightarrow & 0 \end{array}$$

which gives an exact sequence of cokernels:

$$0 \longrightarrow G' := \frac{{P'}^0}{u^{-1}{P'}^0} \longrightarrow G^1 \longrightarrow G^2 \longrightarrow 0. \tag{13}$$

Using the diagram coming from the natural injection, $P^0 \hookrightarrow {P'}^0$,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & P^0(1,0) & \longrightarrow & {P'}^0(1,0) & \longrightarrow & L(1,0) & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle u^{-1}} & & \Big\downarrow{\scriptstyle u^{-1}} & & \Big\downarrow{\scriptstyle u^{-1}} & & \\ 0 & \longrightarrow & P^0 & \longrightarrow & {P'}^0 & \longrightarrow & L & \longrightarrow & 0 \end{array}$$

the snake-lemma yields:

$$0 \longrightarrow U \longrightarrow G^0 \longrightarrow G' \longrightarrow V \longrightarrow 0. \tag{14}$$

By definition of U and V, we have the exact sequence:

$$0 \longrightarrow U \longrightarrow L(1,0) \longrightarrow L \longrightarrow V \longrightarrow 0, \tag{15}$$

which proves (ii).

For $s \leq 1$, the coefficient modules of u^s in $P^0 = \mathbf{R}_e(\mathfrak{m}, N^0)$ and in ${P'}^0$ coincide; hence, the action of u^{-1} on L is nilpotent. Therefore, the dimension of L is at most that of G', which by (13) is bounded by $\dim(N^1)$. Thus, all modules occurring in the exact sequence (15) have dimension at most $\dim(N^1)$, which proves (iii).

Finally, by exact sequences (13), (14) and (15), we have that

$$\ell([G^1]_{s,n}) = \ell([G^0]_{s,n}) + \ell([G^2]_{s,n}) - [\ell(U_{s,n}) - \ell(V_{s,n})],$$

which proves (i). $\square$

The next theorem provides a crucial step in the proof of our main result of this section.

Theorem 6.2. *Let $(R, \mathfrak{m})$ be a local Noetherian ring, A a standard graded Noetherian R-algebra, $I \subseteq J$ A-ideals generated by linear forms, and M a D-dimensional graded A-module generated by finitely many homogeneous elements of degree zero. Assume that $I_\mathfrak{q}$ is a reduction of $(J_\mathfrak{q}, M_\mathfrak{q})$ for every prime $\mathfrak{q}$ of R with $\mathfrak{q} \neq \mathfrak{m}$. Then*

(i) *$c_k^\sharp(I, M) = c_k^\sharp(J, M)$ for all $k = 1, \ldots, D-1$ and $c_0^\sharp(I, M) \geq c_0^\sharp(J, M)$;*

(ii) *Suppose that R is universally catenary, M is equidimensional as an A-module, and $(I_1)_\mathfrak{p} = (A_1)_\mathfrak{p}$ for every prime $\mathfrak{p}$ of R that is the contraction of a minimal prime in* $\mathrm{Supp}_A(M)$. *If $c_0^\sharp(I, M) \leq c_0^\sharp(J, M)$, then I is a reduction of (J, M).*

Proof. We may factor out the annihilator of M to assume that M is a faithful A-module. In particular, A is equidimensional of dimension D

in the setting of (ii). Theorem 4.6 shows that $c_k^\sharp(I, M)$ does not change when we replace I by the ideal generated by all linear forms in J that are integral over I on M. Thus, by our assumptions on I and J we may suppose that $[JM/IM]_1$ has finite length over R.

For any $0 \leq i \leq n-1$ consider the following exact sequences of graded A-modules whose nth piece is described:

(16)
$$0 \longrightarrow \left[\frac{J^{i+1}M}{I^{i+1}M}\right]_n \longrightarrow \left[\frac{J^{i}M}{I^{i+1}M}\right]_n \longrightarrow \left[\frac{J^{i}M}{J^{i+1}M}\right]_n \longrightarrow 0$$

and

(17)
$$0 \longrightarrow \left[\frac{I^{i}M}{I^{i+1}M}\right]_n \longrightarrow \left[\frac{J^{i}M}{I^{i+1}M}\right]_n \longrightarrow \left[\frac{J^{i}M}{I^{i}M}\right]_n \longrightarrow 0.$$

Applying Lemma 6.1 (i) to the exact sequence (16), we have that
(18)
$$\begin{aligned}\sum_{i=0}^{n-1} \ell\left(\left[\frac{J^iM}{\mathfrak{m}^s J^i M + I^{i+1}M}\right]_n\right) &= \sum_{i=0}^{n-1} \ell\left(\left[\frac{J^iM}{\mathfrak{m}^s J^i M + J^{i+1}M}\right]_n\right) \\ &\quad + \sum_{i=0}^{n-1} \ell\left(\left[\frac{J^{i+1}M}{I^{i+1}M}\right]_n\right) \\ &\quad - [h_L^{(1,0)}(s+1, n) - h_L^{(1,0)}(s, n)],\end{aligned}$$

where $L := \oplus_{s,n \in \mathbf{N}} L_{s,n}$ with

$$L_{s,n} := \bigoplus_{i=0}^{n-1} \left[\frac{\mathfrak{m}^s J^i M \cap J^{i+1}M + I^{i+1}M}{\mathfrak{m}^s J^{i+1}M + I^{i+1}M}\right]_n,$$

and where we use the fact that $\ell([J^{i+1}M/I^{i+1}M]_n) < \infty$ and hence, for $s \gg 0$, $\ell([J^{i+1}M/\mathfrak{m}^s J^{i+1}M + I^{i+1}M]_n) = \ell([J^{i+1}M/I^{i+1}M]_n)$.

Applying Lemma 6.1 (i) to the exact sequence (17), we have that (19)

$$\begin{aligned}\sum_{i=0}^{n-1}\ell\bigg(\bigg[\frac{J^iM}{\mathfrak{m}^sJ^iM+I^{i+1}M}\bigg]_n\bigg)&=\sum_{i=0}^{n-1}\ell\bigg(\bigg[\frac{I^iM}{\mathfrak{m}^sI^iM+I^{i+1}M}\bigg]_n\bigg)\\&\quad+\sum_{i=0}^{n-1}\ell\bigg(\bigg[\frac{J^iM}{I^iM}\bigg]_n\bigg)\\&\quad-[h_{L'}^{(1,0)}(s+1,n)-h_{L'}^{(1,0)}(s,n)],\end{aligned}$$

where $L' := \oplus_{s,n\in\mathbf{N}}L'_{s,n}$ with

$$L'_{s,n} := \bigoplus_{i=0}^{n-1}\bigg[\frac{\mathfrak{m}^sJ^iM\cap I^iM+I^{i+1}M}{\mathfrak{m}^sI^iM+I^{i+1}M}\bigg]_n$$

and where we use the fact that $\ell([J^iM/I^iM]_n) < \infty$ and hence, for $i \gg 0$, $\ell([J^iM/\mathfrak{m}^sJ^iM + I^iM]_n) = \ell([J^iM/I^iM]_n)$.

By Lemma 6.1 (iii), L and L' have dimension at most D; hence, $[h_L^{(1,0)}(s+1,n) - h_L^{(1,0)}(s,n)]$ and $[h_{L'}^{(1,0)}(s+1,n) - h_{L'}^{(1,0)}(s,n)]$ are eventually polynomials of degree at most $D-2$.

Notice that

$$\sum_{i=0}^{n-1}\bigg[\ell\bigg(\bigg[\frac{J^{i+1}M}{I^{i+1}M}\bigg]_n\bigg)-\ell\bigg(\bigg[\frac{J^iM}{I^iM}\bigg]_n\bigg)\bigg]=\ell\bigg(\bigg[\frac{J^nM}{I^nM}\bigg]_n\bigg).$$

Hence, the right-hand-side of the above equality is eventually a polynomial in n of degree at most $D-1$ whose leading coefficient we write as $a^\sharp(I/J;M)$.

Therefore, by equalities (18) and (19), we have that

$$\sum_{i=0}^{n-1}\ell\bigg(\bigg[\frac{J^iM}{\mathfrak{m}^sJ^iM+J^{i+1}M}\bigg]_n\bigg)+\ell\bigg(\bigg[\frac{J^nM}{I^nM}\bigg]_n\bigg)$$

and

$$\sum_{i=0}^{n-1}\ell\bigg(\bigg[\frac{I^iM}{\mathfrak{m}^sI^iM+I^{i+1}M}\bigg]_n\bigg)$$

are eventually polynomials of degree at most $D-1$ with the same leading coefficients. Comparing coefficients in degree $D-1$ of these polynomials, one sees that $c_k^\sharp(I,M) = c_k^\sharp(J,M)$ for all $k = 1,\dots,D-1$ and $c_0^\sharp(I,M) = c_0^\sharp(J,M) + a^\sharp(I/J;M)$, which proves (i) since $a^\sharp(I/J;M) \geq 0$.

Under the assumptions of (ii) we have that $a^\sharp(I/J;M) = 0$. This forces I to be a reduction of (J,M) (see the proof of [**17**, Theorem 3.3]). ◻

We are now ready to assemble the proof of the main theorem of this subsection.

Theorem 6.3. *Let $(R,\mathfrak{m})$ be a universally catenary local Noetherian ring, A a standard graded Noetherian R-algebra, $I \subseteq J$ A-ideals generated by linear forms, and M a D-dimensional graded A-module generated by finitely many homogeneous elements of degree zero. Assume that the A-module M is equidimensional locally at every maximal ideal of R, and $(I_1)_\mathfrak{p} = (A_1)_\mathfrak{p}$ for every prime $\mathfrak{p}$ of R that is the contraction of a minimal prime in* $\mathrm{Supp}_A(M)$. *Then the following are equivalent*:

(i) *I is a reduction of (J,M);*

(ii) *$c_0^\sharp(I_\mathfrak{p},M_\mathfrak{p}) = c_0^\sharp(J_\mathfrak{p},M_\mathfrak{p})$ for all $\mathfrak{p} \in \mathrm{Spec}\,(A)$, and*

(iii) *$c_0^\sharp(I_\mathfrak{p},M_\mathfrak{p}) \leq c_0^\sharp(J_\mathfrak{p},M_\mathfrak{p})$ for all $\mathfrak{p} \in \mathrm{Spec}\,(A)$.*

Proof. The implication (i) $\Rightarrow$ (ii) follows from Theorem 4.6, and (ii) $\Rightarrow$ (iii) is trivial. To show that (i) follows by (iii), we let $\mathfrak{q}$ be any prime ideal of A. Assuming that (iii) holds, we prove by induction on $e := \dim A_\mathfrak{q}$ that $I_\mathfrak{q}$ is a reduction of $(J_\mathfrak{q}, M_\mathfrak{q})$. If $e = 0$, then the result follows immediately from Theorem 6.2 applied to the local ring $(A_\mathfrak{q}, \mathfrak{q}_\mathfrak{q})$. Hence, we may suppose that $e > 0$. Let $\mathfrak{p}$ be any prime ideal of A such that $\dim A_\mathfrak{p} = e-1$ and $\mathfrak{p} \subset \mathfrak{q}$. Hence, by induction hypotheses, we have that $I_\mathfrak{p}$ is a reduction of $(J_\mathfrak{p}, M_\mathfrak{p})$. The equidimensional assumption on M is preserved under localization [**7**, Proof of 3.2]. Thus, in the local ring $(A_\mathfrak{q}, \mathfrak{q}_\mathfrak{q})$, all the assumptions of Theorem 6.2 hold; hence, $I_\mathfrak{q}$ is a reduction of $(J_\mathfrak{q}, M_\mathfrak{q})$. ◻

7. Multiplicity sequence for arbitrary modules. We begin by recalling the notion of the Buchsbaum-Rim multiplicity for submodules of finite colength in a free module as introduced and developed in [**4**].

Let $(R, \mathfrak{m})$ be a Noetherian local ring and E a submodule of the free R-module R^p. The symmetric algebra $A := \operatorname{Sym}(R^p) = \oplus S_n(R^p)$ of R^p is a polynomial ring $R[T_1, \dots, T_p]$. If $h = (h_1, \dots, h_p) \in R^p$, then we define the element $w(h) = h_1T_1 + \cdots + h_pT_p \in A$. We denote by $\mathcal{R}(E) := \oplus\mathcal{R}_n(E)$ the subalgebra of A generated in degree one by $\{w(h) : h \in E\}$ and call it *the Rees algebra* of E. Then $\mathcal{R}(E)$ has dimension $d + p$.

If E has finite colength in R^p then, for $n \gg 0$, the function

$$H(n) = \ell_A(S_n(A^p)/\mathcal{R}_n(M))$$

becomes a polynomial in n with rational coefficients and degree $d+p-1$ where the coefficient of n^{d+p-1} could be written as

$$\frac{e_{BR}(E)}{(d+p-1)!}.$$

It is proved that, if $E \neq R^p$, then $e_{BR}(E) > 0$ (see [**4**, page 214]) and is called the *Buchsbaum-Rim multiplicity* of E.

We are now ready to introduce the main object of this paper, the multiplicity sequence of an arbitrary module. Here the ideal of the previous section will be replaced by a module E.

Let $(R, \mathfrak{m})$ be a Noetherian local ring, E a submodule of the free R-module R^p and N a finitely generated R-module of dimension d. Consider the A-ideal I generated by $\mathcal{R}_1(E)$ and the A-module $M := A \otimes_R N$. Notice that I is an A-ideal generated by linear forms and M is a finitely generated graded A-module of dimension $d + p$ that is generated in degree zero.

Definition 7.1. We define the *multiplicity sequence* associated to module E with respect to N by

$$c_k(E, N) := c_k^{\sharp}(I, M), \quad k = 0, \dots, d + p - 1.$$

To be more explicit,

$$\left[\frac{I^iM}{\mathfrak{m}^{s+1}I^iM + I^{i+1}M}\right]_n = \frac{\mathcal{R}_i(E)S_{n-i}(R^p)N}{\mathfrak{m}^{s+1}\mathcal{R}_i(E)S_{n-i}(R^p)N + \mathcal{R}_{i+1}(E)S_{n-i-1}(R^p)N}$$

for $0 \leq i \leq n-1$. Thus, the Hilbert function of $T^{\sharp} = G_{\mathfrak{m}}(A_1 G_I(M))$ is

$$h^{\sharp\,(1,0)}_{(I,M)}(s,n) = \sum_{i=0}^{n-1} \ell_R \left[\frac{\mathcal{R}_i(E) S_{n-i}(R^p) N}{\mathfrak{m}^{s+1} \mathcal{R}_i(E) S_{n-i}(R^p) N + \mathcal{R}_{i+1}(E) S_{n-i-1}(R^p) N} \right],$$

which, for $s, n \gg 0$, becomes a polynomial of degree at most $d+p-1$ whose leading coefficients are

$$\frac{c_k(E,N)}{k!(d+p-1-k)!}, \quad k = 0, \ldots, d+p-1.$$

If $N = R$, we simply write $c_k(E)$ instead of $c_k(E,N)$ for $k = 0, \ldots, d+p-1$.

Remark 7.2. Our multiplicity sequence $c_k(E,N)$ with $k = 0, \ldots, d+p-1$ for the pair (E,N) generalizes the Buchsbaum-Rim multiplicity defined when E has finite colength in R^p as well as the Achilles-Manaresi multiplicity sequence that applies when $E \subseteq R$ is an ideal. In fact, if E has finite colength in R^p, then, for $s \gg 0$, we have that, for all $i \in \mathbf{N}$,

$$\mathfrak{m}^{s+1} S_1(R^p) \mathcal{R}_i(E) \subseteq \mathcal{R}_{i+1}(E),$$

thus

$$\mathfrak{m}^{s+1} \mathcal{R}_i(E) S_{n-i}(R^p) \subseteq \mathcal{R}_{i+1}(E) S_{n-i-1}(R^p),$$

for all $i \leq n$. Hence, in this context, if I and M are as in Definition 7.1 with $N = R$, then

$$\begin{aligned} h^{\sharp\,(1,0)}_{(I,M)}(s,n) &= \sum_{i=0}^{n-1} \ell_R \left[\frac{\mathcal{R}_i(E) S_{n-i}(R^p)}{\mathfrak{m}^{s+1} \mathcal{R}_i(E) S_{n-i}(R^p) + \mathcal{R}_{i+1}(E) S_{n-i-1}(R^p)} \right] \\ &= \sum_{i=0}^{n-1} \ell_R \left[\frac{\mathcal{R}_i(E) S_{n-i}(R^p)}{\mathcal{R}_{i+1}(E) S_{n-i-1}(R^p)} \right] \\ &= \ell_R \left[\frac{S_n(R^p)}{\mathcal{R}_n(E)} \right] \\ &= \frac{e_{BR}(E)}{(d+p-1)!} n^{d+p-1} + \cdots . \end{aligned}$$

Thus, by comparing the coefficients in the above equality, we have in this case that $c_0(E,R) = e_{BR}(E)$ and $c_k(E,R) = 0$ for all $k = 1,\dots,d+p-1$. Thus, our multiplicity sequence $c_k(E,N)$ with $k = 0,\dots,d+p-1$ for the pair (E,N) generalizes the Buchsbaum-Rim multiplicity.

In the case that E is an arbitrary ideal J of R ($p = 1!$), then our multiplicity sequence $c_k(E,N)$ coincides with the Achilles-Manaresi multiplicity sequence $c_k(J,N)$ for all $k = 0,\dots,d$. In fact, in this case, $\mathcal{R}_i(E) = J^i$ and $S_j(R^p) = R$ for all i,j. Thus,

$$\mathcal{R}_i(E)S_{n-i}(R^p)N = J^iN$$

and

$$\mathfrak{m}^{s+1}\mathcal{R}_i(E)S_{n-i}(R^p) + \mathcal{R}_{i+1}(E)S_{n-i-1}(R^p)N = \mathfrak{m}^{s+1}J^iN + J^{i+1}N.$$

Hence,

$$\begin{aligned} h^{\sharp\,(1,0)}_{(I,M)}(s,n) &= \sum_{i=0}^{n-1} \ell_R\left[\frac{\mathcal{R}_i(E)S_{n-i}(R^p)N}{\mathfrak{m}^{s+1}\mathcal{R}_i(E)S_{n-i}(R^p)N + \mathcal{R}_{i+1}(E)S_{n-i-1}(R^p)N}\right] \\ &= \sum_{i=0}^{n-1} \ell_R\left(\frac{J^iN}{\mathfrak{m}^{s+1}J^iN + J^{i+1}N}\right) \\ &= \sum_{i=0}^{n-1} h^{(1,0)}_{(J,N)}(s,i) = h^{(1,1)}_{(J,N)}(s,n-1). \end{aligned}$$

Therefore, by comparing the coefficients in the above equality, we obtain the claim.

Theorem 4.6 immediately gives the following result which says that the multiplicity sequence associated to module E with respect to N is an invariant of E with respect to N up to reduction.

Theorem 7.3. *Let $(R,\mathfrak{m})$ be a Noetherian local ring, let $E \subseteq F \subseteq R^p$ be R-modules and write $I := \mathcal{R}_1(E)A$ for the corresponding ideal of $A := \mathrm{Sym}\,(R^p)$. Let N be a d-dimensional finitely generated R-module, and set $M := A \otimes_R N$. If E is a reduction of (F,N), then $c_k(E,N) = c_k(F,N)$ for all $k = 0,\dots,d+p-1$.*

This result generalizes a result proved by Ciupercă in [**6**, Proposition 2.7] when E is an ideal of R.

Remark 7.4. To close the circle of ideas around Rees's theorem, it will be extremely important to prove that the converse of Theorem 7.3 also holds. This is not even known in the ideal case. The only known result in this direction is due to Gaffney and Gassler [**10**, Corollary 4.9] where they proved that the converse of Theorem 7.3 holds for ideals in the local ring of a pure dimensional analytic germ, but their proof is analytic in nature since it is a consequence of what they call *the principle of specialization of integral dependence.*

The following result shows that sufficiently general hyperplane sections behave well with respect to the multiplicity sequence of a module. Before stating this result, we need some notation.

Let $(R, \mathfrak{m})$ be a Noetherian local ring, $E \subseteq R^p$ an R-module and write $I := \mathcal{R}_1(E)$ for the corresponding ideal of $A := \mathrm{Sym}\,(R^p)$. Let N be a d-dimensional finitely generated R-module. Let $\pi_i : R^p \to R$ be the projection on the ith-factor. Set J to be the ideal of R generated by the union of $\pi_i(I)$. Let $S := G_{\mathfrak{m}}(G_J(N))$. An element $y \in J$ will be called a *superficial element* for (E, N) if $y' \in S_{0,1}$ is a filter regular element with respect to $S_{0,1}$, in the sense of [**1**, Remark 2.3].

Theorem 7.5. *Let $(R, \mathfrak{m})$ be a Noetherian local ring, $E \subseteq R^p$ an R-module, N a d-dimensional finitely generated R-module and write $I := \mathcal{R}_1(E)A$ for the corresponding ideal of $A := \mathrm{Sym}\,(R^p)$. Let y be a superficial element for (E, N) and a nonzero divisor on N. Then*

$$c_k(E, N) = c_k(\overline{E}, \overline{N}), \quad k = 0, \dots, d + p - 2,$$

where $\overline{N} := N/yN$ and $\overline{E} := E \otimes_R \overline{R}$ with $\overline{R} := R/yR$.

The above theorem follows immediately from Proposition 5.4 and Remark 5.5.

Theorem 6.3 immediately gives the following local criterium for reduction of arbitrary modules.

Theorem 7.6. *Let $(R, \mathfrak{m})$ be a universally catenary local Noetherian ring, $E \subseteq F$ submodules of the free module $L := R^p$ and N a d-*

dimensional finitely generated R-module. Assume that the R-module N is locally equidimensional and that $E_{\mathfrak{p}} = L_{\mathfrak{p}}$ for every minimal prime $\mathfrak{p}$ in $\operatorname{Supp}_R(N)$. Then the following are equivalent:

(i) *E is a reduction of (F, N)*;

(ii) *$c_0(E_{\mathfrak{p}}, N_{\mathfrak{p}}) = c_0(F_{\mathfrak{p}}, N_{\mathfrak{p}})$ for all $\mathfrak{p} \in \operatorname{Spec}(R)$, and*

(iii) *$c_0(E_{\mathfrak{p}}, N_{\mathfrak{p}}) \leq c_0(F_{\mathfrak{p}}, N_{\mathfrak{p}})$ for all $\mathfrak{p} \in \operatorname{Spec}(R)$.*

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