ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 43, Number 1, 2013

A NOTE ON SYMMETRY IN THE VANISHING OF EXT

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ABSTRACT. In [**2**] Avramov and Buchweitz proved that for finitely generated modules M and N over a complete intersection local ring R, $\mathrm{Ext}^i_R(M,N) = 0$ for all $i \gg 0$ implies $\mathrm{Ext}^i_R(N,M) = 0$ for all $i \gg 0$. In this note we give some generalizations of this result. Indeed we prove the above-mentioned result when (1) M is finitely generated and N is arbitrary, (2) M is arbitrary and N has finite length and (3) M is complete and N is finitely generated.

1. Introduction. Throughout the paper, R is assumed to be a commutative Noetherian ring with unity and $\dim(R) < \infty$. When R is a local ring, for each R-module M, $\widehat{M}$ denotes the completion of M with respect to the maximal ideal.

In [**2**, Theorem III] Avramov and Buchweitz proved that for finitely generated modules M and N over a complete intersection local ring R, $\mathrm{Ext}^i_R(M,N) = 0$ for all $i \gg 0$ implies $\mathrm{Ext}^i_R(N,M) = 0$ for all $i \gg 0$. They were interested in determining a class of local rings which satisfy this property. Then Huneke and Jorgensen [**6**] defined a class of Gorenstein local rings, which they called AB rings, and they showed that AB rings satisfy the above-mentioned property (see [**6**, Theorem 4.1]).

Using the notation of [**1**], for given nonzero R-modules M and N, we define $p^R(M,N)$ to be

$$p^R(M,N) = \sup\{i \in \mathbf{N} \mid \mathrm{Ext}^i_R(M,N) \neq 0\}.$$

Following [**6**], define the Ext-index of ring R, denoted by Ext $-$ index(R), to be the supremum of finite values of $p^R(M,N)$ for finitely generated R-modules M and N. Furthermore, R is called an AB ring

2010 AMS *Mathematics subject classification.* Primary 13H10, 13D07, 13D02.
Keywords and phrases. Complete intersection ring, complete module, Gorenstein ring.
This research was in part supported by a grant from IPM (No. 88130212).
The first author is the corresponding author.
Received by the editors on July 3, 2009, and in revised form on June 3, 2010.
DOI:10.1216/RMJ-2013-43-1-329

if it is a Gorenstein local ring of finite Ext-index. Notice that finiteness of the Ext-index is a stronger condition than Auslander's condition on vanishing of cohomology that has been studied for several years and recently is discussed in [**4**] (rings that satisfy Auslander's condition, are called AC rings).

Our aim in this note is to give some generalizations of the above-mentioned result of Avramov and Buchweitz.

In Section 2 of this paper we introduce a special class of AB rings and show that every complete intersection local ring belongs to this class. Then we show the following theorem:

Theorem A. *Let R be a d-dimensional complete intersection local ring. Assume that M and N are two R-modules such that M is finitely generated and N is arbitrary. Then*:

$$\operatorname{Ext}^i_R(M,N)=0 \quad \text{for all } i \gg 0 \Longrightarrow \operatorname{Ext}^i_R(N,M)=0 \text{ for all } i>d.$$

In Section 3 we are concerned with the property of symmetry in the vanishing of Ext over complete intersection local rings when the module which appears on the left-hand side is not necessarily finitely generated and the right-hand side module is finitely generated. As we see in [**9**], it is a general feeling that completeness is a kind of finiteness condition. In this direction we prove the following theorem:

Theorem B. *Suppose that R is a d-dimensional complete intersection local ring and M, N are two R-modules. If either M is of finite length and N is arbitrary, or M is finitely generated and N is complete, then*:

$$\operatorname{Ext}^i_R(N,M)=0 \quad \text{for all } i \gg 0 \Longrightarrow \operatorname{Ext}^i_R(M,N)=0 \text{ for all } i \gg 0.$$

2. Preliminaries and Theorem A. Let R be a Gorenstein local ring and M a finitely generated R-module. Let M^* denote the dual R-module $\operatorname{Hom}_R(M,R)$. If M is a maximal Cohen-Macaulay (MCM for short) R-module, then there exists a long exact sequence

$$\mathcal{C}(M):\cdots\longrightarrow F_2\xrightarrow{\partial_2}F_1\xrightarrow{\partial_1}F_0\xrightarrow{\partial_0}F_{-1}\xrightarrow{\partial_{-1}}F_{-2}\xrightarrow{\partial_{-2}}F_{-3}\xrightarrow{\partial_{-3}}\cdots$$

of finitely generated free R-modules such that $M = \operatorname{Ker} \partial_{-1}$ (see [**6**]). Define the syzygies of M by $M_i = \operatorname{Ker} \partial_{i-1}$ for every integer i.

Lemma 2.1. *Let R be a Gorenstein local ring. Suppose that M is an* MCM *R-module and N is an arbitrary R-module. Then for fixed $t \geq 3$ and for $1 \leq i \leq t-2$ we have*

$$\operatorname{Ext}^i_R(M_{-t}, N) \cong \operatorname{Tor}^R_{t-i-1}(M^*, N).$$

Before proving this, we should remark that this lemma was shown in [**6**, Lemma 1.1] when N is a finitely generated R-module.

Proof. Let $t \geq 3$ be an integer. By the above explications, there exists an exact sequence of R-modules

$$0 \longrightarrow M \longrightarrow F_{-1} \xrightarrow{\partial_{-1}} F_{-2} \xrightarrow{\partial_{-2}} \cdots \longrightarrow F_{-t} \longrightarrow M_{-t} \longrightarrow 0,$$

where each F_i is a finitely generated free R-module. Thus, we have an exact sequence

$$F^*_{-t} \xrightarrow{\partial^*_{-t+1}} \cdots \xrightarrow{\partial^*_{-1}} F^*_{-1} \longrightarrow M^* \longrightarrow 0$$

of R-modules. Therefore, the complexes

$$0 \longrightarrow \operatorname{Hom}_R(M_{-t}, N) \longrightarrow \operatorname{Hom}_R(F_{-t}, N) \longrightarrow \cdots \longrightarrow \operatorname{Hom}_R(F_{-1}, N)$$

and

$$F^*_{-t} \otimes_R N \longrightarrow \cdots \longrightarrow F^*_{-1} \otimes_R N \longrightarrow M^* \otimes_R N \longrightarrow 0$$

of R-modules exist. Since each F_i is a finitely generated free R-module, the natural maps

$$h_i : \operatorname{Hom}_R(F_i, R) \otimes_R N \longrightarrow \operatorname{Hom}_R(F_i, N)$$

given by $h_i(f \otimes n) = \{a \mapsto f(a)n\}$ are isomorphisms. For each i it is easy to check that the diagram

$$\begin{tikzcd}
\operatorname{Hom}_R(F_i, R) \otimes_R N \arrow[r, "\partial^*_i \otimes N"] \arrow[d, "h_i"'] & \operatorname{Hom}_R(F_{i+1}, R) \otimes_R N \arrow[d, "h_i+1"] \\
\operatorname{Hom}_R(F_i, N) \arrow[r, "\operatorname{Hom}_R(\partial_{i+1}, N)"'] & \operatorname{Hom}_R(F_{i+1}, N)
\end{tikzcd}$$

is commutative. Hence for $1 \le i \le t-2$ we have

$$\begin{aligned}
&\operatorname{Ext}^i_R(M_{-t}, N)\\
&= \mathrm{H}\big(\operatorname{Hom}_R(F_{-t+i-1}, N) \to \operatorname{Hom}_R(F_{-t+i}, N) \to \operatorname{Hom}_R(F_{-t+i+1}, N)\big)\\
&\cong \mathrm{H}\big(F^*_{-t+i-1} \otimes_R N \longrightarrow F^*_{-t+i} \otimes_R N \longrightarrow F^*_{-t+i+1} \otimes_R N\big)\\
&= \operatorname{Tor}^R_{t-i-1}(M^*, N). \qquad \square
\end{aligned}$$

Definition 2.2. Set $\xi(R)$ to be the supremum of finite values of $p^R(M,N)$ where M and N are R-modules and M is finitely generated, i.e.,

$$\xi(R) = \sup\big\{p^R(M,N) \mid p^R(M,N) < \infty \text{ where } M \text{ is a finitely generated } R\text{-module}\big\}.$$

We say that the ring R has finite ξ (or is of finite ξ) if it satisfies $\xi(R) < \infty$.

The following proposition contains some obvious properties of this type of ring.

Proposition 2.3. (1) *Suppose that R is a ring with $\xi(R) < \infty$. Assume that x is a nonzero divisor on R. Then $\xi(R/xR) < \infty$.*

(2) *If R is a d-dimensional Gorenstein local ring with $\xi(R) < \infty$, then $\xi(R) = d$.*

(3) *Every complete intersection local ring $(R, \mathfrak{m})$ has finite ξ.*

(4) *Every Gorenstein local ring with finite ξ is an* AB *ring.*s

(5) *Suppose that R is a ring with finite ξ. Then, for every $\mathfrak{p} \in$ Spec(R), $R_{\mathfrak{p}}$ is of finite ξ.*

The proofs of (1), (2) and (3) are completely similar to the proofs of [**6**, Propositions 3.3 (1), 3.2 and Corollary 3.5], respectively. However, we give below the proofs for the convenience of the reader.

Proof. (1) Set $n = \xi(R)$. Suppose that M is a finitely generated R/xR-module and N is an arbitrary R/xR-modules such that $\operatorname{Ext}^i_{R/xR}(M,N) = 0$ for all $i \gg 0$. By [**8**, 11.65], we have the following change of rings long exact sequence

$$\begin{aligned}
0 \quad &\longrightarrow \operatorname{Ext}^1_{R/xR}(M,N) \longrightarrow \operatorname{Ext}^1_R(M,N) \longrightarrow \operatorname{Ext}^0_{R/xR}(M,N)\\
&\longrightarrow \operatorname{Ext}^2_{R/xR}(M,N) \longrightarrow \operatorname{Ext}^2_R(M,N) \longrightarrow \operatorname{Ext}^1_{R/xR}(M,N)\\
&\longrightarrow \cdots
\end{aligned}$$

and this gives $\mathrm{Ext}^i_R(M,N)=0$ for all $i\gg 0$. By our assumption we have $\mathrm{Ext}^i_R(M,N)=0$ for all $i>n$. Using again the above change of rings' long exact sequence we obtain $\mathrm{Ext}^i_{R/xR}(M,N)\cong \mathrm{Ext}^{i+2}_{R/xR}(M,N)$ for $i>n-1$. Since by the assumption $\mathrm{Ext}^i_{R/xR}(M,N)=0$ for all $i\gg 0$, we have $\mathrm{Ext}^i_{R/xR}(M,N)=0$ for all $i>n-1$ and this shows that $\xi(R/xR)\le n-1$.

(2) Since $\mathrm{id}\,(R)=d$, there exists a finitely generated R-module K such that $\mathrm{Ext}^d_R(K,R)\neq 0$ and $\mathrm{Ext}^i_R(K,R)=0$ for all $i>d$. Thus, $\xi(R)\ge d$.

Suppose that $\xi(R)>d$. So there exist a finitely generated R-module M and an arbitrary R-module N such that $\mathrm{Ext}^{\xi(R)}_R(M,N)\neq 0$ and $\mathrm{Ext}^i_R(M,N)=0$ for all $i>\xi(R)$. Since M_d is a maximal Cohen-Macaulay R-module, for all $i>d$ we have the isomorphisms

$$\mathrm{Ext}^{i+1}_R((M_d)_{-d-1},N)\cong \mathrm{Ext}^{i-d}_R(M_d,N)\cong \mathrm{Ext}^i_R(M,N).$$

Thus, $\mathrm{Ext}^{\xi(R)+1}_R((M_d)_{-d-1},N)\neq 0$ and $\mathrm{Ext}^i_R((M_d)_{-d-1},N)=0$ for all $i>\xi(R)+1$, which contradicts the definition of $\xi(R)$.

(3) Suppose that M and N are two R-modules such that M is finitely generated, N is arbitrary and $\mathrm{Ext}^i_R(M,N)=0$ for all $i\gg 0$. Since $R\hookrightarrow \widehat{R}$ is a faithfully flat homomorphism, we have $\mathrm{Ext}^i_R(M,N)=0$ for all $i\gg 0$ if and only if $\mathrm{Ext}^i_{\widehat{R}}(\widehat{M},N\otimes_R\widehat{R})=0$ for all $i\gg 0$. So we can suppose that R is complete, i.e., $R=S/(x_1,\dots,x_n)$ where S is local regular and $x_1,\dots,x_n$ is a S-sequence. Every finitely generated S-module has finite projective dimension $\le \dim(S)$ and this shows that $\xi(S)\le\dim(S)$. Now the assertion follows from (1).

(4) is trivial.

(5) Suppose that M is a finitely generated $R_{\mathfrak p}$-module and N is an arbitrary $R_{\mathfrak p}$-module such that $\mathrm{Ext}^i_{R_{\mathfrak p}}(M,N)=0$ for all $i\gg 0$. Write $M=R_{\mathfrak p}y_1+\cdots+R_{\mathfrak p}y_t$. Let $M'=Ry_1+\cdots+Ry_t$. We have $M_{\mathfrak p}\cong M\cong M'_{\mathfrak p}$. Thus, if $\mathbf{F}.\to M'\to 0$ is a free resolution for M' as an R-module, then $\mathbf{F}.\otimes_R R_{\mathfrak p}\to M'_{\mathfrak p}\to 0$ is a free resolution for $M'_{\mathfrak p}$ as an $R_{\mathfrak p}$-module. So, we have $\mathrm{Ext}^i_R(M',N)\cong\mathrm{Ext}^i_R(M',\mathrm{Hom}_{R_{\mathfrak p}}(R_{\mathfrak p},N))\cong \mathrm{Ext}^i_{R_{\mathfrak p}}(M,N)$. Therefore, $\mathrm{Ext}^i_R(M',N)=0$ for all $i\gg 0$. By the assumption, there exists an integer $c\ge 0$ such that $\mathrm{Ext}^i_R(M',N)=0$ for all $i>c$. Thus, $\mathrm{Ext}^i_{R_{\mathfrak p}}(M,N)=0$ for all $i>c$ and this shows that $\xi(R_{\mathfrak p})\le c$. $\square$

Remark 2.4. (1) Suppose that R and S are two rings and T is an additive contravariant left exact functor from the category of R-modules to the category of S-modules. Let $\mathcal{Q} \to M \to 0$ be a left resolution for the R-module M such that for all $i > 0$ and $j \geq 0$ we have $(\mathcal{R}^i T)(Q_j) = 0$, where $\mathcal{R}^i T$ is the ith right derived functor of T. Then for all $i \geq 0$ we have $(\mathcal{R}^i T)(M) \cong \mathrm{H}^i(T(\mathcal{Q}))$.

(2) Let R be a ring and $0 \to M' \to M \to M'' \to 0$ be an exact sequence of R-modules such that $\mathrm{id}_R(M) < \infty$. Then for every R-module N we have $\mathrm{Ext}^i_R(N, M) = 0$ for $i > \mathrm{id}_R(M)$. Thus, using the long exact sequence

$$\mathrm{Ext}^i_R(N, M) \longrightarrow \mathrm{Ext}^i_R(N, M'') \longrightarrow \mathrm{Ext}^{i+1}_R(N, M') \longrightarrow \mathrm{Ext}^{i+1}_R(N, M),$$

we obtain that $\mathrm{Ext}^i_R(N, M'') \cong \mathrm{Ext}^{i+1}_R(N, M')$ for all $i > \mathrm{id}_R(M)$. Consequently, if there exists an integer c (respectively h) such that $\mathrm{Ext}^i_R(N, M'') = 0$ for all $i > c$ (respectively $\mathrm{Ext}^i_R(N, M') = 0$ for all $i > h$), then $\mathrm{Ext}^i_R(N, M') = 0$ for all $i > \sup\{c + 1, \mathrm{id}_R(M)\}$ (respectively $\mathrm{Ext}^i_R(N, M'') = 0$ for all $i > \sup\{h - 1, \mathrm{id}_R(M)\}$).

Theorem 2.5. *Let R be a d-dimensional Gorenstein local ring with $\xi(R) < \infty$. Assume that M and N are two R-modules such that M is finitely generated and N is arbitrary. Then:*

$$\mathrm{Ext}^i_R(M, N) = 0 \quad \text{for all } i \gg 0 \Longrightarrow \mathrm{Ext}^i_R(N, M) = 0 \text{ for all } i > d.$$

Proof. Let L be the dth syzygy of M in a free resolution. We know that L is an MCM R-module and $\mathrm{Ext}^i_R(M, N) \cong \mathrm{Ext}^{i-d}_R(L, N)$ for all $i > d$. This shows that $\mathrm{Ext}^i_R(L, N) = 0$ for all $i \gg 0$. Thus, for each $t \geq 1$, $\mathrm{Ext}^i_R(L_{-t}, N) = 0$ for all $i \gg 0$. Since $\xi(R) < \infty$, then for each $t \geq 1$ and $i > d$, $\mathrm{Ext}^i_R(L_{-t}, N) = 0$. On the other hand $\mathrm{Ext}^i_R(L_{-t}, N) \cong \mathrm{Ext}^1_R(L_{i-t-1}, N)$ for all $i \geq 1$. Thus, for each $t \geq 1$ and $i > d$, $\mathrm{Ext}^1_R(L_{i-t-1}, N) = 0$. Now by suitable changing of i and t, we will have $\mathrm{Ext}^1_R(L_{-t'}, N) = 0$ for each $t' \geq 1$. Therefore, by Lemma 2.1, $\mathrm{Tor}^R_{t'-2}(L^*, N) = 0$ for each $t' \geq 3$.

Therefore, if $\mathbf{F}. \to N \to 0$ is a free resolution for N, then $\mathbf{F}. \otimes_R L^* \to N \otimes_R L^* \to 0$ is an exact sequence. Also, L^* is an MCM R-module. Thus, for $i \geq 1$ and every free R-module F, $\mathrm{Ext}^i_R(F \otimes_R L^*, R) = 0$. So by Remark 2.4, for $i \geq 1$ we have $\operatorname{Ext}^i_R(N \otimes_R L^*, R) = \mathrm{H}^i(\operatorname{Hom}_R(\mathbf{F}. \otimes_R L^*, R))$. Hence for $i \geq 1$ we get the following isomorphisms

$$\begin{aligned} \operatorname{Ext}^i_R(N \otimes_R L^*, R) &\cong \mathrm{H}^i(\operatorname{Hom}_R(\mathbf{F}., \operatorname{Hom}_R(L^*, R))) \\ &= \mathrm{H}^i(\operatorname{Hom}_R(\mathbf{F}., L^{**})) \\ &\cong \mathrm{H}^i(\operatorname{Hom}_R(\mathbf{F}., L)) \\ &= \operatorname{Ext}^i_R(N, L). \end{aligned}$$

But R is Gorenstein, so $\operatorname{Ext}^i_R(N \otimes_R L^*, R) = 0$ for $i > d$. Therefore, $\operatorname{Ext}^i_R(N, L) = 0$ for $i > d$. Now since $\operatorname{id}(R) = d$, by Remark 2.4 (2) we obtain that $\operatorname{Ext}^i_R(N, M) = 0$ for $i > d$. $\square$

3. Theorem B. Let $(R, \mathfrak{m})$ be a local ring and $E(R/\mathfrak{m})$ the injective envelope of the residue class field $R/\mathfrak{m}$. Recall that the Matlis dual of an R-module T is $\operatorname{Hom}_R(T, E(R/\mathfrak{m}))$ and is denoted by $T^\vee$. We say that T is Matlis reflexive if $T^{\vee\vee} \cong T$. Note that if T has finite length, then T is Matlis reflexive. Furthermore, we have the following isomorphisms for R-modules V and W:

$$\operatorname{Tor}^R_i(V, W)^\vee \cong \operatorname{Ext}^i_R(V, W^\vee)$$

and

$$\operatorname{Ext}^i_R(V, W)^\vee \cong \operatorname{Tor}^R_i(V, W^\vee) \quad \text{when } V \text{ is finitely generated.}$$

Proposition 3.1. *Suppose that $(R, \mathfrak{m})$ is a d-dimensional Gorenstein local ring with finite ξ. Then for every R-modules M and N, where M has finite length and N is arbitrary, we have*

$$\operatorname{Ext}^i_R(N, M) = 0 \quad \textit{for all } i \gg 0 \Longrightarrow \operatorname{Ext}^i_R(M, N) = 0 \textit{ for all } i > d.$$

Proof. We have

$$\operatorname{Ext}^i_R(N, M) \cong \operatorname{Ext}^i_R(N, M^{\vee\vee}) \cong \operatorname{Tor}^R_i(N, M^\vee)^\vee \cong \operatorname{Ext}^i_R(M^\vee, N^\vee).$$

Thus, by assumption and Theorem 2.5, $\operatorname{Ext}^i_R(N^\vee, M^\vee) = 0$ for all $i > d$. Since $\operatorname{Ext}^i_R(N^\vee, M^\vee) \cong \operatorname{Tor}^R_i(N^\vee, M)^\vee$, we have $\operatorname{Tor}^R_i(N^\vee, M) = 0$

for all $i > d$. On the other hand $\mathrm{Tor}_i^R(N^\vee, M) \cong \mathrm{Ext}_R^i(M,N)^\vee$. Therefore, $\mathrm{Ext}_R^i(M,N) = 0$ for all $i > d$. □

By Theorem 2.5 and Proposition 3.1 we have the following corollary.

Corollary 3.2. *Let R be an Artinian Gorenstein local ring with $\xi(R) < \infty$. Assume that M and N are two R-modules where M is finitely generated and N is arbitrary. Then*:

$$\mathrm{Ext}_R^i(N,M) = 0 \quad \text{for all } i \gg 0 \Longrightarrow \mathrm{Ext}_R^i(M,N) = 0 \text{ for all } i > 0.$$

Theorem 3.3. *Suppose that R is a d-dimensional Gorenstein local ring with $\xi(R) < \infty$. Assume that M is a finitely generated R-module and N is a complete R-module. Then*:

$$\mathrm{Ext}_R^i(N,M) = 0 \quad \text{for all } i \gg 0 \Longrightarrow \mathrm{Ext}_R^i(M,N) = 0 \text{ for all } i \gg 0.$$

To prove this theorem, we need the following preliminaries.

Definition 3.4 [**9**]. Let $(R, \mathfrak{m})$ be a local ring and N an arbitrary R-module. Let $\tau_N : N \to \widehat{N}$ be the natural morphism. We say that N is quasi-complete if τ_N is surjective and N is separated if τ_N is injective. Now N is complete when τ_N is bijective.

Remark 3.5. Suppose that $(R, \mathfrak{m})$ is a local ring and N is an arbitrary R-module. Let $0 \to K \to L \to L/K \to 0$ be an exact sequence of R-modules. From [**7**, Section 8], recall that

(1) N is separated if and only if $\cap_n \mathfrak{m}^n N = 0$ for all $n \in \mathbf{N} \cup \{0\}$.

(2) L/K is separated if and only if K is closed in L.

(3) Using [**9**, 1.2, Corollary] and (2), we get that if K is closed in L and L is quasi-complete then L/K is complete.

(4) Suppose that J is a set of indices and $\{M_n\}_{n\in J}$ is a set of R-modules. By [**9**, 9.4], the completion of $M = \oplus_{n\in J} M_n$ is

$$\widehat{M} = \Big\{(m_n)_{n\in J} \in \prod_{n\in J} \widehat{M_n} \,\Big|\, \text{ for all } s, \text{ but finitely many } m_n \text{ belong to } \mathfrak{m}^s\widehat{M_n}\Big\}.$$

Lemma 3.6. *Let $(R, \mathfrak{m})$ be a local ring. Then every complete flat R-module is the completion of a free R-module.*

We remark that the converse of this lemma is also true (see [**9**, 2.4]).

Proof. Suppose that F is a flat R-module. Then by [**10**, Proposition 2.1.12 (i)] there exists a free R-submodule $L \subseteq F$ such that the natural injection $\rho : L \to F$ is pure (i.e., $\rho \otimes Id_H : L \otimes_R H \to F \otimes_R H$ is injective for every R-module H) and L is dense in the $\mathfrak{m}$-adic topology of F (i.e., $\cap_{n \geq 1}(L + \mathfrak{m}^n F) = F$ or $L + \mathfrak{m}^n F = F$ for all n). This implies that $L/\mathfrak{m}^n L \cong F/\mathfrak{m}^n F$ for all n. Therefore, when F is a complete flat R-module we have $F \cong \widehat{L}$. □

Lemma 3.7. *Let $(R, \mathfrak{m})$ be a local ring and $x \in \mathfrak{m}$ is a nonzero divisor on R. If F is a complete flat R-module, then F/xF is complete, i.e., xF is closed in F.*

Proof. It is mentioned in Lemma 3.6 that F is the completion of a free R-module L. As we see in Remark 3.5 (4), $\widehat{L}$ is a submodule of $S = \prod_{n \in J} \widehat{R_n}$ where, for each n, $\widehat{R_n} = \widehat{R}$. But $S/xS \cong R/xR \otimes_R S \cong \prod_{n \in J}(\widehat{R_n}/x\widehat{R_n})$. Since $\widehat{R_n}/x\widehat{R_n}$ is complete, by [**9**, 1.5], $\prod_{n \in J}(\widehat{R_n}/x\widehat{R_n})$ is complete. Hence S/xS is complete and by Remark 3.5 (1), we have $\cap_{n \geq 0}\mathfrak{m}^n(S/xS) = 0$. On the other hand, by [**9**, 2.4], the map $\widehat{L}/x\widehat{L} \to S/xS$ is one to one. Now since $\cap_{n \geq 0}\mathfrak{m}^n(\widehat{L}/x\widehat{L}) \to \cap_{n \geq 0}\mathfrak{m}^n(S/xS)$ is one to one, we have $\cap_{n \geq 0}\mathfrak{m}^n(\widehat{L}/x\widehat{L}) = 0$. Thus, by Remark 3.5 (1), (2) and (3), $F/xF \cong \widehat{L}/x\widehat{L}$ is a complete module. □

Lemma 3.8. *Let $(R, \mathfrak{m})$ be a local ring and M a complete R-module in $\mathfrak{m}$-adic topology. Suppose that $x \in \mathfrak{m}$ is a nonzero divisor on both R and M. Let*

$$0 \longrightarrow T \longrightarrow F \longrightarrow M \longrightarrow 0$$

be an exact sequence of R-modules where F is a complete flat R-module. Then both T and T/xT are complete in their $\mathfrak{m}$-adic topology.

Before proving the lemma, we should remark that M/xM is not necessarily complete, because xM is not necessarily closed in M. The following is an example of A.M. Simon.

Example 3.9. Let $R = k[[X,Y,Z]]$, where k is a field. Put $M_n = R/(XY - Z^n)$, and let M be the completion of $\oplus_{n=1}^{\infty} M_n$ as described in Remark 3.5 (3). In fact

$$M = \Big\{ (m_n)_{n \geq 1} \in \prod_{n=1}^{\infty} M_n \mid \quad \text{for all } s, \\ \text{all but finitely many } m_n \text{ belong to } \mathfrak{m}^s M_n \Big\}.$$

Thus, $M \subset \prod_{n=1}^{\infty} M_n$. Note that X is regular on R and M. Denote the images of X, Y, Z in M_n with x_n, y_n, z_n. Let $w_t = (z_1, z_2^2, z_3^3, \ldots, z_t^t, 0, \ldots)$ for each t. We have that $w_t = X.v_t$, where $(v_t)_i = y_i$ if $i \leq t$ and $(v_t)_i = 0$ otherwise. Thus, $w_t \in XM$. The Cauchy sequence w_t has its limit in $M - XM$; indeed, we have

$$\lim_{t \to \infty} w_t = (z_1, z_2^2, z_3^3, \ldots, z_t^t, z_{t+1}^{t+1}, \ldots)$$

and $(z_1, z_2^2, z_3^3, \ldots, z_t^t, z_{t+1}^{t+1}, \ldots) = X(y_1, y_2, y_3, \ldots, y_t, \ldots)$ which is not in XM because by the above-mentioned structure of M, $(y_1, y_2, y_3, \ldots, y_t, \ldots)$ is not an element of M.

Proof. Since M is complete, T is closed in F and thus complete (see [**9**, 1.3, Proposition]). With our hypothesis, we also have an exact sequence

$$0 \longrightarrow T/xT \longrightarrow F/xF \longrightarrow M/xM \longrightarrow 0.$$

Thus, $xT = T \cap xF$ and xT is closed in T because $T \to F$ is continuous. Consequently, by Remark 3.5 (3), T/xT is complete. $\square$

Lemma 3.10 [**5**, page 85, Corollary 3.2.7]. *Suppose that S is a Noetherian ring with* $\dim(S) < \infty$. *Then for every flat S-module F we have* $\mathrm{pd}_S(F) \leq \dim(S)$.

Lemma 3.11 [**3**, page 79, 3.3.4]. *Let R be a Gorenstein local ring. Then an R-module X has finite flat dimension if and only if it has finite injective dimension.*

Lemma 3.12 [**9**, 1.5, Lemma]. *Let S be a ring and $\mathfrak{a}$ an ideal of S. Let M be a complete S-module in $\mathfrak{a}$-adic topology. For each S-module N and for some i, if* $\mathrm{Ext}_S^i(N,M) = \mathfrak{a}\mathrm{Ext}_S^i(N,M)$, *then* $\mathrm{Ext}_S^i(N,M) = 0$.

Now we can give the proof of Theorem 3.3:

Proof. We proceed by induction on d. The case $d = 0$ has been proved in a stronger form in 3.2.

Assume that $d \geq 1$. Suppose that $\mathbf{P}. \to M \to 0$ is a free resolution of M. Consider the short exact sequence $\Lambda : 0 \to M_1 \to P_0 \to M \to 0$. Since $\mathrm{id}_R(P_0) = d$, using the short exact sequence Λ, hypothesis and by Remark 2.4 (2), we have $\mathrm{Ext}^i_R(N, M_1) = 0$ for all $i \gg 0$.

On the other hand, by [**9**, 2.5, Proposition], there exists a complete flat resolution $\mathbf{F}. \to N \to 0$ for the R-module N. By Lemma 3.8, N_2 is a complete R-module. Also by Lemma 3.10, for all j we have $\mathrm{pd}_R(F_j) \leq d$. Using the exact sequences $\Omega_j : 0 \to N_j \to F_{j-1} \to N_{j-1} \to 0$ for $j = 1, 2$ and the fact that $\mathrm{pd}_R(F_{j-1}) \leq d$, we obtain $\mathrm{Ext}^i_R(N_2, M_1) = 0$ for all $i \gg 0$. Since $\operatorname{depth}(R) \geq 1$, there exists an element x of $\mathfrak{m}$ which is a non-zero divisor on R, M_1 and N_2. Thus, we have the long exact sequence

$$\mathrm{Ext}^i_R(N_2, M_1) \longrightarrow \mathrm{Ext}^{i+1}_R(N_2/xN_2, M_1) \longrightarrow \mathrm{Ext}^{i+1}_R(N_2, M_1)$$

obtained from the short exact sequence

$$0 \longrightarrow N_2 \xrightarrow{x.} N_2 \longrightarrow N_2/xN_2 \longrightarrow 0 \tag{‡}$$

By hypothesis, we have $\mathrm{Ext}^i_R(N_2/xN_2, M_1) = 0$ for all $i \gg 0$. Therefore, by [**7**, page 140, Lemma 2], $\mathrm{Ext}^i_{R/xR}(N_2/xN_2, M_1/xM_1) = 0$ for all $i \gg 0$.

Now R/xR is a $(d-1)$-dimensional Gorenstein local ring with finite ξ (see Proposition 2.3). Also by Lemma 3.8, all N_i/xN_i are complete for $i \geq 2$ and consequently by the inductive hypothesis we have $\mathrm{Ext}^i_{R/xR}(M_1/xM_1, N_2/xN_2) = 0$ for all $i \gg 0$. Therefore, again by [**7**, page 140, Lemma 2], $\mathrm{Ext}^i_R(M_1, N_2/xN_2) = 0$ for all $i \gg 0$. Using again the short exact sequence (‡), we obtain the long exact sequence

$$\begin{aligned}\mathrm{Ext}^i_R(M_1, N_2/xN_2) \longrightarrow \mathrm{Ext}^{i+1}_R(M_1, N_2) \xrightarrow{x.} &\mathrm{Ext}^{i+1}_R(M_1, N_2)\\ \longrightarrow &\mathrm{Ext}^{i+1}_R(M_1, N_2/xN_2).\end{aligned}$$

So, we have $\mathrm{Ext}^i_R(M_1, N_2) = x\mathrm{Ext}^i_R(M_1, N_2)$ for all $i \gg 0$. Therefore, by Lemma 3.12, $\mathrm{Ext}^i_R(M_1, N_2) = 0$ for all $i \gg 0$. Now, because

R is a Gorenstein ring, by Lemma 3.11, $\mathrm{id}_R(F_j) < \infty$. So, using the exact sequences Ω_j $(j = 1, 2)$ and by Remark 2.4 (2), we have $\mathrm{Ext}^i_R(M_1, N) = 0$ for all $i \gg 0$. Now using again the exact sequence Λ, we obtain $\mathrm{Ext}^i_R(M, N) = 0$ for all $i \gg 0$. □

As another application of Lemma 3.12, we close this note by proving the following proposition with the same method as above.

Proposition 3.13. *Let $(R, \mathfrak{m})$ be a d-dimensional Gorenstein complete local ring with finite ξ. Set*

$$\xi'(R) = \sup\{p^R(N, M) \mid p^R(N, M) < \infty \\ \text{where } M \text{ is finitely generated and } N \text{ is arbitrary}\}.$$

Then we have $\xi'(R) = d$.

Proof. By Corollary 3.2 and Theorem 2.5, the claim obviously holds for $d = 0$. Suppose that $d > 0$. Since $\mathrm{id}\,(R) = d$, there exists an R-module L such that $\mathrm{Ext}^d_R(L, R) \neq 0$ and $\mathrm{Ext}^i_R(L, R) = 0$ for all $i > d$. Thus, $\xi'(R) \geq d$.

Let M be a finitely generated R-module, and let N be an arbitrary R-module such that $\mathrm{Ext}^i_R(N, M) = 0$ for all $i \gg 0$. Since $\mathrm{id}\,(R) = d$, by Remark 2.4 (2), we can replace M and N by their first syzygies in their R-free resolutions. Thus, there exists a nonzero divisor x on R, M and N. Also using the short exact sequence $0 \to M \xrightarrow{x} M \to M/xM \to 0$, we obtain $\mathrm{Ext}^i_R(N, M/xM) = 0$ for all $i \gg 0$. Therefore, by [**7**, page 140, Lemma 2], $\mathrm{Ext}^i_{R/xR}(N/xN, M/xM) = 0$ for all $i \gg 0$. Now by the inductive hypothesis we have $\mathrm{Ext}^i_{R/xR}(N/xN, M/xM) = 0$ for all $i > d - 1$. Therefore, using again the above exact sequence, we have $\mathrm{Ext}^i_R(N, M) = x\mathrm{Ext}^i_R(N, M)$ for all $i > d$. But M is a complete R-module, so by Lemma 3.12, $\mathrm{Ext}^i_R(N, M) = 0$ for all $i > d$. This shows that $\xi'(R) \leq d$. □

Acknowledgments. The authors are grateful to professor A.M. Simon for Example 3.9 and her other valuable comments on Section 3 of this paper.

REFERENCES

1. T. Araya and Y. Yoshino, *Remarks on a depth formula, a grade inequality and a conjecture of Auslander*, Comm. Algebra **26** (1998), 3793–3806.

2. L.L. Avramov and R.-O. Buchweitz, *Support varieties and cohomology over complete intersections*, Invent. Math. **142** (2000), 285–318.

3. L.W. Christensen, *Gorenstein dimensions*, Lect. Notes Math. **1747**, Springer-Verlag, Berlin, 2000.

4. L.W. Christensen and H. Holm, *Algebras that satisfy Auslander's condition on vanishing of cohomology*, available at arXiv:0801.0401v2 [Math RA] 21 Jan 2009.

5. L. Gruson and M. Raynaud, *Critères de platitude et de projectivité Techniques de "platification" d'un module*, Invent. Math. **13** (1971), 1–89.

6. C. Huneke and D.A. Jorgensen, *Symmetry in the vanishing of Ext over Gorenstein rings*, Math. Scand. **93** (2003), 161–184.

7. H. Matsumura, *Commutative ring theory*, Cambr. Stud. Adv. Math. **8**, Cambridge University Press, Cambridge, 1989.

8. J. Rotman, *An introduction to homological algebra*, Academic Press, Inc., New York, 1979.

9. A.M. Simon, *Some homological properties of complete modules*, Math. Proc. Camb. Phil. Soc. **108** (1990), 231–246.

10. J.R. Strooker, *Homological questions in local algebra*, London Math. Soc. Lect. Note Ser. **145**, Cambridge University Press, Cambridge, 1990.

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