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POSITIVE AND NEGATIVE RESULTS FOR EINSTEIN METRICS ON QUOTIENT MANIFOLDS OF $S^3 \times S^5$

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ABSTRACT. We use Riemannian submersions to explicitly construct an Einstein metric on Einstein-Witten manifolds, viewing Einstein-Witten manifolds as quotient manifolds of $S^3 \times S^5$. For generalized Einstein-Witten manifolds we show that, in certain cases, Riemannian submersions cannot be used to construct an Einstein metric. In particular, we show the metric constructed using Riemannian submersions, and constant length orthogonal basis vectors is not an Einstein metric. We then show that a totally geodesic metric constructed using Riemannian submersions is not an Einstein metric.

1. Introduction. Riemannian geometry is distinguished from topology in part by the Riemannian metric. The Riemannian metric induces geometric structure that gives rise to sectional curvature, Ricci curvature and scalar curvature. Sectional curvature is a generalization of the Gaussian curvature of Riemannian surfaces. It is the Gaussian curvature of two-dimensional submanifolds. Ricci curvature is an average of sectional curvatures and scalar curvature is the trace of Ricci curvature.

A natural question arising in the study of curvature is the question of constant curvature: Are there metrics that give rise to constant curvature of some form? Sectional curvature is a strong measure, and constant sectional curvature is too restrictive to give rise to many interesting examples. The common simply connected examples (dimension $n \geq 3$) with constant sectional curvature 0, 1 and -1, respectively, are $\mathbf{R}^n$, S^n or $\mathbf{H}^n$. (See do Carmo, [**3**, Chapter 6].) There are many manifolds that do not admit constant sectional curvature metrics (see Petersen, [**4**, Chapter 6]). So, the requirement of constant sectional curvature is too strong. Constant scalar curvature is a considerably weaker condition than constant sectional curvature, but turns out to be too weak. We focus our attention on examples with constant Ricci curvature.

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Metrics that have constant Ricci curvature arise naturally in general relativity as solutions to Einstein's field equations for gravity. As such, the construction of metrics with constant Ricci curvature (called Einstein metrics) is of interest in physics as well as mathematics. A Riemannian manifold endowed with an Einstein metric is called an Einstein manifold. Although no topological obstructions are known for dimension $n \geq 5$, Einstein metrics are relatively rare among Riemannian metrics.

This paper presents a computational approach to producing Einstein manifolds using quotient manifolds. In 1990, Wang and Ziller published a paper on the existence of Einstein metrics on principal torus bundles [**5**], including results for circle bundles. The main result for circle bundles shows that the principal S^1-bundle over a product of Kähler-Einstein manifolds with positive first Chern class admits a totally geodesic Einstein metric that is unique up to scaling. In this approach, Einstein-Witten manifolds can be visualized as the total space of the principal S^1-bundle over $\mathbf{CP}^1 \times \mathbf{CP}^2$ and thus, by [**5**], admits a totally geodesic Einstein metric. The metric is determined by the property that the projection onto the base is a Riemannian submersion with totally geodesic fibers. Two special cases of these metrics (the circle bundles over $\mathbf{CP}^1 \times \mathbf{CP}^2$ and over $\mathbf{CP}^1 \times \mathbf{CP}^1 \times \mathbf{CP}^1$) were first discovered by the physicists Castellani, D'Auria and Fré in an attempt to construct an effective Kaluza-Klein supergravity theory in dimension 11 [**2**].

In this paper, we demonstrate a technique to explicitly construct Einstein metrics that attempts to "reverse" the process used by Wang and Ziller through the construction of Einstein metrics using quotient manifolds and Riemannian submersions. This process allows us to explicitly recover the Einstein metric of Wang and Ziller on Einstein-Witten manifolds, viewed instead as quotient manifolds of $S^3 \times S^5$.

We begin with a construction of Einstein-Witten manifolds as quotient manifolds and construction of a basis of vector fields on $S^3 \times S^5$ to be used in the computation. We then show in Theorem 1 the construction of an Einstein metric on the orbit space of the quotient manifold. In Theorems 2 and 3 we explore the application of the construction technique to generalized Einstein-Witten manifolds and find that, in certain cases, it does not produce Einstein metrics.

2. Construction of Einstein-Witten manifolds. An Einstein-Witten manifold can be visualized as a quotient manifold of an S^1-action on $S^3 \times S^5$ given by

$$(x, y) \longmapsto (e^{il\theta}x, e^{ik\theta}y)$$

where $k, l \in \mathbf{Z}$ are relatively prime and $\theta \in [0, 2\pi)$. We view $S^3 \subseteq \mathbf{C}^2$ and $S^5 \subseteq \mathbf{C}^3$ so that the S^1-action is scalar multiplication on each factor with orbit space $M_{k,l}$.

Theorem 1. *Let the metric on $S^3 \times S^5$ be given by the product metric $g = \alpha_1^2 g_0^{S^3} + g_{\alpha_2^2}^{S^5}$ where $\alpha_1^2 g_0^{S^3}$ is the metric with constant sectional curvature* 1 *on S^3 scaled by the parameter α_1, and $g_{\alpha_2^2}^{S^5}$ is the canonical variation of the metric on S^5 with the parameter α_2 so that the quotient map $S^1 \to S^5 \to \mathbf{CP}^2$ of the Hopf circle action is a Riemannian submersion.*

There exists a unique choice of parameters α_1 and α_2 so that the metric on $M_{k,l}$ that makes the quotient map of $S^1 \to S^3 \times S^5 \to M_{k,l}$ a Riemannian submersion is an Einstein metric with totally geodesic fibers.

To prove the theorem, we need a basis of vector fields on $S^3 \times S^5$ for the purpose of curvature computations.

3. Background information.

3.1. A basis of vector fields on $S^3 \times S^5$. Let (M, g) and (N, h) be Riemannian manifolds, and let $\phi : (M, g) \to (N, h)$ be a submersion. ϕ is a Riemannian submersion under the following condition: if $v, w \in T_pM$ are perpendicular to the kernel of $D\phi : T_pM \to T_{\phi(p)}N$, then $g(v, w) = h(D\phi(v), D\phi(w))$. (See Petersen, [**4**].)

Since the metric on $M_{k,l}$ is chosen to make the quotient map a Riemannian submersion, we may use O'Neill's curvature equations [**1**] to explicitly compute curvatures on $M_{k,l}$ and $S^3 \times S^5$, and thus demonstrate that the given metric on $M_{k,l}$ is indeed an Einstein metric with the appropriate choice of α_1 and α_2.

For the curvature computations, we use a "nice basis" for $S^3 \times S^5$. "Nice" in this sense means a basis consisting of vectors that are purely vertical (tangent to the orbit of the S^1-action) or purely horizontal (orthogonal to the orbit of the S^1-action). Since the action is one-dimensional, the basis needs to consist of one vertical vector and seven horizontal vectors. Since $S^3 \times S^5$ is endowed with a product metric, all mixed curvature terms (those terms involving vectors from both S^3 and S^5) will be zero, thereby simplifying the curvature computations. The use of the product metric allows us to begin the construction of a nice basis by considering S^3 and S^5 individually.

3.1.1. The vertical direction on $S^3 \times S^5$. On S^3, we use the identification of S^3 with the Lie group $SU(2)$. As a subset of $\mathbf{C}^2$, S^3 is the set $\{(z,w) \in \mathbf{C}^2 \big| |z|^2 + |w|^2 = 1\}$. S^3 is then naturally identified with

$$SU(2) = \left\{ \begin{pmatrix} z & -w \\ \overline{w} & \overline{z} \end{pmatrix} \;\Big|\; z, w \in \mathbf{C} \text{ with } |z|^2 + |w|^2 = 1 \right\}$$

by the assignment $(z,w) \mapsto \left(\begin{smallmatrix} z & -w \\ \overline{w} & \overline{z} \end{smallmatrix}\right)$.

$SU(2)$ has the associated Lie algebra (containing vector fields on S^3)

$$\mathfrak{su}(2) = \left\{ \begin{pmatrix} i\alpha & \beta + i\gamma \\ -\beta + i\gamma & -i\alpha \end{pmatrix} \;\Big|\; \alpha, \beta, \gamma \in \mathbf{R} \right\}.$$

The standard basis for $\mathfrak{su}(2)$ is given by the matrices $\mathbf{X}_1 = \left(\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right)$, $\mathbf{X}_2 = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ and $\mathbf{X}_3 = \left(\begin{smallmatrix} 0 & i \\ i & 0 \end{smallmatrix}\right)$. The basis vectors $\mathbf{X}_1$, $\mathbf{X}_2$ and $\mathbf{X}_3$ are left-invariant vector fields on $S^3 \equiv SU(2)$ via multiplication on the left by $\mathbf{X}_i$. That is, $\mathbf{X}_{i(z,w)} = \mathbf{X}_i \cdot \left(\begin{smallmatrix} z & -w \\ \overline{w} & \overline{z} \end{smallmatrix}\right)$.

By declaring $\mathbf{X}_1$, $\mathbf{X}_2$, $\mathbf{X}_3$ to be orthonormal, we generate a left-invariant metric on S^3. This metric is the standard metric g_0 on S^3 with constant sectional curvature 1. Scaling the metric on S^3 by α_1 does not change the orthogonality of the basis $\{\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3\}$.

The Hopf circle action on S^3 acts by scalar multiplication as $z \mapsto e^{i\theta} z$ and $\mathbf{X}_1$ is tangent to the orbit of the action. The S^1-action used to generate $M_{k,l}$ acts on S^3 by scalar multiplication as $z \mapsto e^{il\theta} z$. It follows (and can be shown directly through computation) that $l\mathbf{X}_1$ is tangent to the orbit of the S^1-action on the S^3 factor of $S^3 \times S^5$.

On S^5, the metric is chosen to be a canonical variation of the metric on S^5 so that the quotient map $S^1 \to S^5 \to \mathbf{CP}^2$ of the Hopf circle action is a Riemannian submersion. There is an orthogonal basis $\{\mathbf{Y}_1, \mathbf{Y}_2, \mathbf{Y}_3, \mathbf{Y}_4, \mathbf{Y}_5\}$ of S^5 such that $\mathbf{Y}_5$ is tangent to the Hopf circle action. Using this basis for S^5, the S^1-action used to generate $M_{k,l}$ acts on S^5 by scalar multiplication as $w \mapsto e^{ik\theta}w$. Since $\mathbf{Y}_5$ is tangent to the Hopf circle action, $k\mathbf{Y}_5$ is tangent to the orbit of the S^1-action on the S^5 factor of $S^3 \times S^5$. Therefore, the vertical basis vector we need on $S^3 \times S^5$ is the combination given by $\mathbf{V} = l\mathbf{X}_1 + k\mathbf{Y}_5$.

3.1.2. The horizontal directions on $S^3 \times S^5$. For the horizontal basis vectors on $S^3 \times S^5$, we begin with the horizontal vectors on each factor. On S^3, the vectors $\mathbf{X}_2$ and $\mathbf{X}_3$ are horizontal vectors since they are orthogonal to $\mathbf{X}_1$. On S^5, the vectors $\mathbf{Y}_1$, $\mathbf{Y}_2$, $\mathbf{Y}_3$ and $\mathbf{Y}_4$ are horizontal vectors since they are orthogonal to $\mathbf{Y}_5$. Thus, it remains only to find one additional horizontal basis vector, say $\mathbf{Z}$, on $S^3 \times S^5$.

Writing $\mathbf{Z}$ in terms of the orthogonal basis $\{\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3, \mathbf{Y}_1, \mathbf{Y}_2, \mathbf{Y}_3, \mathbf{Y}_4, \mathbf{Y}_5\}$ on $S^3 \times S^5$ and using the requirement that $\mathbf{Z}$ be orthogonal to $\mathbf{V}$ leads to $\mathbf{Z} = -k\alpha_2^2\mathbf{X}_1 + l\alpha_1^2\mathbf{Y}_5$. Therefore, we take the basis on $S^3 \times S^5$ to be $\{\mathbf{V}, \mathbf{X}_2, \mathbf{X}_3, \mathbf{Y}_1, \mathbf{Y}_2, \mathbf{Y}_3, \mathbf{Y}_4, \mathbf{Z}\}$, and so the basis of the horizontal projection, $M_{k,l}$, is $\mathcal{B} = \{\mathbf{X}_2, \mathbf{X}_3, \mathbf{Y}_1, \mathbf{Y}_2, \mathbf{Y}_3, \mathbf{Y}_4, \mathbf{Z}\}$.

3.2. O'Neill's curvature equations for Riemannian submersions. For the curvature computations on $M_{k,l}$, the necessary O'Neill equations in this setting are given in Appendix A. These equations relate the curvatures of the fibers, horizontal projection and total space and further give an avenue of computation that is purely in terms of the horizontal and vertical directions. O'Neill's equations make use of the tensorial invariants T and A of a Riemannian submersion on a manifold M defined by

$$T_{E_1}E_2 = \mathcal{H}\left(\nabla_{\mathcal{V}E_1}\mathcal{V}E_2\right) + \mathcal{V}\left(\nabla_{\mathcal{V}E_1}\mathcal{H}E_2\right)$$

and

$$A_{E_1}E_2 = \mathcal{H}\left(\nabla_{\mathcal{H}E_1}\mathcal{V}E_2\right) + \mathcal{V}\left(\nabla_{\mathcal{H}E_1}\mathcal{H}E_2\right)$$

for all $E_1, E_2 \in \mathfrak{X}(M)$ with $\mathcal{H}E_1$ and $\mathcal{V}E_1$ denoting the horizontal and vertical components of E_1. If E_1 and E_2 are purely vertical, then $T_{E_1}E_2 = \mathcal{H}(\nabla_{E_1}E_2)$. So, if $T \equiv 0$, then each fiber is totally geodesic. That is, the directional derivative in the vertical direction remains vertical.

The metrics on S^3 and S^5, respectively, are totally geodesic. This means that $\nabla_{X_1}X_1 = 0$ and $\nabla_{Y_5}Y_5 = 0$. So, the connection (directional derivative) along the vertical direction of $S^3 \times S^5$ is

$$\nabla_{\mathbf{V}}\mathbf{V} = \nabla_{l\mathbf{X}_1+k\mathbf{Y}_5}(l\mathbf{X}_1 + k\mathbf{Y}_5) = l^2\nabla_{\mathbf{X}_1}\mathbf{X}_1 + k^2\nabla_{\mathbf{Y}_5}\mathbf{Y}_5 = 0.$$

Thus, the metric on $S^3 \times S^5$ is totally geodesic on the fibers.

4. Ricci curvature on $M_{k,l}$. The Ricci curvature computations can be broken into four parts: computations involving only a single vector $\mathbf{X}_i$ from S^3; computations involving only a single vector $\mathbf{Y}_j$ from S^5; computations involving $\mathbf{Z}$ (the mixed horizontal direction) and the computations of mixed curvature terms on S^3 and S^5, respectively. In what follows, sectional curvature and Ricci curvature are denoted by *sec* and *Ric*, respectively.

Since the metric on S^3 is scaled by α_1, we have $\langle \mathbf{X}_i, \mathbf{X}_i \rangle = \alpha_1^2$. Since the metric on S^5 is a canonical variation, we have $\langle \mathbf{Y}_j, \mathbf{Y}_j \rangle = 1$ for $j = 1, \dots, 4$ and $\langle \mathbf{Y}_5, \mathbf{Y}_5 \rangle = \alpha_2^2$. Therefore, using the product metric, the basis $\mathcal{B}$ consists of orthogonal vectors with

$$\begin{aligned}
\langle \mathbf{X}_2, \mathbf{X}_2 \rangle &= \langle \mathbf{X}_3, \mathbf{X}_3 \rangle = \alpha_1^2 \\
\langle \mathbf{Y}_j, \mathbf{Y}_j \rangle &= 1 \quad \text{for } j = 1, \dots, 4 \\
\langle \mathbf{Z}, \mathbf{Z} \rangle &= \alpha_1^2\alpha_2^2\left(l^2\alpha_1^2 + k^2\alpha_2^2\right)
\end{aligned}$$

4.1. Ricci curvature for $\mathbf{X}_i$. For $Ric_{M_{k,l}}(\mathbf{X}_i/\|\mathbf{X}_i\|, \mathbf{X}_i/\|\mathbf{X}_i\|)$, we use the identification of S^3 with $SU(2)$ and the properties of $\mathfrak{su}(2)$ to compute any curvatures needed for the S^3 factor of $S^3 \times S^5$. (See [**4**] for further details, including the values of the Lie bracket and Levi-Civita connection on S^3.)

For $Ric_{M_{k,l}}(\mathbf{X}_i/\|\mathbf{X}_i\|, \mathbf{X}_i/\|\mathbf{X}_i\|)$, we use the O'Neill equation

$$Ric_{S^3\times S^5}(\mathbf{X}, \mathbf{Y}) = Ric_{M_{k,l}}(\mathbf{X}, \mathbf{Y}) - 2\langle A_{\mathbf{X}}, A_{\mathbf{Y}} \rangle,$$

where $\mathbf{X}$ and $\mathbf{Y}$ are orthonormal horizontal vector fields. We must compute $Ric_{S^3\times S^5}(\mathbf{X}_i/\|\mathbf{X}_i\|, \mathbf{X}_i/\|\mathbf{X}_i\|)$ and $\langle A_{\mathbf{X}_i/\|\mathbf{X}_i\|}, A_{\mathbf{X}_i/\|\mathbf{X}_i\|}\rangle$. To give a flavor of the computations, we include some steps here for $Ric_{S^3\times S^5}(\mathbf{X}_2/\|\mathbf{X}_2\|, \mathbf{X}_2/\|\mathbf{X}_2\|)$.

$$\begin{aligned}
Ric_{S^3\times S^5}&\left(\frac{\mathbf{X}_2}{\|\mathbf{X}_2\|}, \frac{\mathbf{X}_2}{\|\mathbf{X}_2\|}\right)\text{(an average of sectional curvatures)}\\
&= sec_{S^3\times S^5}\left(\frac{\mathbf{V}}{\|\mathbf{V}\|}, \frac{\mathbf{X}_2}{\|\mathbf{X}_2\|}\right) + sec_{S^3\times S^5}\left(\frac{\mathbf{X}_3}{\|\mathbf{X}_3\|}, \frac{\mathbf{X}_2}{\|\mathbf{X}_2\|}\right)\\
&\quad + \sum_{j=1}^{4} sec_{S^3\times S^5}\left(\mathbf{Y}_j, \frac{\mathbf{X}_2}{\|\mathbf{X}_2\|}\right) + sec_{S^3\times S^5}\left(\frac{\mathbf{Z}}{\|\mathbf{Z}\|}, \frac{\mathbf{X}_2}{\|\mathbf{X}_2\|}\right)\\
&= \frac{1}{\|\mathbf{X}_2\|^2\|\mathbf{V}\|^2}\langle R(\mathbf{X}_2, \mathbf{V})\mathbf{V}, \mathbf{X}_2\rangle\\
&\quad + \frac{1}{\|\mathbf{X}_2\|^2\|\mathbf{X}_3\|^2}\langle R(\mathbf{X}_2, \mathbf{X}_3)\mathbf{X}_3, \mathbf{X}_2\rangle\\
&\quad + \frac{1}{\|\mathbf{X}_2\|^2\|\mathbf{Z}\|^2}\langle R(\mathbf{X}_2, \mathbf{Z})\mathbf{Z}, \mathbf{X}_2\rangle\\
&= \frac{l^2}{\alpha_1^2\left(l^2\alpha_1^2 + k^2\alpha_2^2\right)}\alpha_1^2 + \frac{1}{\alpha_1^4}\alpha_1^2 + \frac{k^2\alpha_2^4}{\alpha_1^4\alpha_2^2\left(l^2\alpha_1^2 + k^2\alpha_2^2\right)}\alpha_1^2\\
&= \frac{2}{\alpha_1^2}.
\end{aligned}$$

With only one vertical direction, $\langle A_{\mathbf{X}_2/\|\mathbf{X}_2\|}, A_{\mathbf{X}_2/\|\mathbf{X}_2\|}\rangle$ becomes

$$\begin{aligned}
\left\langle A_{\mathbf{X}_2/\|\mathbf{X}_2\|}, A_{\mathbf{X}_2/\|\mathbf{X}_2\|}\right\rangle &= \frac{1}{\|\mathbf{X}_2\|^2\|\mathbf{V}\|^2}\langle A_{\mathbf{X}_2}\mathbf{V}, A_{\mathbf{X}_2}\mathbf{V}\rangle\\
&= \frac{l^2\alpha_1^2}{\alpha_1^2\left(l^2\alpha_1^2 + k^2\alpha_2^2\right)} = \frac{l^2}{l^2\alpha_1^2 + k^2\alpha_2^2}.
\end{aligned}$$

The computations for $\mathbf{X}_3$ are no different in substance, yield the same result and are omitted. Therefore, we have

$$Ric_{M_{k,l}}\left(\frac{\mathbf{X}_i}{\|\mathbf{X}_i\|}, \frac{\mathbf{X}_i}{\|\mathbf{X}_i\|}\right) = \frac{2}{\alpha_1^2} + \frac{2l^2}{l^2\alpha_1^2 + k^2\alpha_2^2}.$$

4.2. Ricci curvature for $\mathbf{Y}_j$. For $Ric_{M_{k,l}}(\mathbf{Y}_j, \mathbf{Y}_j)$, we begin with the canonical variation of the metric that makes the quotient map $S^1 \to$

$(S^5, g_0) \to \mathbf{CP}^2$, a Riemannian submersion with $\mathbf{Y}_5$ tangent to the S^1-action. It is known that the orthonormal basis $\{\mathbf{Y}_1, \mathbf{Y}_2, \mathbf{Y}_3, \mathbf{Y}_4, \mathbf{Y}_5\}$ can be chosen so that $sec_{(S^5,g_0)}(\mathbf{Y}_i, \mathbf{Y}_j) = 1, i \neq j$ and

$$\begin{aligned} &sec_{\mathbf{CP}^2}(\mathbf{Y}_1, \mathbf{Y}_2) = 1 \qquad &&sec_{\mathbf{CP}^2}(\mathbf{Y}_2, \mathbf{Y}_3) = 4 \\ &sec_{\mathbf{CP}^2}(\mathbf{Y}_1, \mathbf{Y}_3) = 1 &&sec_{\mathbf{CP}^2}(\mathbf{Y}_2, \mathbf{Y}_4) = 1 \\ &sec_{\mathbf{CP}^2}(\mathbf{Y}_1, \mathbf{Y}_4) = 4 &&sec_{\mathbf{CP}^2}(\mathbf{Y}_3, \mathbf{Y}_4) = 1 \end{aligned}$$

(see [**4**]). Using O'Neill's equations for $S^1 \to (S^5, g_0) \to \mathbf{CP}^2$ for horizontal vector fields $\mathbf{X}$, $\mathbf{Y}$, we have

$$sec_{S^5}(\mathbf{X}, \mathbf{Y}) = sec_{\mathbf{CP}^2}(\mathbf{X}, \mathbf{Y}) - 3\|A_{\mathbf{X}}\mathbf{Y}\|^2.$$

Thus, we deduce the following values for $\|A_{\mathbf{Y}_i}\mathbf{Y}_j\|_{g_0}$.

$$\begin{aligned} &\|A_{\mathbf{Y}_1}\mathbf{Y}_2\| = 0 \qquad &&\|A_{\mathbf{Y}_2}\mathbf{Y}_3\| = 1 \\ &\|A_{\mathbf{Y}_1}\mathbf{Y}_3\| = 0 &&\|A_{\mathbf{Y}_2}\mathbf{Y}_4\| = 0 \\ &\|A_{\mathbf{Y}_1}\mathbf{Y}_4\| = 1 &&\|A_{\mathbf{Y}_3}\mathbf{Y}_4\| = 0 \end{aligned}$$

Note that $A^{\alpha_2^2}_{\mathbf{X}}\mathbf{Y} = A_{\mathbf{X}}\mathbf{Y}$, since the canonical variation is on the vertical direction and X and Y are horizontal. Also, $\|A^{\alpha_2^2}_{\mathbf{X}}\mathbf{Y}\|^2 = \alpha_2^2\|A_{\mathbf{X}}\mathbf{Y}\|^2_{g_0}$ since, with X and Y horizontal, $A_X Y = \mathcal{V}(\nabla_X Y)$. With the sectional curvatures on $\mathbf{CP}^2$ and values for A relying solely on the single vertical direction, the sectional curvatures on $(S^5, g_{\alpha_2^2})$ are as given below:

$$\begin{aligned} &sec_{(S^5,g_{\alpha_2^2})}(\mathbf{Y}_1, \mathbf{Y}_2) = 1 \qquad &&sec_{(S^5,g_{\alpha_2^2})}(\mathbf{Y}_2, \mathbf{Y}_3) = 4 - 3\alpha_2^2 \\ &sec_{(S^5,g_{\alpha_2^2})}(\mathbf{Y}_1, \mathbf{Y}_3) = 1 &&sec_{(S^5,g_{\alpha_2^2})}(\mathbf{Y}_2, \mathbf{Y}_4) = 1 \\ &sec_{(S^5,g_{\alpha_2^2})}(\mathbf{Y}_1, \mathbf{Y}_4) = 4 - 3\alpha_2^2 &&sec_{(S^5,g_{\alpha_2^2})}(\mathbf{Y}_3, \mathbf{Y}_4) = 1. \end{aligned}$$

Using the sectional curvatures for (S^5, g_0) and $(S^5, g_{\alpha_2^2})$, we complete the Ricci curvature calculations on S^5 and find that

$$Ric_{M_{k,l}}(\mathbf{Y}_j, \mathbf{Y}_j) = 6 - 2\alpha_2^2 + \frac{2k^2\alpha_2^4}{l^2\alpha_1^2 + k^2\alpha_2^2}.$$

4.3. Ricci curvature for Z. The remaining nonzero Ricci curvature computation is for $\mathbf{Z}$. Since $\mathbf{Z} = -k\alpha_2^2\mathbf{X}_1 + l\alpha_1^2\mathbf{Y}_5$, we must incorporate results from both S^3 and S^5.

$$\begin{aligned} Ric_{S^3\times S^5}\left(\frac{\mathbf{Z}}{\|\mathbf{Z}\|}, \frac{\mathbf{Z}}{\|\mathbf{Z}\|}\right) &= \sum_{i=2}^{3} sec_{S^3\times S^5}\left(\frac{\mathbf{X}_i}{\|\mathbf{X}_i\|}, \frac{\mathbf{Z}}{\|\mathbf{Z}\|}\right) \\ &\quad + sec_{S^3\times S^5}\left(\frac{\mathbf{V}}{\|\mathbf{V}\|}, \frac{\mathbf{Z}}{\|\mathbf{Z}\|}\right) \\ &\quad + \sum_{j=1}^{4} sec_{S^3\times S^5}\left(\mathbf{Y}_j, \frac{\mathbf{Z}}{\|\mathbf{Z}\|}\right) \end{aligned}$$

The computation necessary on the S^3 component of $\mathbf{Z}$ is no different in substance than those given above. We find that

$$\sec_{S^3\times S^5}\left(\frac{\mathbf{V}}{\|\mathbf{V}\|}, \frac{\mathbf{Z}}{\|\mathbf{Z}\|}\right) = 0.$$

This is confirmed with a brief, direct calculation of $\langle R(\mathbf{Z},\mathbf{V})\mathbf{V},\mathbf{Z}\rangle$. This result is unsurprising as $\mathbf{V}$ is vertical and $\mathbf{Z}$ is horizontal. What is not obvious is the value of the S^5 component

$$\sum_{j=1}^{4} sec_{S^3\times S^5}\left(Y_j, \frac{\mathbf{Z}}{\|\mathbf{Z}\|}\right).$$

The computation of this term requires some cleverness in the use of the canonical variation metric on S^5.

To begin,

$$\sum_{j=1}^{4} sec_{S^3\times S^5}\left(Y_j, \frac{\mathbf{Z}}{\|\mathbf{Z}\|}\right) = \frac{l^2\alpha_1^4}{\|\mathbf{Z}\|^2}\sum_{j=1}^{4}\langle R(\mathbf{Y}_j,\mathbf{Y}_5)\mathbf{Y}_5,\mathbf{Y}_j\rangle.$$

To compute the sum $\sum_{j=1}^{4}\langle R(\mathbf{Y}_j,\mathbf{Y}_5)\mathbf{Y}_5,\mathbf{Y}_j\rangle$, we again use O'Neill's equations. On S^5, we have

$$\left\langle R\left(\mathbf{Y}_j, \frac{\mathbf{Y}_5}{\|\mathbf{Y}_5\|}\right)\frac{\mathbf{Y}_5}{\|\mathbf{Y}_5\|}, \mathbf{Y}_j\right\rangle = sec_{S^3\times S^5}\left(\mathbf{Y}_j, \frac{\mathbf{Y}_5}{\|\mathbf{Y}_5\|}\right) = \left\| A^{\alpha_2^2}_{\mathbf{Y}_j}\frac{\mathbf{Y}_5}{\|\mathbf{Y}_5\|}\right\|^2$$

since $\mathbf{Y}_j$ is horizontal and $\mathbf{Y}_5$ is vertical. (See O'Neill's equations as given in Appendix A). Thus,

$$sec_{S^3\times S^5}\left(\mathbf{Y}_j, \frac{\mathbf{Y}_5}{\|\mathbf{Y}_5\|}\right) = \left\|A^{\alpha_2^2}_{\mathbf{Y}_j}\frac{\mathbf{Y}_5}{\|\mathbf{Y}_5\|}\right\|^2 = \left\|\alpha_2^2 A^{g_0}_{\mathbf{Y}_j}\frac{\mathbf{Y}_5}{\alpha_2}\right\|^2$$
$$= \left\|\alpha_2 A_{\mathbf{Y}_j}\mathbf{Y}_5\right\|^2_{g_{\alpha_2^2}}.$$

Since $A_{\mathbf{Y}_j}\mathbf{Y}_5 = \mathcal{H}\nabla_{\mathbf{Y}_j}\mathbf{Y}_5$, the canonical variation metric $g_{\alpha_2^2}$ does not change the value of $\|\alpha_2 A_{\mathbf{Y}_j}\mathbf{Y}_5\|^2_{g_{\alpha_2^2}}$. So, $\|\alpha_2 A_{\mathbf{Y}_j}\mathbf{Y}_5\|^2_{g_{\alpha_2^2}} = \|\alpha_2 A_{\mathbf{Y}_j}\mathbf{Y}_5\|^2_{g_0}$. Returning again to O'Neill's equations and using known curvatures for (S^5, g_0), we have

$$1 = sec_{(S^5,g_0)}(\mathbf{Y}_j, \mathbf{Y}_5) = \|A_{\mathbf{Y}_j}\mathbf{Y}_5\|^2_{g_0}.$$

Therefore,

$$sec_{S^3\times S^5}\left(\mathbf{Y}_j, \frac{\mathbf{Y}_5}{\|\mathbf{Y}_5\|}\right) = \left\|\alpha_2 A_{\mathbf{Y}_j}\mathbf{Y}_5\right\|^2_{g_{\alpha_2^2}} = \left\|\alpha_2 A_{\mathbf{Y}_j}\mathbf{Y}_5\right\|^2_{g_0}$$
$$= \alpha_2^2\left\|A_{\mathbf{Y}_j}\mathbf{Y}_5\right\|^2_{g_0} = \alpha_2^2.$$

Finally,

$$\left\langle R\left(\mathbf{Y}_j, \frac{\mathbf{Y}_5}{\|\mathbf{Y}_5\|}\right)\frac{\mathbf{Y}_5}{\|\mathbf{Y}_5\|}, \mathbf{Y}_j\right\rangle = sec_{S^3\times S^5}\left(\mathbf{Y}_j, \frac{\mathbf{Y}_5}{\|\mathbf{Y}_5\|}\right)$$
$$= \alpha_2^2$$
$$\Longrightarrow \frac{1}{\|\mathbf{Y}_5\|^2}\langle R(\mathbf{Y}_j, \mathbf{Y}_5)\mathbf{Y}_5, \mathbf{Y}_j\rangle = \alpha_2^2$$
$$\Longrightarrow \langle R(\mathbf{Y}_j, \mathbf{Y}_5)\mathbf{Y}_5, \mathbf{Y}_j\rangle = \alpha_2^4$$
$$\Longrightarrow \sum_{j=1}^{4}\langle R(\mathbf{Y}_j, \mathbf{Y}_5)\mathbf{Y}_5, \mathbf{Y}_j\rangle = 4\alpha_2^4$$

For $\langle A_{\mathbf{Z}/\|\mathbf{Z}\|}, A_{\mathbf{Z}/\|\mathbf{Z}\|}\rangle$, the computation is straightforward since the metric on $S^3 \times S^5$ is totally geodesic.

$$\begin{aligned}
\langle A_{\mathbf{Z}/\|\mathbf{Z}\|}, A_{\mathbf{Z}/\|\mathbf{Z}\|}\rangle &= \left\langle A_{\mathbf{Z}/\|\mathbf{Z}\|}\frac{\mathbf{V}}{\|\mathbf{V}\|}, A_{\mathbf{Z}/\|\mathbf{Z}\|}\frac{\mathbf{V}}{\|\mathbf{V}\|}\right\rangle \\
&= \frac{1}{\|\mathbf{V}\|^2\|\mathbf{Z}\|^2}\left\|A_{\mathbf{Z}}\mathbf{V}\right\|^2 \\
A_{\mathbf{Z}}\mathbf{V} &= \mathcal{H}\nabla_{\mathbf{Z}}\mathbf{V} \\
&= \mathcal{H}\left(-k\alpha_2^2\nabla_{\mathbf{X}_1}\mathbf{V} + l\alpha_1^2\nabla_{\mathbf{Y}_5}\mathbf{V}\right) \\
&= \mathcal{H}\left(-kl\alpha_2^2\nabla_{\mathbf{X}_1}\mathbf{X}_1 + kl\alpha_1^2\nabla_{\mathbf{Y}_5}\mathbf{Y}_5\right) \\
&= 0.
\end{aligned}$$

So, using established quantities from S^3 and S^5, we conclude that

$$Ric_{M_{k,l}}\left(\frac{\mathbf{Z}}{\|\mathbf{Z}\|}, \frac{\mathbf{Z}}{\|\mathbf{Z}\|}\right) = \frac{2k^2\alpha_2^2}{\alpha_1^2\left(l^2\alpha_1^2 + k^2\alpha_2^2\right)} + \frac{4l^2\alpha_1^2\alpha_2^2}{l^2\alpha_1^2 + k^2\alpha_2^2}.$$

4.4. Mixed Ricci curvature terms.

4.4.1. $Ric_{M_{k,l}}(\mathbf{X}_2/\|\mathbf{X}_2\|, \mathbf{X}_3/\|\mathbf{X}_3\|)$**.** Since $R(\mathbf{X}_i, \mathbf{X}_j)\mathbf{X}_k = 0$ for i, j, k distinct (see [**4**]), we turn our attention to $Ric_{M_{k,l}}(\mathbf{X}_2/\|\mathbf{X}_2\|, \mathbf{X}_3/\|\mathbf{X}_3\|)$.

$$\begin{aligned}
Ric_{S^3\times S^5}\left(\frac{\mathbf{X}_2}{\|\mathbf{X}_2\|}, \frac{\mathbf{X}_3}{\|\mathbf{X}_3\|}\right) &= Ric_{S^3}\left(\frac{\mathbf{X}_2}{\|\mathbf{X}_2\|}, \frac{\mathbf{X}_3}{\|\mathbf{X}_3\|}\right) \\
&= \sum_{i=2}^{3}\left\langle R\left(\frac{\mathbf{X}_i}{\|\mathbf{X}_i\|}, \frac{\mathbf{X}_2}{\|\mathbf{X}_2\|}\right)\frac{\mathbf{X}_3}{\|\mathbf{X}_3\|}, \frac{\mathbf{X}_i}{\|\mathbf{X}_i\|}\right\rangle.
\end{aligned}$$

Since $R(\mathbf{X}_2, \mathbf{X}_2)\mathbf{X}_3 = 0$, $R(\mathbf{X}_3, \mathbf{X}_2)\mathbf{X}_3 = -\mathbf{X}_2$ and $\langle \mathbf{X}_2, \mathbf{X}_3\rangle = 0$, the sum is zero. This leaves only $\langle A_{\mathbf{X}_2/\|\mathbf{X}_2\|}, A_{\mathbf{X}_3/\|\mathbf{X}_3\|}\rangle$ to compute.

$$\begin{aligned}
A_{\mathbf{X}_2}\mathbf{V} &= \mathcal{H}\left(\nabla_{\mathbf{X}_2} l\mathbf{X}_1\right) = \mathcal{H}(-l\mathbf{X}_3) = -l\mathbf{X}_3 \\
A_{\mathbf{X}_3}\mathbf{V} &= \mathcal{H}\left(\nabla_{\mathbf{X}_3} l\mathbf{X}_1\right) = \mathcal{H}(-l\mathbf{X}_3) = -l\mathbf{X}_2.
\end{aligned}$$

So, $\langle A_{\mathbf{X}_2}\mathbf{V}, A_{\mathbf{X}_3}\mathbf{V}\rangle = \langle -l\mathbf{X}_3, -l\mathbf{X}_2\rangle = 0$ and $Ric_{M_{k,l}}(\mathbf{X}_2/\|\mathbf{X}_2\|, \mathbf{X}_3/\|\mathbf{X}_3\|) = 0$.

4.4.2. $Ric_{M_{k,l}}(\mathbf{Y}_i, \mathbf{Y}_j), i \neq j$. In this case, some preliminary notes on sectional curvatures are useful before launching into the Ricci curvature computations.

The tensorial invariant A has the property that $A_{\mathbf{X}}\mathbf{Y} = (1/2)\mathcal{V}[\mathbf{X}, \mathbf{Y}]$ when $\mathbf{X}$ and $\mathbf{Y}$ are horizontal vector fields. Since $\mathbf{Y}_5$ is the sole vertical direction on (S^5, g_0), we may write $A_{\mathbf{X}}\mathbf{Y} = (1/2)\mathcal{V}[\mathbf{X}, \mathbf{Y}] = (1/2)\langle[\mathbf{X}, \mathbf{Y}], \mathbf{Y}_5\rangle\mathbf{Y}_5$. Using this in conjunction with another of O'Neill's equations, we have

$$sec_{S^5}(\mathbf{X}, \mathbf{Y}) = sec_{\mathbf{CP}^2}(\mathbf{X}, \mathbf{Y}) - \frac{3}{4}\langle[\mathbf{X}, \mathbf{Y}], \mathbf{Y}_5\rangle^2.$$

Therefore, $\langle[\mathbf{X}, \mathbf{Y}], \mathbf{Y}_5\rangle^2 = [4(sec_{\mathbf{CP}^2}(\mathbf{X}, \mathbf{Y}) - 1)]/3$. Using the known values for the sectional curvatures on $\mathbf{CP}^2$, we find $\langle[\mathbf{Y}_i, \mathbf{Y}_k], \mathbf{Y}_5\rangle = \pm 2\delta_{5-i,k}$. Therefore, when $i \neq j$, $\langle[\mathbf{Y}_i, \mathbf{Y}_k], \mathbf{Y}_5\rangle \cdot \langle[\mathbf{Y}_j, \mathbf{Y}_k], \mathbf{Y}_5\rangle = 0$.

Since

$$\begin{aligned}\langle A_{\mathbf{Y}_i}, A_{\mathbf{Y}_j}\rangle &= \frac{1}{\|\mathbf{V}\|^2}\langle A_{\mathbf{Y}_i}\mathbf{V}, A_{\mathbf{Y}_j}\mathbf{V}\rangle \\ &= \frac{k^2}{\|\mathbf{V}\|^2}\langle\mathcal{H}\left(\nabla_{\mathbf{Y}_i}\mathbf{Y}_5\right), \mathcal{H}\left(\nabla_{\mathbf{Y}_j}\mathbf{Y}_5\right)\rangle,\end{aligned}$$

we need to compute $\mathcal{H}(\nabla_{\mathbf{Y}_i}\mathbf{Y}_5)$.

Using the horizontal vectors $\mathbf{Y}_1, \ldots, \mathbf{Y}_4$ on $(S^5, g_{\alpha_2^2})$,

$$\mathcal{H}\left(\nabla_{\mathbf{Y}_i}\mathbf{Y}_5\right) = \sum_{k=1}^{4}\langle\nabla_{\mathbf{Y}_i}\mathbf{Y}_5, \mathbf{Y}_k\rangle\mathbf{Y}_k.$$

Since the connection is torsion-free, $\langle\nabla_{\mathbf{Y}_i}\mathbf{Y}_5, \mathbf{Y}_k\rangle = -\langle\nabla_{\mathbf{Y}_i}\mathbf{Y}_k, \mathbf{Y}_5\rangle$, and we observe that $\langle\nabla_{\mathbf{Y}_i}\mathbf{Y}_k, \mathbf{Y}_5\rangle\mathbf{Y}_5/\|\mathbf{Y}_5\|^2 = \mathcal{V}(\nabla_{\mathbf{Y}_i}\mathbf{Y}_k) = A_{\mathbf{Y}_i}\mathbf{Y}_k =$

$(1/2)\mathcal{V}[\mathbf{Y}_i, \mathbf{Y}_k]$. Thus,

$$\begin{aligned}
\langle \nabla_{\mathbf{Y}_i}\mathbf{Y}_k, \mathbf{Y}_5\rangle \frac{\mathbf{Y}_5}{\|\mathbf{Y}_5\|^2} &= \frac{1}{2}\mathcal{V}[\mathbf{Y}_i, \mathbf{Y}_k] \\
&= \frac{1}{2}\langle [\mathbf{Y}_i, \mathbf{Y}_k], \mathbf{Y}_5\rangle \frac{\mathbf{Y}_5}{\|\mathbf{Y}_5\|^2} \\
\Longrightarrow \langle \nabla_{\mathbf{Y}_i}\mathbf{Y}_k, \mathbf{Y}_5\rangle &= \frac{1}{2}\langle [\mathbf{Y}_i, \mathbf{Y}_k], \mathbf{Y}_5\rangle \\
\Rightarrow \mathcal{H}\left(\nabla_{\mathbf{Y}_i}\mathbf{Y}_5\right) &= \sum_{k=1}^{4}\langle \nabla_{\mathbf{Y}_i}\mathbf{Y}_5, \mathbf{Y}_k\rangle \mathbf{Y}_k \\
&= -\sum_{k=1}^{4}\langle \nabla_{\mathbf{Y}_i}\mathbf{Y}_k, \mathbf{Y}_5\rangle \mathbf{Y}_k \\
&= -\frac{1}{2}\sum_{k=1}^{4}\langle [\mathbf{Y}_i, \mathbf{Y}_k], \mathbf{Y}_5\rangle \mathbf{Y}_k.
\end{aligned}$$

Returning now to $\langle A_{\mathbf{Y}_i}, A_{\mathbf{Y}_j}\rangle$,

$$\begin{aligned}
\langle A_{\mathbf{Y}_i}, A_{\mathbf{Y}_j}\rangle &= \frac{k^2}{\|\mathbf{V}\|^2}\langle \mathcal{H}\left(\nabla_{\mathbf{Y}_i}\mathbf{Y}_5\right), \mathcal{H}\big(\nabla_{\mathbf{Y}_j}\mathbf{Y}_5\big)\rangle \\
&= \frac{k^2}{4\|\mathbf{V}\|^2}\left\langle \sum_{k=1}^{4}\langle [\mathbf{Y}_i, \mathbf{Y}_k], \mathbf{Y}_5\rangle \mathbf{Y}_k, \sum_{l=1}^{4}\langle [\mathbf{Y}_j, \mathbf{Y}_l], \mathbf{Y}_5\rangle \mathbf{Y}_l \right\rangle \\
&= \frac{k^2}{4\|\mathbf{V}\|^2}\sum_{k=1}^{4}\langle [\mathbf{Y}_i, \mathbf{Y}_k], \mathbf{Y}_5\rangle \cdot \langle [\mathbf{Y}_j, \mathbf{Y}_k], \mathbf{Y}_5\rangle = 0.
\end{aligned}$$

Finally,

$$\begin{aligned}
Ric_{\mathbf{CP}^2}(\mathbf{Y}_i, \mathbf{Y}_j) &= Ric_{(S^5, g_{\alpha_2^2})}(\mathbf{Y}_i, \mathbf{Y}_j) + 2\langle A_{\mathbf{Y}_i}, A_{\mathbf{Y}_j}\rangle \\
&= Ric_{(S^5, g_{\alpha_2^2})}(\mathbf{Y}_i, \mathbf{Y}_j).
\end{aligned}$$

Since $\mathbf{CP}^2$ is Einstein with $Ric_{\mathbf{CP}^2}(\mathbf{Y}_i, \mathbf{Y}_i) = 6$ and $Ric_{\mathbf{CP}^2}(\mathbf{Y}_i, \mathbf{Y}_j) = 0$, we find that $Ric_{(S^5, g_{\alpha_2^2})}(\mathbf{Y}_i, \mathbf{Y}_j) = 0$ and therefore $Ric_{M_{k,l}}(\mathbf{Y}_i, \mathbf{Y}_j) = 0$.

We are now ready to prove Theorem 1.

5. Proof of Theorem 1.

Proof. To show that $M_{k,l}$ is an Einstein manifold, we must show that there exists a scalar λ, called the Einstein constant, such that

(1)
$$Ric_{M_{k,l}}\left(\frac{\mathbf{X}_i}{\|\mathbf{X}_i\|}, \frac{\mathbf{X}_i}{\|\mathbf{X}_i\|}\right) = \frac{2}{\alpha_1^2} + \frac{2l^2}{l^2\alpha_1^2 + k^2\alpha_2^2} = \lambda$$

(2)
$$Ric_{M_{k,l}}(\mathbf{Y}_j, \mathbf{Y}_j) = 6 - 2\alpha_2^2 + \frac{2k^2\alpha_2^4}{l^2\alpha_1^2 + k^2\alpha_2^2} = \lambda$$

(3)
$$Ric_{M_{k,l}}\left(\frac{\mathbf{Z}}{\|\mathbf{Z}\|}, \frac{\mathbf{Z}}{\|\mathbf{Z}\|}\right) = \frac{2k^2\alpha_2^2}{\alpha_1^2\,(l^2\alpha_1^2 + k^2\alpha_2^2)} + \frac{4l^2\alpha_1^2\alpha_2^2}{l^2\alpha_1^2 + k^2\alpha_2^2} = \lambda.$$

Therefore, we must find α_1, α_2 and λ so that equations (1), (2) and (3) are solved simultaneously.

For computational simplicity, we make the substitutions $a_1 = \alpha_1^2$ and $a_2 = \alpha_2^2$ and clear all denominators by multiplying each equation by $a_1(l^2a_1 + k^2a_2)$. This yields

(4)
$$2a_1(l^2a_1 + k^2a_2) + 2l^2a_1 = \lambda a_1(l^2a_1 + k^2a_2)$$

(5)
$$(6 - 2a_2)a_1(l^2a_1 + k^2a_2) + 2k^2a_1a_2^2 = \lambda a_1(l^2a_1 + k^2a_2)$$

(6)
$$2k^2a_2 + 4l^2a_1^2a_2 = \lambda a_1(l^2a_1 + k^2a_2)$$

Subtracting equation (6) from equation (4) and factoring gives $4l^2a_1(1-a_1a_2) = 0$. Since $a_1 = \alpha_1^2 = \langle \mathbf{X}_2, \mathbf{X}_2\rangle = \langle \mathbf{X}_3, \mathbf{X}_3\rangle \neq 0$, $a_2 = 1/a_1$. Now, subtracting equation (5) from equation (4) and substituting $1/a_1$ for a_2, we derive the cubic equation

$$3l^2a_1^3 - 3l^2a_1^2 + 3k^2a_1 - k^2 = 0.$$

Therefore, a_1 is a root of the cubic polynomial $f(x) = 3l^2x^3 - 3l^2x^2 + 3k^2x - k^2$.

Note that $f(x) = 3l^2x^3 - 3l^2x^2 + 3k^2x - k^2 = 3l^2x^2(x-1) + k^2(3x-1)$. If x is negative, then so are $x - 1$ and $3x - 1$, forcing $f(x) < 0$. If x is at least 1, then $x - 1$ and $3x - 1$ are both positive, forcing $f(x) > 0$. Therefore, all real roots of $f(x)$ must fall in the interval $(0, 1)$.

To determine the number of real roots, consider $f'(x)$. The derivative $f'(x) = 9l^2x^2 - 6l^2x + 3k^2$ has roots

$$x_1 = \frac{1}{3} - \frac{\sqrt{l^2 - 3k^2}}{3l} \quad \text{and} \quad x_2 = \frac{1}{3} + \frac{\sqrt{l^2 - 3k^2}}{3l}.$$

With consideration of the discriminant, we have three cases.

(i) $l^2 < 3k^2$. If $l^2 < 3k^2$, the roots x_1 and x_2 are both complex and hence, f has exactly one real root as there are no local extrema and f is always increasing.

(ii) $l^2 = 3k^2$. Since $l, k \in \mathbf{Z}$ with $\gcd(l, k) = 1$, $l^2 = 3k^2$ is impossible.

(iii) $l^2 > 3k^2$. In this case, the roots x_1 and x_2 are both real roots. Since the graph of f' is an upward pointing parabola, f will have a local maximum at the point $(x_1, f(x_1))$ since x_1 is the smaller of the two roots of f'. We consider the value of $f(x_1)$. After some simplification, we see that

$$f(x_1) = \frac{(-4l^2 + 6k^2)(l - \sqrt{l^2 - 3k^2})}{9l}.$$

Since $l^2 > 3k^2$, $2l^2 > 3k^2$ and $-4l^2 + 6k^2 < 0$. Since $l > \sqrt{l^2 - 3k^2}$, the sign of $l - \sqrt{l^2 - 3k^2}$ and the sign of $9l$ will be the same. Therefore, $f(x_1) < 0$. Since the local maximum value is negative, f has only one real root.

Finally, by choosing a_1 to be the unique root of the cubic polynomial $f(x) = 3l^2x^3 - 3l^2x^2 + 3k^2x - k^2$ and $a_2 = 1/a_1$, λ is determined, the Ricci curvature equations (1)–(3) are satisfied, and the metric on $M_{k,l}$ is therefore an Einstein metric. Earlier computation directly showed that the metric is totally geodesic. □

6. Construction of generalized Einstein-Witten manifolds. A natural question arising from Theorem 1 is whether the technique used yields other Einstein metrics. One class of manifolds constructed in the same manner as that used with $M_{k,l}$ is the class of generalized Einstein-Witten manifolds, denoted $N_{k,l}$. Generalized Einstein-Witten manifolds are obtained through the use of a more general S^1-action on $S^3 \times S^5$. $N_{k,l}$ is the seven-dimensional quotient manifold obtained by taking the orbit space of $S^1 \to S^3 \times S^5 \to N_{k,l}$ under the S^1-action given by

$$\big((z, w), y\big) \longmapsto \big((e^{il_1\theta}z, e^{il_2\theta}w), e^{ik\theta}y\big),$$

where $k, l \in \mathbf{Z}$, $\theta \in [0, 2\pi)$, $\gcd(l, k) = 1$ and $\gcd(l_1, l_2) = l$ with $l_1 \neq l_2$. We again view $S^3 \subseteq \mathbf{C}^2$ and $S^5 \subseteq \mathbf{C}^3$ so that $(z, w) \in S^3 \subseteq \mathbf{C}^2$ and S^1 acts on S^5 by scalar multiplication.

Theorem 2. *Let the metric on $S^3 \times S^5$ be given by the product metric $g = \alpha_1^2 g_0^{S^3} + g_{\alpha_2^2}^{S^5}$ where $\alpha_1^2 g_0^{S^3}$ is the metric with constant sectional curvature* 1 *on S^3 scaled by the parameter α_1, and $g_{\alpha_2^2}^{S^5}$ is the canonical variation of the metric on S^5 with the parameter α_2 so that the quotient map $S^1 \to S^5 \to \mathbf{CP}^2$ of the Hopf circle action is a Riemannian submersion.*

Equip $N_{k,l}$ with the metric that makes the quotient map of $S^1 \to S^3 \times S^5 \to N_{k,l}$ a Riemannian submersion. If the basis vectors on $S^3 \times S^5$ have constant length, then $N_{k,l}$ with this metric is not Einstein.

For the proof of Theorem 2, as with Theorem 1, we need a basis of vector fields on $S^3 \times S^5$ consisting of one vertical direction and seven horizontal directions.

7. Background information.

7.1. A basis on $S^3 \times S^5$.

7.1.1. The vertical direction on $S^3 \times S^5$. Since the S^1-action on S^5 is unchanged, we use the same orthogonal basis $\{\mathbf{Y}_1, \mathbf{Y}_2, \mathbf{Y}_3, \mathbf{Y}_4, \mathbf{Y}_5\}$ with $k\mathbf{Y}_5$ tangent to the orbit of the S^1-action.

To find the vertical direction on S^3, we again utilize the identification of S^3 with $SU(2)$. Translating the S^1-action on S^3 to $SU(2)$ gives the action as

$$\begin{pmatrix} z & -w \\ \overline{w} & \overline{z} \end{pmatrix} \longmapsto \begin{pmatrix} e^{il_1\theta} z & -e^{il_2\theta} w \\ e^{-il_2\theta}\overline{w} & e^{-il_1\theta}\overline{z} \end{pmatrix}.$$

This corresponds to the multiplication of $\begin{pmatrix} z & -w \\ \overline{w} & \overline{z} \end{pmatrix}$ on the left by the matrix

$$A(\theta) = \begin{pmatrix} e^{il_1\theta}|z|^2 + e^{il_2\theta}|w|^2 & (e^{il_1\theta} - e^{il_2\theta}) zw \\ (e^{-il_2\theta} - e^{-il_1\theta})\overline{zw} & e^{-il_1\theta}|z|^2 + e^{-il_2\theta}|w|^2 \end{pmatrix}.$$

Note that $A(\theta) \in SU(2)$ and

$$A'(\theta) = \begin{pmatrix} il_1e^{il_1\theta}|z|^2 + il_2e^{il_2\theta}|w|^2 & (il_1e^{il_1\theta} - il_2e^{il_2\theta})zw \\ (-il_2e^{il_2\theta} + il_1e^{il_1\theta})\overline{zw} & -il_1e^{-il_1\theta}|z|^2 - il_2e^{-il_2\theta}|w|^2 \end{pmatrix}$$
$$\in \mathfrak{s}u(2).$$

We see that $A(0)$ is the identity matrix, so $\begin{pmatrix} z & -w \\ \overline{w} & \overline{z} \end{pmatrix} = A(0) \cdot \begin{pmatrix} z & -w \\ \overline{w} & \overline{z} \end{pmatrix}$ and the S^3 component of the vertical direction on $S^3 \times S^5$ is given by $A'(0)$ as it is tangent to the orbit of the action on S^3. In terms of the standard basis $\mathbf{X}_1$, $\mathbf{X}_2$, $\mathbf{X}_3$ on $\mathfrak{s}u(2)$, we have

$$\begin{aligned} A'(0) &= \begin{pmatrix} i(l_1|z|^2 + l_2|w|^2) & i(l_1 - l_2)zw \\ i(l_1 - l_2)\overline{zw} & -i(l_1|z|^2 + l_2|w|^2) \end{pmatrix} \\ &= (l_1|z|^2 + l_2|w|^2)\mathbf{X}_1 - (l_1 - l_2)\,\mathrm{Im}\,(zw)\mathbf{X}_2 + (l_1 - l_2)\,\mathrm{Re}\,(zw)\mathbf{X}_3. \end{aligned}$$

So, the vector tangent to the S^1-action on $S^3 \times S^5$ is $\mathbf{V} = \mathbf{B}_1 + k\mathbf{Y}_5$ where $\mathbf{B}_1 = A'(0)$.

7.1.2. The horizontal directions on $S^3 \times S^5$. The horizontal directions on S^5 are, as with $M_{k,l}$, the vectors $\mathbf{Y}_1, \mathbf{Y}_2, \mathbf{Y}_3, \mathbf{Y}_4$. We must find the two horizontal vectors on S^3 and then the remaining seventh horizontal vector on $S^3 \times S^5$. Let the two horizontal vectors on S^3 be denoted by $\mathbf{B}_2$ and $\mathbf{B}_3$. We assume $\mathbf{X}_1$, $\mathbf{X}_2$ and $\mathbf{X}_3$ are orthogonal with constant length. Setting $\mathbf{B}_2 = \sum_{i=1}^3 b_{2i}\mathbf{X}_i$ and requiring that $\mathbf{B}_2$ be orthogonal to $\mathbf{B}_1$, we may choose b_{21}, b_{22} and b_{23} so that $\mathbf{B}_2 = (l_1 - l_2)\,\mathrm{Im}\,(zw)\langle\mathbf{X}_2, \mathbf{X}_2\rangle\mathbf{X}_1 + (l_1|z|^2 + l_2|w|^2)\langle\mathbf{X}_1, \mathbf{X}_1\rangle\mathbf{X}_2$. Following the same procedure with $\mathbf{B}_3$ and with the requirement that $\mathbf{B}_3$ be orthogonal to both $\mathbf{B}_1$ and $\mathbf{B}_2$, we may choose b_{31}, b_{32} and b_{33} so that

$$\begin{aligned} \mathbf{B}_3 &= (l_1|z|^2 + l_2|w|^2)(l_1 - l_2)\,\mathrm{Re}\,(zw)\langle\mathbf{X}_3, \mathbf{X}_3\rangle\mathbf{X}_1 \\ &\quad - (l_1|z|^2 + l_2|w|^2)(l_1 - l_2)\,\mathrm{Im}\,(zw)\langle\mathbf{X}_3, \mathbf{X}_3\rangle\mathbf{X}_2 \\ &\quad - \big((l_1|z|^2 + l_2|w|^2)^2\langle\mathbf{X}_1, \mathbf{X}_1\rangle + (l_1 - l_2)^2(\mathrm{Im}\,(zw))^2\langle\mathbf{X}_2, \mathbf{X}_2\rangle\big)\mathbf{X}_3. \end{aligned}$$

For computational and notational simplicity, we define the following functions. Let

$$\begin{aligned} f_1 &= l_1|z|^2 + l_2|w|^2 \\ f_2 &= -(l_1 - l_2)\,\mathrm{Im}\,(zw) \\ f_3 &= (l_1 - l_2)\,\mathrm{Re}\,(zw). \end{aligned}$$

Then, the basis on S^3 may be written as

$$\begin{aligned}\mathbf{B}_1 &= f_1\mathbf{X}_1 + f_2\mathbf{X}_2 + f_3\mathbf{X}_3 \\ \mathbf{B}_2 &= -f_2\langle\mathbf{X}_2,\mathbf{X}_2\rangle\mathbf{X}_1 + f_1\langle\mathbf{X}_1,\mathbf{X}_1\rangle\mathbf{X}_2 \\ \mathbf{B}_3 &= f_1f_3\langle\mathbf{X}_3,\mathbf{X}_3\rangle\mathbf{X}_1 + f_2f_3\langle\mathbf{X}_3,\mathbf{X}_3\rangle\mathbf{X}_2 \\ &\quad - \left(f_1^2\langle\mathbf{X}_1,\mathbf{X}_1\rangle + f_2^2\langle\mathbf{X}_2,\mathbf{X}_2\rangle\right)\mathbf{X}_3.\end{aligned}$$

The remaining horizontal direction $\mathbf{Z}$ is found in the same manner as on $M_{k,l}$, and we have $\mathbf{Z} = -k\langle\mathbf{Y}_5,\mathbf{Y}_5\rangle\mathbf{B}_1 + \langle\mathbf{B}_1,\mathbf{B}_1\rangle\mathbf{Y}_5$.

The basis on $N_{k,l}$ is now given by $\{\mathbf{B}_2,\mathbf{B}_3,\mathbf{Y}_1,\mathbf{Y}_2,\mathbf{Y}_3,\mathbf{Y}_4,\mathbf{Z}\}$. Since $\mathbf{B}_2$ and $\mathbf{B}_3$ are constructed from the standard basis of $\mathfrak{s}u(2)$ with functional coefficients rather than constant coefficients, we need to compute the derivatives of the coefficient functions for use in the curvature equations.

7.2. Derivatives on S^3. For the derivatives of the coefficient functions, we use integral curves. An integral curve is a curve passing through a given point p tangent to a given direction X_p. That is, an integral curve is the solution to a second order differential equation with initial conditions given as a point and a direction. The derivative of a function on the manifold in the direction of a given vector is, then, the derivative of the composition of the function with the integral curve.

Taking advantage of the identification of S^3 with $SU(2)$, we use the following integral curves γ for $\mathbf{X}_1$, $\mathbf{X}_2$ and $\mathbf{X}_3$, respectively.

$$\begin{aligned}\gamma_{\mathbf{X}_1}(t) &= \begin{pmatrix}\cos(t)+i\sin(t) & 0 \\ 0 & \cos(t)-i\sin(t)\end{pmatrix} \\ \gamma_{\mathbf{X}_2}(t) &= \begin{pmatrix}\cos(t) & \sin(t) \\ -\sin(t) & \cos(t)\end{pmatrix} \\ \gamma_{\mathbf{X}_3}(t) &= \begin{pmatrix}\cos(t) & i\sin(t) \\ i\sin(t) & \cos(t)\end{pmatrix}\end{aligned}$$

Each of these curves is the identity matrix when $t = 0$ and is translated to $(z,w) \in S^3$ through multiplication on the right by $\begin{pmatrix} z & -w \\ \overline{w} & \overline{z}\end{pmatrix}$. That is, $\gamma_i(t) = \gamma_{\mathbf{X}_i}(t)\cdot\begin{pmatrix} z & -w \\ \overline{w} & \overline{z}\end{pmatrix}$. Note that $\mathbf{X}_1$, $\mathbf{X}_2$ and $\mathbf{X}_3$ are right-invariant in this case.

To compute the derivatives of f_i on S^3, we compute $(d/dt)(f_i \circ \gamma_j(t))|_{t=0}$ for $i, j \in \{1, 2, 3\}$. We demonstrate with $\mathbf{X}_2(f_1)$.

$$\begin{aligned}
\mathbf{X}_2(f_1)_{(z,w)} &= \frac{d}{dt}\left(f_1 \circ \gamma_2(t)\right)\big|_{t=0} \\
&= \frac{d}{dt}\left[f_1\begin{pmatrix} z\cos(t) + \overline{w}\sin(t) & -w\cos(t) + \overline{z}\sin(t) \\ -z\sin(t) + \overline{w}\cos(t) & w\sin(t) + \overline{z}\cos(t)\end{pmatrix}\right]\Bigg|_{t=0} \\
&= \frac{d}{dt}\big[l_1\left(z\cos(t) + \overline{w}sin(t)\right)\left(w\sin(t) + \overline{z}\cos(t)\right) \\
&\qquad - l_2\left(-w\cos(t) + \overline{z}\sin(t)\right)\left(-z\sin(t) + w\cos(t)\right)\big]\big|_{t=0} \\
&= \big[2(l_1 - l_2)\operatorname{Re}(zw)\left(\cos^2(t) - \sin^2(t)\right) \\
&\qquad - 2\left(l_1|z|^2 + l_2|w|^2\right)\cos(t)\sin(t) \\
&\qquad + 2\left(l_1|w|^2 + l_2|z|^2\right)\cos(t)\sin(t)\big]\big|_{t=0} \\
&= 2(l_1 - l_2)Re(zw) = 2f_3.
\end{aligned}$$

The remaining derivatives are found in a similar fashion. The table below summarizes the results.

$$\begin{array}{lll}
\mathbf{X}_1(f_1) = 0 & \mathbf{X}_1(f_2) = -2f_3 & \mathbf{X}_1(f_3) = 2f_2 \\
\mathbf{X}_2(f_1) = 2f_3 & \mathbf{X}_2(f_2) = 0 & \mathbf{X}_2(f_3) = -2f_1 + (l_1 + l_2) \\
\mathbf{X}_3(f_1) = -2f_2 & \mathbf{X}_3(f_2) = 2f_1 - (l_1 + l_2) & \mathbf{X}_3(f_3) = 0.
\end{array}$$

Now declaring $\mathbf{X}_1$, $\mathbf{X}_2$ and $\mathbf{X}_3$ to be orthogonal with $\langle \mathbf{X}_1, \mathbf{X}_1\rangle = a^2$, $\langle \mathbf{X}_2, \mathbf{X}_2\rangle = b^2$, $\langle \mathbf{X}_3, \mathbf{X}_3\rangle = c^2$, where a, b and c are constants, we may use the Koszul formula and the values of the right-invariant brackets to compute $\nabla_{\mathbf{X}_i}\mathbf{X}_j$ for $i, j \in \{1, 2, 3\}$. With this information, we may also compute the values of the curvature tensor $R(\mathbf{X}_i, \mathbf{X}_j)\mathbf{X}_k$, $i, j, k \in \{1, 2, 3\}$. See Appendix B for a listing of these quantities.

7.3. The metric with constant length basis vectors is not totally geodesic. The simplest case for O'Neill's equations occurs when the fibers are totally geodesic. That is, when $\nabla_{\mathbf{V}}\mathbf{V} = 0$. On

$N_{k,l}$, we have $\mathbf{V} = \mathbf{B}_1 + k\mathbf{Y}_5$.

$$\begin{aligned}\nabla_{\mathbf{V}}\mathbf{V} &= \nabla_{\mathbf{B}_1+k\mathbf{Y}_5}(\mathbf{B}_1 + k\mathbf{Y}_5) = \nabla_{\mathbf{B}_1}\mathbf{B}_1 + k^2\nabla_{\mathbf{Y}_5}\mathbf{Y}_5 \\ &= \nabla_{\mathbf{B}_1}\mathbf{B}_1 \text{ since the metric on } S^5 \text{ is totally geodesic} \\ &= \sum_{i=1}^{3} f_i \nabla_{\mathbf{X}_i}(f_1\mathbf{X}_1 + f_2\mathbf{X}_2 + f_3\mathbf{X}_3) \\ &= \sum_{i=1}^{3} f_i \big[f_1\nabla_{\mathbf{X}_i}\mathbf{X}_1 + (\mathbf{X}_i f_1)\mathbf{X}_1 + f_2\nabla_{\mathbf{X}_i}\mathbf{X}_2 \\ &\quad + (\mathbf{X}_i f_2)\mathbf{X}_2 + f_3\nabla_{\mathbf{X}_i}\mathbf{X}_3 + (\mathbf{X}_i f_3)\mathbf{X}_3\big] \\ &= \frac{2f_2f_3(b^2-c^2)}{a^2}\mathbf{X}_1 + \left(\frac{2f_1f_3(c^2-a^2)}{b^2} - (l_1+l_2)f_3\right)\mathbf{X}_2 \\ &\quad + \left(\frac{2f_1f_3(a^2-b^2)}{c^2} + (l_1+l_2)f_2\right)\mathbf{X}_3.\end{aligned}$$

When $l_1 \neq l_2$, $\nabla_{\mathbf{V}}\mathbf{V}$ will not be zero for all $(z, w) \in S^3$, so this metric with $\langle \mathbf{X}_1, \mathbf{X}_1\rangle = a^2$, $\langle \mathbf{X}_2, \mathbf{X}_2\rangle = b^2$ and $\langle \mathbf{X}_3, \mathbf{X}_3\rangle = c^2$ is not totally geodesic.

7.4. O'Neill's equations for $N_{k,l}$. Since the metric on $N_{k,l}$ is not totally geodesic, we must use an expanded form of O'Neill's equations for curvature. The necessary equation is

$$\begin{aligned}Ric_{N_{k,l}}(\mathbf{X},\mathbf{Y}) &= Ric_{S^3\times S^5}(\mathbf{X},\mathbf{Y}) + 2\langle A_{\mathbf{X}}, A_{\mathbf{Y}}\rangle + \langle T\mathbf{X}, T\mathbf{Y}\rangle \\ &\quad - \frac{1}{2}\left(\langle \nabla_{\mathbf{X}}N, \mathbf{Y}\rangle - \langle \nabla_{\mathbf{Y}}N, \mathbf{X}\rangle\right),\end{aligned}$$

where T is the tensorial invariant for Riemannian submersion on a manifold. Let $\{U_j\}_{j\in J}$ be an orthonormal basis of the vertical distribution $\mathcal{V}_p$. Then $\langle T\mathbf{X}, T\mathbf{Y}\rangle = \sum_j \langle T_{U_j}\mathbf{X}, T_{U_j}\mathbf{Y}\rangle$ and $\mathbf{N} = \sum_j T_{U_j}U_j$ is the mean curvature vector.

8. Ricci curvature equations on $N_{k,l}$. The Ricci curvature equations for $N_{k,l}$ are considerably more complicated than those for $M_{k,l}$, not in small part due to the functional coefficients of the basis on S^3. As a result, the computations necessary to find the Ricci curvature equations are lengthy. The computations are not included here, but are a straightforward, if somewhat messy, application of O'Neill's equation and associated tensors using the values of the connection and curvature tensor found in Appendix B and the derivatives summarized above in subsection 7.2.

The Ricci curvature equations themselves are lengthy and not included in full generality here. The equations presented here are in the simplified cases when the points on S^3 are in the form $(0,w)$ with $w \in S^1 \subseteq \mathbf{C}$ or $(z,0)$ with $z \in S^1 \subseteq \mathbf{C}$. We will see from these simplified equations that the metric on $N_{k,l}$ as described in Theorem 2 is not an Einstein metric.

9. Proof of Theorem 2.

Proof. For the class of points $(S^1,0) \subseteq S^3$, we have $f_1 = l_1$ and $f_2 = f_3 = 0$. This yields $\|\mathbf{B}_1\|^2 = a^2 l_1^2$, $\|\mathbf{B}_2\|^2 = a^4 b^2 l_1^2$, $\|\mathbf{B}_3\| = a^4 c^2 l_1^4$, $\|\mathbf{V}\|^2 = a^2 l_1^2 + k^2\alpha_2^2$ and $\|\mathbf{Z}\|^2 = \alpha_2^2 a^2 l_1^2 (a^2 l_1^2 + k^2 \alpha_2^2)$. Using these quantities, the Ricci curvature equations become:

(†)

$$
\begin{aligned}
Ric_{N_{k,l}}\left(\frac{\mathbf{B}_2}{\|\mathbf{B}_2\|}, \frac{\mathbf{B}_2}{\|\mathbf{B}_2\|}\right) &= \frac{2(b^4 - (a^2 - c^2)^2)}{a^2 b^2 c^2} + \frac{2(c^2 - a^2) l_1 (l_2 - l_1)}{b^2 (a^2 l_1^2 + k^2 \alpha_2^2)} \\
&\quad + \frac{2\big((b^2 + c^2 - a^2) l_1 - c^2 (l_1 + l_2)\big)^2}{b^2 c^2 (a^2 l_1^2 + k^2 \alpha_2^2)} \\
&\quad + \frac{l_1^2 - l_2^2}{a^2 l_1^2 + k^2 \alpha_2^2} \\
Ric_{N_{k,l}}\left(\frac{\mathbf{B}_3}{\|\mathbf{B}_3\|}, \frac{\mathbf{B}_3}{\|\mathbf{B}_3\|}\right) &= \frac{2\big(c^4 - (a^2 - b^2)^2\big)}{a^2 b^2 c^2} + \frac{4(a^2 - b^2) l_1^2}{c^2 (a^2 l_1^2 + k^2 \alpha_2^2)} \\
&\quad + \frac{2\big((a^2 - b^2 - c^2) l_1 + b^2 (l_1 + l_2)\big)}{a^2 b^2 c^2 l_1 (a^2 l_1^2 + k^2 \alpha_2^2)} \\
&\quad - \frac{2(a^2 - b^2)(l_1 + l_2)}{c^2 l_1 (a^2 l_1^2 + k^2 \alpha_2^2)} + \frac{l_1^2 - l_2^2}{a^2 l_1^2 + k^2 \alpha_2^2} \\
Ric_{N_{k,l}}\left(\frac{\mathbf{Z}}{\|\mathbf{Z}\|}, \frac{\mathbf{Z}}{\|\mathbf{Z}\|}\right) &= \frac{2k^2 \alpha_2^2 \big(a^4 - (b^2 - c^2)^2\big)}{a^2 b^2 c^2 (a^2 l_1^2 + k^2 \alpha_2^2)} + \frac{4\alpha_2^2 a^2 l_1^2}{a^2 l_1^2 + k^2 \alpha_2^2} \\
Ric_{N_{k,l}}(\mathbf{Y}_i, \mathbf{Y}_i) &= 6 - 2\alpha_2^2 + \frac{2k^2 \alpha_2^6}{a^2 l_1^2 + k^2 \alpha_2^2}.
\end{aligned}
$$

For the class of points $(0, S^1) \subseteq S^3$, we have $f_1 = l_2$ and $f_2 = f_3 = 0$. Also, $\|\mathbf{B}_1\|^2 = a^2 l_2^2$, $\|\mathbf{B}_2\|^2 = a^4 b^2 l_2^2$, $\|\mathbf{B}_3\| = a^4 c^2 l_2^4$, $\|\mathbf{V}\|^2 = a^2 l_2^2 + k^2\alpha_2^2$ and $\|\mathbf{Z}\|^2 = \alpha_2^2 a^2 l_2^2 (a^2 l_2^2 + k^2\alpha_2^2)$. With these quantities, we have the following Ricci curvature equations:

$(\star)$

$$\begin{aligned}
Ric_{N_{k,l}}\left(\frac{\mathbf{B}_2}{\|\mathbf{B}_2\|}, \frac{\mathbf{B}_2}{\|\mathbf{B}_2\|}\right) &= \frac{2(b^4 - (a^2 - c^2)^2)}{a^2 b^2 c^2} + \frac{2(c^2 - a^2)(l_1 - l_2)}{b^2(a^2 l_2^2 + k^2\alpha_2^2)} \\
&\quad + \frac{2\big((b^2 + c^2 - a^2) l_2 - c^2(l_1 + l_2)\big)^2}{b^2 c^2 (a^2 l_2^2 + k^2\alpha_2^2)} \\
&\quad - \frac{l_1^2 - l_2^2}{a^2 l_2^2 + k^2\alpha_2^2} \\
Ric_{N_{k,l}}\left(\frac{\mathbf{B}_3}{\|\mathbf{B}_3\|}, \frac{\mathbf{B}_3}{\|\mathbf{B}_3\|}\right) &= \frac{2\big(c^4 - (a^2 - b^2)^2\big)}{a^2 b^2 c^2} + \frac{4(a^2 - b^2) l_2^2}{c^2(a^2 l_2^2 + k^2\alpha_2^2)} \\
&\quad + \frac{2\big((a^2 - b^2 - c^2) l_2 + b^2(l_1 + l_2)\big)^2}{b^2 c^2 (a^2 l_2^2 + k^2\alpha_2^2)} \\
&\quad - \frac{2(a^2 - b^2)(l_1 + l_2) l_2}{c^2(a^2 l_2^2 + k^2\alpha_2^2)} - \frac{l_1^2 - l_2^2}{a^2 l_2^2 + k^2\alpha_2^2} \\
Ric_{N_{k,l}}\left(\frac{\mathbf{Z}}{\|\mathbf{Z}\|}, \frac{\mathbf{Z}}{\|\mathbf{Z}\|}\right) &= \frac{2k^2\alpha_2^2\big(a^4 - (b^2 - c^2)^2\big)}{a^2 b^2 c^2 (a^2 l_2^2 + k^2\alpha_2^2)} + \frac{4\alpha_2^2 a^2 l_2^2}{a^2 l_2^2 + k^2\alpha_2^2} \\
Ric_{N_{k,l}}\left(\mathbf{Y}_i, \mathbf{Y}_i\right) &= 6 - 2\alpha_2^2 + \frac{2k^2\alpha_2^6}{a^2 l_2^2 + k^2\alpha_2^2}.
\end{aligned}$$

If the given metric on $N_{k,l}$ is an Einstein metric, then we must, at least, be able to find values for the parameters to simultaneously satisfy the eight Ricci curvature equations given above. We begin by setting the above equations equal to λ, and we focus on the Ricci curvature equations $(\dagger)$ and $(\star)$. Subtracting these two equations gives

$$\frac{2k^2\alpha_2^6}{a^2 l_1^2 + k^2\alpha_2^2} - \frac{2k^2\alpha_2^6}{a^2 l_2^2 + k^2\alpha_2^2} = 0.$$

This yields $l_1^2 - l_2^2 = 0$. So, l_1 must be equal to l_2 or $-l_2$. By assumption, $l_1 \neq l_2$; therefore, $l_1 = -l_2$, in which case, we find that $f_1 = l_2(1 - 2|z|^2)$. We now consider the point $(z_0, w_0) = (1/\sqrt{2}, 1/\sqrt{2}) \in S^3$. At (z_0, w_0), $f_1 = l_2(1 - 2(1/2)) = 0$ and $f_2 = 2l_2 \operatorname{Im}(1/2) = 0$. Thus, $\|\mathbf{B}_2\|^2 = a^4 b^2 f_1^2 + a^2 b^4 f_2^2 = 0$.

Since a basis vector cannot be length zero at any point, it cannot be the case that $l_1 = -l_2$. This is a contradiction. Therefore, $N_{k,l}$ with this metric is not Einstein for any choice of a, b, c, α_2^2. Earlier computation (see subsection 7.3) showed the metric is not totally geodesic. □

10. A totally geodesic metric on $N_{k,l}$.

Since the necessary Ricci curvature equation (subsection 7.4) on $N_{k,l}$ is complicated by the fact that the "natural" metric arising from the generalization of the construction on $M_{k,l}$ is not totally geodesic, it is also natural to ask whether $N_{k,l}$ would admit an Einstein metric if it were endowed with a totally geodesic metric.

Theorem 3. *Let the metric on $S^3 \times S^5$ be given by the product metric $g = \alpha_1^2 g_0^{S^3} + g_{\alpha_2^2}^{S^5}$ where $\alpha_1^2 g_0^{S^3}$ is the metric with constant sectional curvature 1 on S^3 scaled by the parameter α_1 and $g_{\alpha_2^2}^{S^5}$ is the canonical variation of the metric on S^5 with the parameter α_2 so that the quotient map $S^1 \to S^5 \to \mathbf{CP}^2$ of the Hopf circle action is a Riemannian submersion.*

Equip $N_{k,l}$, the metric that makes the quotient map of $S^1 \to S^3 \times S^5 \to N_{k,l}$ a Riemannian submersion. If the metric on $N_{k,l}$ is totally geodesic, then $N_{k,l}$ does not admit an Einstein metric.

10.1. Background information.

In order for the metric on $N_{k,l}$ defined by the horizontal projection of the basis $\{\mathbf{V}, \mathbf{B}_2, \mathbf{B}_3, \mathbf{Y}_1, \mathbf{Y}_2, \mathbf{Y}_3, \mathbf{Y}_4, \mathbf{Z}\}$ on $S^3 \times S^5$ to be totally geodesic, we must have $\nabla_{\mathbf{V}}\mathbf{V} = 0$. That is, we need $\nabla_{\mathbf{B}_1}\mathbf{B}_1 = 0$.

We consider the length of $\mathbf{B}_1$. Assuming that the standard basis vectors $\mathbf{X}_1$, $\mathbf{X}_2$, $\mathbf{X}_3$ for $\mathfrak{su}(2)$ are orthogonal, $\|\mathbf{B}_1\|^2 = f_1^2\langle \mathbf{X}_1, \mathbf{X}_1\rangle + f_2^2\langle \mathbf{X}_2, \mathbf{X}_2\rangle + f_3^2\langle \mathbf{X}_3, \mathbf{X}_3\rangle$. We have seen that the metric cannot be Einstein if the lengths of the $\mathbf{X}_i$ are constant. So, we consider the possibility that the lengths of the $\mathbf{X}_i$ are functions of the complex variables z and w. The metric on S^5 remains the same.

If $\|\mathbf{B}_1\|^2 = \alpha_1$, the metric will be totally geodesic. For ease of computation, we assume that $\langle \mathbf{X}_1, \mathbf{X}_1 \rangle = \langle \mathbf{X}_2, \mathbf{X}_2 \rangle = \langle \mathbf{X}_3, \mathbf{X}_3 \rangle = g(z, w)$. Then, $\|\mathbf{B}_1\|^2 = g(z,w)(f_1^2 + f_2^2 + f_3^2) = \alpha_1$. Thus,

$$g(z,w) = \frac{\alpha_1}{f_1^2 + f_2^2 + f_3^2} = \frac{\alpha_1}{l_1^2|z|^2 + l_2^2|w|^2}.$$

Notice that $g(z,w)$ is a positive function since $\alpha_1 > 0$ and $l_1|z|^2 + l_2|w|^2 > 0$ since z and w cannot simultaneously be 0.

Now, using the derivatives on S^3 computed in subsection 7.2, we compute the values of the connection for this metric using the Koszul formula. It is then a straightforward exercise to show that $\nabla_{\mathbf{V}}\mathbf{V} = 0$ and this is, indeed, a totally geodesic metric. We now prove Theorem 3.

Proof. Using O'Neill's equations for curvature in their simplified form, the Ricci curvature equations for $N_{k,l}$ with this totally geodesic metric can be found. A similar argument to the one given above in the proof of Theorem 2, considering the two classes of points $(S^1, 0) \subseteq S^3$ and $(0, S^1) \subseteq S^3$ once again yields the condition that $l_1 = -l_2$. This condition further simplifies the Ricci curvature equations, but the simplified equations at the point $(1/\sqrt{2}, 1/\sqrt{2}) \in S^3$ force the requirement that l_1 be equal to 0. Therefore, this totally geodesic metric on $N_{k,l}$ cannot be Einstein. □

APPENDIX

A. O'Neill's equations for $S^3 \times S^5$ and $M_{k,l}$. Let $\mathbf{U}, \mathbf{V}$ be orthonormal vertical vector fields, and let $\mathbf{X}, \mathbf{Y}$ be orthonormal horizontal vector fields. Then,

$$sec_{S^3\times S^5}(\mathbf{U},\mathbf{V}) = sec_{F_p}(\mathbf{U},\mathbf{V}) \tag{7}$$

$$sec_{S^3\times S^5}(\mathbf{X},\mathbf{U}) = \|A_{\mathbf{X}}\mathbf{U}\|^2 \tag{8}$$

$$sec_{S^3\times S^5}(\mathbf{X},\mathbf{Y}) = sec_{M_{k,l}}(\mathbf{X},\mathbf{Y}) - 3\|A_{\mathbf{X}}\mathbf{Y}\|^2, \tag{9}$$

and

$$Ric_{S^3\times S^5}(\mathbf{U},\mathbf{V}) = Ric_{F_p}(\mathbf{U},\mathbf{V}) + \langle A\mathbf{U}, A\mathbf{V}\rangle \tag{10}$$

$$Ric_{S^3\times S^5}(\mathbf{X},\mathbf{U}) = -\langle \left(\hat{\delta}A\right)\mathbf{X},\mathbf{U}\rangle \tag{11}$$

$$Ric_{S^3\times S^5}(\mathbf{X},\mathbf{Y}) = Ric_{M_{k,l}}(\mathbf{X},\mathbf{Y}) - 2\langle A_{\mathbf{X}}, A_{\mathbf{Y}}\rangle, \tag{12}$$

where *sec* denotes the sectional curvature and *Ric* denotes the Ricci curvature. In addition, F_p is the fiber at p and A is the tensorial invariant of a Riemannian submersion on a manifold M defined by

$$A_{E_1}E_2 = \mathcal{H}\left(\nabla_{\mathcal{H}E_1}\mathcal{V}E_2\right) + \mathcal{V}\left(\nabla_{\mathcal{H}E_1}\mathcal{H}E_2\right),$$

for all $E_1, E_2 \in \mathfrak{X}(M)$ with $\mathcal{H}E_1$ and $\mathcal{V}E_1$ denoting the horizontal and vertical components of E_1.

The definitions of $\langle A\mathbf{U}, A\mathbf{V}\rangle$, $\langle(\widehat{\delta}A)\mathbf{X}, \mathbf{U}\rangle$ and $\langle A_{\mathbf{X}}, A_{\mathbf{Y}}\rangle$ are given below. Let $\{X_i\}_{i\in I}$ be an orthonormal basis of $\mathcal{H}_p$ and $\{U_j\}_{j\in J}$ be an orthonormal basis of $\mathcal{V}_p$. Then

$$\langle A\mathbf{U}, A\mathbf{V}\rangle = \sum_i \langle A_{X_i}\mathbf{U}, A_{X_i}\mathbf{V}\rangle$$
$$\left(\widehat{\delta}A\right)\mathbf{X} = -\sum_j \left(\nabla_{U_j}\mathbf{X}\right)U_j$$
$$\langle A_{\mathbf{X}}, A_{\mathbf{Y}}\rangle = \sum_j \langle A_{\mathbf{X}}U_j, A_{\mathbf{Y}}U_j\rangle.$$

B. Right-invariant connections and curvatures on $N_{k,l}$. Using the Koszul formula and the right-invariant brackets on S^3 with $\langle \mathbf{X}_1, \mathbf{X}_1\rangle = a^2$, $\langle \mathbf{X}_2, \mathbf{X}_2\rangle = b^2$ and $\langle \mathbf{X}_3, \mathbf{X}_3\rangle = c^2$, we find that $\nabla_{\mathbf{X}_i}\mathbf{X}_i = 0$ and

$$\begin{aligned}
\nabla_{\mathbf{X}_1}\mathbf{X}_2 &= \left(\frac{a^2-b^2-c^2}{c^2}\right)\mathbf{X}_3 & \nabla_{\mathbf{X}_1}\mathbf{X}_3 &= \left(\frac{-a^2+b^2+c^2}{b^2}\right)\mathbf{X}_2\\
\nabla_{\mathbf{X}_2}\mathbf{X}_1 &= \left(\frac{a^2-b^2+c^2}{c^2}\right)\mathbf{X}_3 & \nabla_{\mathbf{X}_2}\mathbf{X}_3 &= \left(\frac{-a^2+b^2-c^2}{a^2}\right)\mathbf{X}_1\\
\nabla_{\mathbf{X}_3}\mathbf{X}_1 &= \left(\frac{-a^2-b^2+c^2}{b^2}\right)\mathbf{X}_2 & \nabla_{\mathbf{X}_3}\mathbf{X}_2 &= \left(\frac{a^2+b^2-c^2}{a^2}\right)\mathbf{X}_1.
\end{aligned}$$

The values of the curvature tensor R on the basis vectors $\mathbf{X}_i$ are listed below.

$$R(\mathbf{X}_i, \mathbf{X}_j)\mathbf{X}_k = 0 \quad \text{if } i, j, k \text{ distinct}$$
$$R(\mathbf{X}_1, \mathbf{X}_2)X_1 = \frac{c^4-(a^2-b^2)^2-2c^2(a^2+b^2-c^2)}{b^2c^2}\mathbf{X}_2$$

$$R(\mathbf{X}_1,\mathbf{X}_2)X_2 = \frac{(a^2-b^2)^2-c^4+2c^2(a^2+b^2-c^2)}{a^2c^2}\mathbf{X}_1$$
$$R(\mathbf{X}_1,\mathbf{X}_3)X_1 = \frac{b^4-(a^2-c^2)^2-2b^2(a^2-b^2+c^2)}{b^2c^2}\mathbf{X}_3$$
$$R(\mathbf{X}_1,\mathbf{X}_3)X_3 = \frac{(a^2-c^2)^2-b^4+2b^2(a^2-b^2+c^2)}{a^2b^2}\mathbf{X}_1$$
$$R(\mathbf{X}_2,\mathbf{X}_1)X_1 = \frac{(a^2-b^2)^2-c^4+2c^2(a^2+b^2-c^2)}{b^2c^2}\mathbf{X}_2$$
$$R(\mathbf{X}_2,\mathbf{X}_1)X_2 = \frac{c^4-(a^2-b^2)^2-2c^2(a^2+b^2-c^2)}{a^2c^2}\mathbf{X}_1$$
$$R(\mathbf{X}_2,\mathbf{X}_3)X_2 = \frac{a^4-(b^2-c^2)^2+2a^2(a^2-b^2-c^2)}{a^2c^2}\mathbf{X}_3$$
$$R(\mathbf{X}_2,\mathbf{X}_3)X_3 = \frac{(b^2-c^2)^2-a^4+2a^2(b^2+c^2-a^2)}{a^2b^2}\mathbf{X}_2$$
$$R(\mathbf{X}_3,\mathbf{X}_1)X_1 = \frac{(a^2-c^2)^2-b^4+2b^2(a^2-b^2+c^2)}{b^2c^2}\mathbf{X}_3$$
$$R(\mathbf{X}_3,\mathbf{X}_1)X_3 = \frac{b^4-(a^2-c^2)^2-2b^2(a^2-b^2+c^2)}{a^2b^2}\mathbf{X}_1$$
$$R(\mathbf{X}_3,\mathbf{X}_2)X_2 = \frac{(b^2-c^2)^2-a^4-2a^2(a^2-b^2-c^2)}{a^2c^2}\mathbf{X}_3$$
$$R(\mathbf{X}_3,\mathbf{X}_2)X_3 = \frac{a^4-(b^2-c^2)^2-2a^2(b^2+c^2-a^2)}{a^2b^2}\mathbf{X}_2.$$

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