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# ON AN APPLICATION OF POINCARÉ SERIES

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ABSTRACT. Let $\varrho : G \hookrightarrow GL\,(V)$ be a faithful representation of a finite group $G$ over the field $\mathbf{F}$ and $V \cong \mathbf{F}^n$ be a $\mathbf{F}(G)$-module. In the nonmodular case, if $\mathbf{F}[V]^G$ is a polynomial algebra, then the finite dimensional $G$-representation $\mathbf{F}[V]_G$ is isomorphic to a regular representation, denoted as $\mathrm{Reg}_{\mathbf{F}}(G)$. In the modular case, this result is no longer valid. In this note, we explore the relationship between the isomorphism classes of $[\mathbf{F}[V]_G]$ and $[\mathrm{Reg}_{\mathbf{F}}(G)]$ in the Grothendieck group over any field of positive characteristic.

**1. Introduction.** Let $G$ be a finite group, and let $V$ be a vector space of dimension $n$ over a field $\mathbf{F}$ and $\mathbf{F}[V]$ denote the symmetric algebra of $V^*$, where $V^*$ is the dual of $V$. If $\{x_1, \dots , x_n\}$ is a $\mathbf{F}$-basis of $V^*$, we can identify $\mathbf{F}[V]$ with the graded polynomial algebra $\mathbf{F}[x_1, \dots , x_n]$. Let $G \hookrightarrow GL(V)$ be a faithful representation of $G$ over $\mathbf{F}$. The subalgebra $\mathbf{F}[V]^G = \{f \in \mathbf{F}[V] \mid \sigma f = f \mathrm{forall} \sigma \in G\} \subseteq \mathbf{F}[V]$ fixed under the $G$-action is called the ring of invariants of $G$. The ring $\mathbf{F}[V]^G$ is as $\mathbf{F}$-algebra finitely generated by at least $n$ generators. That is, there exists a homogeneous system of parameters, say $f_1, \dots , f_n$, in $\mathbf{F}[V]^G$ such that $\mathbf{F}[V]^G$ is a finitely generated module over $\mathbf{F}[f_1, \dots , f_n]$. Such generators $f_1, \dots , f_n$ are called primary invariants, the other generators secondary invariants. If $d_1, \dots , d_n$ are the degrees of primary invariants of $\mathbf{F}[V]^G$, then the product $\prod_{i=1}^n d_i$ is divisible by the order $|G|$ of $G$. The ideal, denoted as $h(G)$, in $\mathbf{F}[V]$ generated by all $G$-invariant homogeneous polynomials of strictly positive degree is a primary ideal and invariant under the action of $G$ on $\mathbf{F}[V]$ so $G$ acts on the graded quotient algebra $\mathbf{F}[V]_G = \mathbf{F}[V]/h(G)$ which is called the ring of coinvariants. The ring of coinvariants $\mathbf{F}[V]_G$ is a finite dimensional representation of $G$ and has the Krull dimension zero. As a convenient reference for invariant theory see [**5**].

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An $\mathbf{F}(G)$-module is called a *regular representation* of a group $G$ over a field $\mathbf{F}$, denoted as $\mathrm{Reg}_{\mathbf{F}}(G)$, if there is an element $v \in \mathrm{Reg}_{\mathbf{F}}(G)$ such that $\{\sigma \cdot v \mid \sigma \in G\}$ is a $\mathbf{F}$-basis of $\mathrm{Reg}_{\mathbf{F}}(G)$, where $\sigma \cdot v$ are all distinct. In the nonmodular case, i.e., the order of $G$ is prime to the characteristic of $\mathbf{F}$, it is well known that

**Theorem 1.1.** [**1**]. *Let $G \hookrightarrow GL(V)$ be a representation of $G$ over $\mathbf{F}$. If $\mathbf{F}[V]^G$ is a polynomial algebra and the order of $G$ is relatively prime to the characteristic of $\mathbf{F}$, then $\mathbf{F}[V]_G \cong \mathrm{Reg}_{\mathbf{F}}(G)$.*

However, in the modular case $\mathbf{F}[V]_G$ may not be a regular representation even though $\mathbf{F}[V]^G$ is polynomial. In this note we discuss the relationship between the isomorphic classes $[\mathbf{F}[V]_G]$ of the ring of coinvariants and $[\mathrm{Reg}_{\mathbf{F}}(G)]$ of regular representation in the Grothendieck group of finitely generated $\mathbf{F}(G)$-modules. Namely, we prove:

**Theorem 1.2.** *Let $G \hookrightarrow GL(V)$ be a representation of $G$ over $\mathbf{F}$. If $\mathbf{F}[V]_G$ is a complete intersection, then $[\mathbf{F}[V]_G]$ is a multiple of $[\mathrm{Reg}_{\mathbf{F}}(G)]$ in $R_{\mathbf{F}}(G)$.*

Recall that the ring of coinvariants $\mathbf{F}[V]_G$ is a complete intersection if the ideal $h(G)$ is generated by all primary invariants of $\mathbf{F}[V]^G$. In general, if the quotient $\mathbf{F}[V]_G$ is a complete intersection, the ring of invariants $\mathbf{F}[V]^G$ may not be polynomial, that is, $\mathbf{F}[V]^G$ may contain secondary invariants as generators.

**2. Modular Poincaré series.** Let $\mathbf{F}$ be a field of positive characteristic $p$. Recall that the Grothendieck group of finitely generated $\mathbf{F}(G)$-modules, denoted as $\mathrm{R}_{\mathbf{F}}(G)$, is an abelian group with isomorphic classes, denoted by $[-]$, of finitely generated $\mathbf{F}(G)$-modules as group elements and the relation $[M] = [L] + [N]$ holds for each exact sequence $0 \to L \to M \to N \to 0$ of finitely generated $\mathbf{F}(G)$-modules. Let $\mathrm{S}_{\mathbf{F}}(G)$ be the subset of $\mathrm{R}_{\mathbf{F}}(G)$ consisting of isomorphic classes of irreducible $\mathbf{F}(G)$-modules. An element $\sigma \in G$ is called $p$-regular if the order of $\sigma$ is relatively prime to $p$. We denote the set of all $p$-regular elements of $G$ by $G_{\mathrm{reg}}$. The elements of $G_{\mathrm{reg}}$ are also diagonalizable. Suppose that $\mathbf{F}$ is a splitting field for $G$ (i.e., each irreducible $\mathbf{F}(G)$-module is absolutely irreducible). For each $\mathbf{F}(G)$-module $M$ with $\dim_{\mathbf{F}}(M) = m$ the modular character $\chi_M$ of $M$ is defined by $\chi_M : G_{\text{reg}} \to \mathbf{F}$, $\sigma \mapsto \chi_M(\sigma) = \sum_{i=1}^{m} \lambda_i(\sigma) = \operatorname{trac}(\sigma)$, where $\lambda_1(\sigma), \dots, \lambda_m(\sigma)$ are eigenvalues to the endomorphism $\sigma$ of $M$. The modular character $\chi_M$ is a class function on $G_{\text{reg}}$ and $\chi_M(1) = \dim_{\mathbf{F}}(M)1_{\mathbf{F}}$, where $1_{\mathbf{F}}$ is the identity of the field $\mathbf{F}$. We extend the field $\mathbf{F}$ to its algebraic closure $\overline{\mathbf{F}}$ which contains the group $\mu_{\overline{\mathbf{F}}}$ of $k$th roots of unity, where $k$ is the least common multiple of the orders of elements of $G_{reg}$. And there is a cyclic group isomorphism $\phi : \mu_{\overline{\mathbf{F}}} \to \mu_{\mathbf{C}}$. Since all the eigenvalues $\lambda_1(\sigma), \dots, \lambda_m(\sigma)$ are elements in $\mu_{\overline{\mathbf{F}}}$, there are corresponding elements in $\mu_{\mathbf{C}}$. Thus, we may rewrite the definition of modular character $\chi_M$ as $\chi_M(\sigma) = \phi(\lambda_1(\sigma)) + \cdots + \phi(\lambda_m(\sigma)) \in \mathbf{C}$. Hence, the value $\chi_M(1)$ may identify with the number $\dim_{\mathbf{F}}(M)$. Let the inner products $\langle \chi_M, \chi_N \rangle := 1/|G| \sum_{\sigma \in G_{\text{reg}}} \chi_N(\sigma^{-1})\chi_M(\sigma)$ and $\langle M, N \rangle := \dim_{\mathbf{F}}(\operatorname{Hom}_{\mathbf{F}(G)}(M, N))$ for finitely generated $\mathbf{F}(G)$-modules $M$ and $N$. If $P$ is a finitely generated projective $\mathbf{F}(G)$-module, then $\langle M, P \rangle = \langle \chi_M, \chi_P \rangle$. Let $E_1$ and $E_2$ be two distinct irreducible $\mathbf{F}(G)$-modules with projective covers $P_{E_1}$ and $P_{E_2}$, respectively. Then there is an orthogonality relation $\langle P_{E_i}, E_j \rangle = \delta_{ij}$, $i, j = 1, 2$. Therefore, $\langle M, E_i \rangle = \langle P_{E_i}, M \rangle$ for an arbitrary finitely generated $\mathbf{F}(G)$-module $M$. Every irreducible isomorphic class $[E]$ in $S_{\mathbf{F}}(G)$ gives rise to a modular character $\chi_E$, which is called an irreducible modular character of $G$. If $M$ is a finite-dimensional $\mathbf{F}(G)$-module, then the element $[M]$ in $R_{\mathbf{F}}(G)$ decomposes uniquely as the direct sum of the $E_i$-isotypical components of $[M]$

$$[M] = \bigoplus_{[E_i] \in S_{\mathbf{F}}(G)} [M]|_{E_i},$$

where the $E_i$-isotypical component $[M]|_{E_i} \cong m_i[E_i]$ is the direct sum of all irreducible components of $[M]$ which are isomorphic to $[E_i]$ and $m_i$ is the multiplicity of $[E_i]$. As a reference for the character theory, see [**2, 4**].

To determine the multiplicity $m_i$ we reformulate the Poincaré series in terms of the Grothendieck group.

**Definition 2.1.** Let $S_{\mathbf{F}}(G) = \{[E_1], \dots, [E_k]\}$ be finite, let $P_{E_i}$ be the projective cover of $E_i$ for all $1 \le i \le k$, and let $[M] = \oplus_{i \ge 0}[M_i]$ be a graded finitely generated $\mathbf{F}(G)$-module in $R_{\mathbf{F}}(G)$ such that $[M_i] =$

$\langle \chi_{P_{E_1}}, \chi_{M_i} \rangle [E_1] \oplus \cdots \oplus \langle \chi_{P_{E_k}}, \chi_{M_i} \rangle [E_k]$, for all $i$. The modular Poincaré series of $[M]$ and $[M] \mid_E$ are defined by

$$\begin{aligned} P([M], t) &= \sum_{i=0}^{\infty} \sum_{j} \langle \chi_{P_{E_j}}, \chi_{M_i} \rangle \cdot \chi_{E_j}(1) \cdot t^i \\ P([M] \mid_E, t) &= \sum_{i=0}^{\infty} \sum_{j} \langle \chi_{P_{E_j}}, \chi_{M_i} \rangle \cdot \langle \chi_{E_j}, \chi_E \rangle \cdot t^i \\ &= \sum_{i=0}^{\infty} \sum_{j} \langle \chi_{P_{E_j}}, \chi_{M_i} \rangle \cdot t^i. \end{aligned}$$

In particular,

$$\begin{aligned} P([\mathbf{F}[V]]|_E, t) &= \frac{1}{|G|} \sum_{\sigma \in G_{\mathrm{reg}}} \chi_{P_E}(\sigma^{-1}) \cdot \sum_{i=0}^{\infty} \chi_{M_i}(\sigma) \cdot t^i \\ &= \frac{1}{|G|} \sum_{\sigma \in G_{\mathrm{reg}}} \chi_{P_E}(\sigma^{-1}) \cdot \frac{1}{\det(1 - \sigma^{-1} t)}. \end{aligned}$$

*Proof of Theorem* 1.2. If an extension field $\mathbf{K}$ of the field $\mathbf{F}$ is a splitting field for $G$, then $\langle M, N \rangle = \langle M_{\mathbf{K}}, N_{\mathbf{K}} \rangle$ for finitely generated (projective) $\mathbf{F}(G)$-modules $M$, $N$, where $M_{\mathbf{K}} = M \otimes_{\mathbf{F}} \mathbf{K}$. Thus, we verify the proof under the assumption that $\mathbf{F}$ is a splitting field for $G$. Assume that the ideal $h(G)$ is generated by primary invariants $f_1, \ldots, f_n$. Then $\mathbf{F}[V] \cong \mathbf{F}[f_1, \ldots, f_n] \otimes \mathbf{F}[V]_G$ as both an $\mathbf{F}[f_1, \ldots, f_n]$- and $\mathbf{F}(G)$-module. Thus, for an irreducible $\mathbf{F}(G)$-module $E$ the modular Poincaré series of the $E$-isotypical component $[\mathbf{F}[V]_G]|_E$ is

$$\begin{aligned} P([\mathbf{F}[V]_G]|_E, t) &= \frac{P([\mathbf{F}[V]]|_E, t)}{P(\mathbf{F}[f_1, \ldots, f_n], t)} \\ &= \frac{1}{|G|} \prod_{i=1}^{n} (1 - t^{d_i}) \\ &\qquad \sum_{\sigma \in G_{\mathrm{reg}}} \chi_{P_E}(\sigma^{-1}) \cdot \frac{1}{\det(1 - \sigma^{-1} t)}. \end{aligned}$$

Also,

$$P([\mathbf{F}[V]_G]|_E, t)|_{t=1} = \frac{1}{|G|}\chi_{P_E}(1)\prod_{i=1}^{n} d_i$$
$$= d\chi_{P_E}(1),$$

where $d = \prod_{i=1}^{n} d_i/|G|$ and $d_i = \deg f_i$ for all $i = 1, \dots, n$. Since $[\mathrm{Reg}_{\mathbf{F}}(G)] \cong \oplus_{[E]\in S_{\mathbf{F}}(G)}\chi_E(1)[E]$, the result holds. $\square$

**Corollary 2.2** [**3**]. *Let* $G \hookrightarrow GL(V)$ *be a representation of* $G$ *over* $\mathbf{F}$. *If* $\mathbf{F}[V]^G$ *is a polynomial algebra, then* $[\mathbf{F}[V]_G] \cong [\mathrm{Reg}_{\mathbf{F}}(G)]$ *in* $R_{\mathbf{F}}(G)$.

*Proof.* If $\mathbf{F}[V]^G \cong \mathbf{F}[f_1, \dots, f_n]$ is a polynomial ring and $d_i$ is the degree of $f_i$, then the product $\prod_{i=1}^{n} d_i = |G|$. Hence, the result. $\square$

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