ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 42, Number 6, 2012

PECULIARITIES IN POWER TYPE COMPARISON RESULTS FOR HALF-LINEAR DYNAMIC EQUATIONS

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ABSTRACT. We look for conditions guaranteeing that oscillatory properties of the half-linear dynamic equation

$$\left(r(t)|y^{\Delta}|^{p-1}\mathrm{sgn}\, y^{\Delta}\right)^{\Delta} + c(t)|y^{\sigma}|^{p-1}\mathrm{sgn}\, y^{\sigma} = 0, \quad p > 1,$$

are preserved when the power p is changed. In particular, we discuss discrepancies (which indeed occur) between the results on different time scales. We provide an example showing an optimality of one of the assumptions. We also present one generalization of a standard Sturm-Picone type comparison theorem and give an important note on the condition $1/r \in C_{\mathrm{rd}}$ which concerns general theory of second order dynamic equations. Many of our observations are new also in the differential and difference equations cases.

1. Introduction. Classical types of comparison results in the theory of half-linear dynamic equations, like the Sturm-Picone one or the Hille-Wintner one, deal with two equations of the form

$$\left(r(t)\Phi_p(y^{\Delta})\right)^{\Delta} + c(t)\Phi_p(y^{\sigma}) = 0, \tag{1}$$

which have different coefficients, see e.g., [**1, 3, 9, 10**]. In this paper, we present results of a different kind: Two equations of form (1) are compared, where their coefficients are the same, but the powers in nonlinearities differ. Among others, we may compare a nonlinear equation with a linear one. The nonlinearity is defined as $\Phi_\lambda(u) = |u|^{\lambda-1}\mathrm{sgn}\, u$, and in (1) we assume $p > 1$. A time scale $\mathbf{T}$ (i.e., a closed subset of $\mathbf{R}$) is unbounded from above. The half-linear dynamic equation (1) is assumed to have the coefficients $r(t) > 0$ and $c(t)$ defined on a time scale interval $[a, \infty)$, $a \in \mathbf{T}$, with $1/r$ and c being

2010 AMS *Mathematics subject classification.* Primary 34C10, 34N05, 39A10, 39A12, 39A13.
Keywords and phrases. Half-linear dynamic equation, time scale, comparison theorem.
Received by the editors on March 26, 2008, and in revised form on April 2, 2010.
DOI:10.1216/RMJ-2012-42-6-1995

rd-continuous on $[a, \infty)$. In the next section, we give an important note on the assumption '$1/r$ is rd-continuous'. Equation (1) covers a large variety of equations, in particular, a half-linear differential equation (provided $\mathbf{T} = \mathbf{R}$ in (1)), a half-linear difference equation (provided $\mathbf{T} = \mathbf{Z}$ in (1)), a linear dynamic equation (provided $p = 2$ in (1)), and a half-linear q-difference equation (provided $\mathbf{T} = \{q^k : k \in \mathbf{N}_0\}$ with $q > 1$ in (1)). Basic results on the existence and oscillation theory of (1) can be found in [**1, 9**].

The principal aim of this paper is to establish conditions guaranteeing that oscillatory properties of (1) are preserved when the power in the nonlinearity is changed. Moreover, we show the discrepancies between the results on different time scales: The statements with a "small" graininess require certain additional conditions on the coefficient r, which is not needed when the graininess is "sufficiently large." An example will be given demonstrating this fact for the differential equations case, including not just the essentiality but also an optimality of that condition. We will also show that, under the assumption $\mu(t) \geq 1$, the statement can be proved in a very general setting, where the coefficients can be nearly arbitrary.

Related comparison results may be found in [**6, 12, 14**] for the differential equations case and in [**10, 11**] for the dynamic equations case. We stress that many subsequent results are new in the standard discrete case ($\mathbf{T} = \mathbf{Z}$) and some of them are new even in the continuous case ($\mathbf{T} = \mathbf{R}$). For another type of nonlinear comparison in the theory of half-linear differential equations, see [**4, 14**].

The paper is organized as follows. In the next section we mention and recall some important facts and state preliminaries that are key to prove the main results. Comparison theorems, for both cases $\int^\infty r^{1/(1-p)}(s)\,\Delta s = \infty$ and $\int^\infty r^{1/(1-p)}(s)\,\Delta s < \infty$, are established in Sections 3 and 4, respectively. Section 5 presents improvements in special cases and discusses peculiarities and discrepancies which occur when we compar the results on different time scales. The paper concludes with an integral comparison theorem which generalizes the Sturm-Picone type result.

2. Preliminaries. We assume that the reader is familiar with the notion of time scales. Thus, note that just $\mathbf{T}$, σ, f^σ, μ, f^Δ,

$\int_a^b f(s)\,\Delta s$ and $C_{\rm rd}$ stand for time scale, forward jump operator, $f \circ \sigma$, graininess, delta derivative of f, delta integral of f from a to b, and the class of rd-continuous functions, respectively. Recall that, for instance, $f^\Delta(t) = f'(t)$ when $\mathbf{T} = \mathbf{R}$, $f^\Delta(t) = \Delta f(t)$ when $\mathbf{T} = \mathbf{Z}$, and $f^\Delta(t) = D_q f(t)$ when $\mathbf{T} = \{q^k : k \in \mathbf{N}_0\}$ with $q > 1$, where D_q denotes the Jackson derivative. See [**7**], which is the initiating paper of the time scale theory written by Hilger, and the monograph [**5**] by Bohner and Peterson containing a lot of information on time scale calculus. Time scale intervals will be denoted as usual real intervals, and from the context it will always be clear whether the interval under consideration is real or of time scale type.

We will proceed with some essentials of oscillation theory of (1). First note that we are interested only in nontrivial solutions of (1). We say that a solution y of (1) has a *generalized zero* at t in case $y(t) = 0$. If $\mu(t) > 0$, then we say that y has a *generalized zero* in $(t, \sigma(t))$ in case $y(t)y^\sigma(t) < 0$. A *solution* y of (1) is called *oscillatory* if it has infinitely many generalized zeros; note that the uniqueness of IVP excludes the existence of a cluster point which is less than ∞. Otherwise a solution is said to be *nonoscillatory.* In view of the fact that the Sturm type separation theorem extends to (1) (see e.g., [**9**]), we have the following equivalence: One solution of (1) is oscillatory if and only if every solution of (1) is oscillatory. Hence we may speak about *oscillation* or *nonoscillation of equation* (1). Fundamental results about qualitative theory of (1) can be found in [**1, 9**].

One may wonder why we assume $1/r \in C_{\rm rd}$ and not $r \in C_{\rm rd}$ as is usual. There are at least two reasons. First, since we want to assume conditions in terms of $\int r^{1/(1-p)}(s)\Delta s$, we need integrability of $r^{1/(1-p)}$ to be guaranteed. Note that $r \in C_{\rm rd}$ does not imply $1/r \in C_{\rm rd}$ in contrast to the usual continuity. Indeed, for $r \in C_{\rm rd}$, at a left-dense $t_0 \in \mathbf{T}$, it may happen that $\lim_{t\to t_0-} r(t) = 0$ and $r(t_0) > 0$ (the author thanks R. Šimon Hilscher for drawing his attention to such possible behavior). The second reason is again related to the above described behavior, but seems to be more serious. It goes back even to the basic theory of linear formally self-adjoint dynamic equations of the form $(r(t)y^\Delta)^\Delta + c(t)y^\sigma = 0$, see e.g., [**5**], which are usually considered under the assumptions $r, c \in C_{\rm rd}$ with $r \neq 0$ or $r > 0$. To show the solvability of such an equation, we rewrite it as a first order system, and then we use the existence theory for systems which utilizes, in particular, an rd-continuity of the right hand side. But, in our concrete case, one of the system coefficients has the form $1/r$, which may not be rd-continuous. Thus, not $r \in C_{\mathrm{rd}}$, but $1/r \in C_{\mathrm{rd}}$ is needed. A reader may see a parallel with the usual differential equations case, where the assumptions of the continuity of r and c is relaxed to the local Lebesgue integrability of $1/r$ and c. A similar observation holds also for half-linear and some other similar second order dynamic equations. In fact, this reasoning shows that the assumption $r \in C_{\mathrm{rd}}$ *should be corrected* to $1/r \in C_{\mathrm{rd}}$ *in dozens of existing works* (including the author's ones) which deal with such types of second order dynamic equations. See also [**8**], where the condition $\inf_{t\in[a,b]} |r(t)| > 0$ for all $b \in [a,\infty)$ was introduced, when deriving existence results for the equation $(r(t)y^{\Delta})^{\Delta} + f(t, x^{\sigma}) = 0$. This condition, under the assumption $r \in C_{\mathrm{rd}}$, is strictly related to $1/r \in C_{\mathrm{rd}}$. Finally, note that by a solution of (1), we mean a function y such that y and $r\Phi_p(y^{\Delta})$ are rd-continuously delta differentiable, and y satisfies (1).

A very important role in the oscillation theory of (1) is played by the so-called *Riccati technique*, described in the next lemma. Lemmata 2 and 3 are certain refinements of this method. First we introduce the function $\mathcal{S}$, which occurs in the Riccati type equation, by

$$\mathcal{S}(x,y,z) = \lim_{\lambda\to\mu} \frac{x}{\lambda}\left(1 - \frac{y}{\Phi_z(\Phi^{-1}{}_z(y) + \lambda\Phi^{-1}{}_z(x))}\right),$$

where $\Phi^{-1}{}_z$ stands for the inverse of Φ_z. The conjugate number to p is denoted as q, i.e., $1/p + 1/q = 1$. Note that $\Phi^{-1}{}_p = \Phi_q$. Observe that

$$\mathcal{S}(x,y,z)(t) = \begin{cases} \left\{\frac{z-1}{\Phi^{-1}{}_z(y)}|x|^{z/(z-1)}\right\}(t) & \text{at right dense } t, \\ \left\{\frac{x}{\mu}\left(1 - \frac{y}{\Phi_z(\Phi^{-1}{}_z(y)+\mu\Phi^{-1}{}_z(x))}\right)\right\}(t) & \text{at right-scattered } t. \end{cases}$$

Lemma 1 [**1, 9**]. *The following statements are equivalent*:

(i) *Equation* (1) *is nonoscillatory.*

(ii) *There is a function w satisfying*

$$w^{\Delta}(t) + c(t) + \mathcal{S}(w,r,p)(t) = 0 \tag{2}$$

and

$$\{\Phi^{-1}{}_p(r) + \mu\Phi^{-1}{}_p(w)\}(t) > 0 \tag{3}$$

for large t.

(iii) *There is a function w satisfying $w^\Delta(t) + c(t) + \mathcal{S}(w, r, p)(t) \le 0$ and* (3) *for large t.*

Under somewhat stronger assumptions, a solution of (2) satisfies an integral inequality (equation) and, moreover, can be effectively estimated.

Lemma 2. *Let $\int^\infty r^{1-q}(s)\,\Delta s = \infty$, $\int^\infty c(s)\,\Delta s$ converge, and $\int_t^\infty c(s)\,\Delta s \ge 0$ ($\not\equiv 0$) for large t. Then* (1) *is nonoscillatory if and only if there is a* (*positive*) *function w satisfying*

$$w(t) = \int_t^\infty c(s)\,\Delta s + \int_t^\infty \mathcal{S}(w, r, p)(s)\,\Delta s \tag{4}$$

for large t. In the if part, the equation can be replaced by the inequality $w(t) \ge \int_t^\infty c(s)\,\Delta s + \int_t^\infty \mathcal{S}(w, r, p)(s)\,\Delta s$. If, in addition, $c(t) \ge 0$, then $w(t) \le R_D^{1-p}(t, t_0)$ for large t, say $t > t_0$, where

$$R_D(t, t_0) = \int_{t_0}^t r^{1-q}(s)\,\Delta s. \tag{5}$$

Proof. The proof can be found in [**1, 11**]. Here we just note that the 'only if' part was proved there only in the case of inequality in (4). The proof of necessity with the equality required $c(t) \ge 0$ there. However, the necessity holds also under the more general sign condition $\int_t^\infty c(s)\,\Delta s \ge 0$. Indeed, nonoscillation of (1) implies the existence of u with $u(t) \ge \int_t^\infty c(s)\,\Delta s + \int_t^\infty \mathcal{S}(u, r, p)(s)\,\Delta s =: u^*(t)$. Now we can introduce the set $\Omega = \{v \in {C_{\mathrm{rd}}}^B[a, \infty) : 0 \le v(t) \le u^*(t)\}$ and the operator $\mathcal{T} : \Omega \to {C_{\mathrm{rd}}}^B[a, \infty)$ defined by $\mathcal{T}(v)(t) = \int_t^\infty c(s)\,\Delta s + \int_t^\infty \mathcal{S}(v, r, p)(s)\,\Delta s$, where ${C_{\mathrm{rd}}}^B$ is the space of all rd-continuous functions $f : [a, \infty) \to \mathbf{R}$ such that $\sup_{t \in [a,\infty)} |f(t)| < \infty$. The norm is defined as $\|f\| = \sup_{t\in[a,\infty)} |f(t)|$. Applying the Schauder fixed point theorem, it follows that there is a $w \in \Omega$ such that $\mathcal{T}(w) = w$. Note that, in showing the assumptions of the Schauder theorem are satisfied, we use the monotone nature of $\mathcal{S}$ with respect to the first variable, the Ascoli-Arzela type result (see [**10**]) and the Lebesgue type dominated convergence theorem (see [**2**]). Alternatively, we can apply the function sequence technique from [**11**]: Define the sequence $\{\varphi_k(t)\}$ by $\varphi_0(t) = \int_t^\infty c(s)\,\Delta s$, $\varphi_k(t) = \varphi_0(t) + \int_t^\infty \mathcal{S}(\varphi_{k-1}, r, p)(s)\Delta s$, $k = 1, 2, \dots$. Nonoscillation of (1) implies that $\lim_{k\to\infty} \varphi_k(t) = \varphi(t) < \infty$, $t \geq a$. Thanks to the Lebesgue type monotone convergence theorem, we then get $\varphi(t) = \varphi_0(t) + \int_t^\infty \mathcal{S}(\varphi, r, p)(s)\Delta s$. $\square$

Remark 1. (i) In the continuous case, i.e., $\mathbf{T} = \mathbf{R}$, the proof of the equivalence between nonoscillation of (1) and a solvability of the Riccati type integral equation $w(t) = \int_t^\infty c(s)\,ds + (p-1)\int_t^\infty r^{1-q}(s)|w(s)|^q\,ds$ does not require any sign condition on c, see e.g., [**6**].

(ii) We present two methods of proof since both of them may play an important role in proving comparison results, see Remark 4 (iv).

The next lemma gives an effective estimation of a solution of (2) in the complementary case to the previous one, i.e., when $\int^\infty r^{1-q}(s)\,\Delta s < \infty$. Note that, also in this case, nonoscillation of (1) can be characterized in terms of certain Riccati type integral equations (inequality) with weights, but here we do not need such a result and we just describe asymptotic behavior of a solution to the Riccati type dynamic equation.

Lemma 3. *Let $\int^\infty r^{1-q}(s)\,\Delta s < \infty$ and $c(t) \geq 0$ for large t. Assume that* (1) *is nonoscillatory and y is a nontrivial solution. Set $w = r\Phi_p(y^\Delta/y)$. Then y and $R_C^{p-1}w$ are bounded, where*

$$R_C(t) = \int_t^\infty r^{1-q}(s)\,\Delta s. \tag{6}$$

Moreover, $w(t) \geq -R_C^{1-p}(t)$ for large t and

$$\limsup_{t\to\infty} R_C^{p-1}(t)w(t) \leq 0. \tag{7}$$

Proof. Without loss of generality, we can assume $y(t) > 0$ for $t \in [a, \infty)$. From (1), the function $r\Phi_p(y^\Delta)$ is nonincreasing. Hence, y^Δ is eventually of constant sign, i.e., either $y^\Delta(t) > 0$ for $t \geq a$ or there is a $t_0 \geq a$ such that $y^\Delta(t) < 0$ for $t \geq t_0$. Further, $r^{q-1}(s)y^\Delta(s) \leq r^{q-1}(t)y^\Delta(t)$, $s \geq t$. Dividing this inequality by $r^{q-1}(s)$ and integrating it over $[t, \tau]$, we obtain

$$y(\tau) \leq y(t) + r^{q-1}(t)y^\Delta(t) \int_t^\tau r^{1-q}(s)\,\Delta s. \tag{8}$$

If $y^\Delta(t) > 0$ for $t \geq a$, then from (8), $y(\tau) \leq y(t) + r^{q-1}(t)y^\Delta(t)R_C(t)$, and so y is bounded on $[a, \infty)$. If $y^\Delta(t) < 0$ for $t \geq t_0$, then y is clearly bounded, and letting $\tau \to \infty$ in (8), we obtain $0 \leq y(t) + r^{q-1}(t)y^\Delta(t)R_C(t)$. In either case, we have $R_C(t)r^{q-1}(t)y^\Delta(t)/y(t) \geq -1$, i.e., $R_C^{p-1}(t)w(t) \geq -1$. Inequality (7) trivially holds if $y^\Delta(t) < 0$, since $w(t) < 0$. If $y^\Delta(t) > 0$, then $M > 0$ and $N > 0$ exist such that $y(t) \geq M$ and $r(t)\Phi_p(y^\Delta(t)) \leq N$ for $t \geq a$. This implies that $w(t) \leq NM^{1-p}$, $t \geq a$. Since $R_C(t) \to 0$ as $t \to \infty$, we conclude that $\lim_{t\to\infty} R_C^{p-1}(t)w(t) = 0$. $\square$

In the following lemma, we describe monotone properties of the function $\mathcal{S}$, appearing in Riccati type equations, with respect to the third variable.

Lemma 4. *The function $F(x) = \mathcal{S}(w, r, x)$, $x > 1$, is nondecreasing provided*:

(i) $w > 0$, $r > 0$ *and*

$$f_\mu(z) := \lim_{\lambda\to\mu} \frac{(1 + \lambda z)\ln(1 + \lambda z) - \lambda z \ln z}{\lambda} \geq 0,$$

or

(ii) $w < 0$, $r > 0$, ${\Phi^{-1}}_x(r) + \mu {\Phi^{-1}}_x(w) > 0$ *and*

$$\widetilde{f}_\mu(z) := \lim_{\lambda\to\mu} \frac{(\lambda z - 1)\ln(1 - \lambda z) - \lambda z \ln z}{\lambda} \geq 0,$$

where $z := (|w|/r)^{1/(x-1)}$.

Proof. With $w \gtrless 0$, the function $\mathcal{S}$ can be written as

$$\mathcal{S}(w,r,x) = \lim_{\lambda\to\mu} \frac{w}{\lambda}\left[1-\left(1\pm\lambda(|w|/r)^{1/(x-1)}\right)^{1-x}\right].$$

Differentiating with respect to x we obtain, for $w \gtrless 0$,

$$F'(x) = \lim_{\lambda\to\mu} \frac{|w|}{\lambda}\cdot\frac{(\lambda z\pm 1)\ln(1\pm\lambda z)-\lambda z\ln z}{(1\pm\lambda z)^x}, \tag{9}$$

from which the statement follows. □

Remark 2. (i) Note that, if $\mu = 0$, then using L'Hospital's rule, (9) yields $F'(x) = |w|(z-\ln z)$, which can also be obtained by differentiating the expression $(x-1)|w|(|w|/r)^{1/(x-1)}$. Moreover, $f_0(z) = \widetilde{f}_0(z) = z-\ln z$.

(ii) Recall that we define f_μ on $\mathbf{R}^+$. A closer examination of the function f_μ with $\mu \equiv h$ fixed shows that it is concave, and $f_h(z) > 0$ for all $z > 0$ provided $h \ge 1$. If $h \in [0,1)$, then f_h has exactly one positive zero $z_0 \in [e,\infty)$, $f(z) > 0$ for $z \in (0,z_0)$, and its maximum is at $z = 1/(1-h)$, e being the basis of natural logarithm. Hence, $z_0 > 1/(1-h)$. If $h = 0$, then $z_0 = e$. Moreover, z_0 increases as h increases. Hence, $f_\mu(z) \ge 0$ for $z \in (0,e]$, whatever μ is.

(iii) Recall that we define $\widetilde{f}_\mu$ on $\mathbf{R}^+$. A closer examination of the function $\widetilde{f}_\mu$ with $\mu \equiv h$ fixed shows that it is concave, and $\widetilde{f}_h(z) > 0$ for all (admissible) $z > 0$ provided $h \ge 1$. If $h \in [0,1)$, then f_h has the only positive zero $\widetilde{z}_0 \in (1,e]$, $f(z) > 0$ for $z \in (0,\widetilde{z}_0)$. If $h = 0$, then $\widetilde{z}_0 = e$. Moreover, $\widetilde{z}_0$ increases as h decreases. Hence, $\widetilde{z}_\mu(z) \ge 0$ for (admissible) $z \in (0,1]$, whatever (admissible) μ is.

We conclude this section with a description of the so-called *reciprocity principle.*

Lemma 5. *Assume that $\mu(t) \equiv h \ge 0$ and $c(t) > 0$ for large t. Then* (1) *is oscillatory if and only if its reciprocal equation*

$$\left(c^{1-q}(t)\Phi_q(u^\Delta)\right)^\Delta + (r^\sigma(t))^{1-q}\Phi_q(u^\sigma) = 0 \tag{10}$$

is oscillatory.

Proof. The statement follows from the fact that (1) and (10) are related by the substitution $u = r\Phi_p(y^\Delta)$; if y solves (1), then u solves (10). Note that here we need the commutativity of the delta derivative and the forward jump operator, which is guaranteed by $\mu(t) \equiv h$. $\square$

Remark 3. Note that, with $\mu(t) \equiv h \geq 0$, clearly we have $r \in C_{\mathrm{rd}}$ if and only if $1/r \in C_{\mathrm{rd}}$ and $c \in C_{\mathrm{rd}}$ if and only if $1/c \in C_{\mathrm{rd}}$.

3. The case $\int^\infty r^{1-q}(s)\,\Delta s = \infty$. Along with (1), consider the equation

$$\left(r(t)\Phi_\alpha(x^\Delta)\right)^\Delta + c(t)\Phi_\alpha(x^\sigma) = 0, \tag{11}$$

where $\alpha > 1$. The conjugate number to α will be denoted as β. In the main theorems, equations (1) and (11) are compared, and conditions are established guaranteeing preservation of nonoscillation. A result of this kind has already been established in [**10, 11**] for the case $\int^\infty r^{1-q}(s)\,\Delta s = \infty$ under the additional condition (12). We recall it here, including a (new) simplified proof, see also Remark 4 (iv) for a comment on the original proofs. The second part of the following theorem with additional condition (13) is new. That additional condition, which may depend upon μ, is the result of different behavior of the function $\mathcal{S}$ on various time scales, and will play an important role in showing discrepancies between the statements on different time scales. The results in the complementary case $\int^\infty r^{1-q}(s)\,\Delta s < \infty$ are completely new, and are presented in the next section.

Theorem 1. *Let $\int^\infty r^{1-q}(s)\,\Delta s = \infty$ and $\int^\infty c(s)\,\Delta s$ converge with $\int_t^\infty c(s)\,\Delta s \geq 0$ ($\not\equiv 0$) for large t. Assume that*

$$\liminf_{t\to\infty} r(t) > 0. \tag{12}$$

Condition (12) *may be dropped provided $\mu(t) \geq 1$ eventually. If $\alpha \leq p$ and* (1) *is nonoscillatory, then* (11) *is nonoscillatory.*

If $c(t) \geq 0$, then (12) *may be replaced by the weaker condition*

$$\limsup_{t\to\infty} \frac{r^{1-q}(t)}{R_D(t,a)} < z_0^{(\alpha-1)(q-1)}, \tag{13}$$

where R_D *is defined by* (5) *and* z_0 *is the positive root of* $f_h(z) = 0$, f_h *being defined in Lemma* 4; *here we assume that the condition* $\mu(t) \geq 1$ *is not satisfied and* $h \in [0, 1)$ *is such that* $h \leq \mu(t)$ *for large* t. *See also subsequent Remark* 4 (i)–(iii) *for important comments on additional conditions* (12) *and* (13).

Proof. First assume that $\int_t^\infty c(s)\,\Delta s \geq 0$ and (12). If (1) is nonoscillatory, then there is a w satisfying (2) with (3) for large t, by Lemma 1. Moreover, by Lemma 2, this w satisfies (4), it is positive, and $w(t) \to 0$ as $t \to \infty$. Now taking into account that (12) holds, we have $w(t)/r(t) \leq 1$ for large t. From Lemma 4 (i) we now have that $\mathcal{S}(w, r, \alpha)(t) \leq \mathcal{S}(w, r, p)(t)$ for large t, since $f_\mu(z) \geq 0$. The conclusion of the theorem now follows from Lemma 1 since w satisfies $w(t) + c(t) + S(w, r, \alpha)(t) \leq 0$ and is positive for large t. The note about the omission of (12) when $\mu(t) \geq 1$ follows from Remark 2 (ii).

Now assume $c(t) \geq 0$ and (13). In addition to the properties of w described in the previous part, this w also satisfies $w(t) \leq R_D^{1-p}(t, t_0)$ for $t > t_0$, where t_0 is sufficiently large, see Lemma 2. We have

$$\frac{r^{1-q}(t)}{R_D(t, t_0)} = \frac{r^{1-q}(t)}{R_D(t, a) - R_D(t_0, a)} = \frac{r^{1-q}(t)/R_D(t, a)}{1 - R_D(t_0, a)/R_D(t, a)}.$$

Hence, $r^{1-q}(t)/R_D(t, t_0) \leq z_0^{(\alpha-1)(q-1)}$ for large t, in view of $\int^\infty r^{1-q}(s)\,\Delta s = \infty$ and (13). Thus, we get $w(t)/r(t) \leq (r^{1-q}(t)/R_D(t, t_0))^{p-1} \leq z_0^{(\alpha-1)(q-1)(p-1)} = z_0^{\alpha-1} \leq z_0^{x-1}$ for large t and $x \geq \alpha$; see also Remark 2 (ii). Consequently, $f_\mu(z) \geq 0$, and $\mathcal{S}(w, r, p)(t) \geq \mathcal{S}(w, r, \alpha)(t)$ by Lemma 4. The (positive) function w now satisfies $w^\Delta(t) + c(t) + \mathcal{S}(w, r, \alpha)(t) \leq 0$, and so (11) is nonoscillatory by Lemma 1. ◻

Remark 4. (i) Observe that, for $\mathbf{T} = \mathbf{Z}$ (and also when $\mu(t) \geq 1$ eventually), the function F from Lemma 4 always has the desired monotone properties, and hence (12) and (13) can be omitted.

(ii) If $\mathbf{T} = \mathbf{R}$, then condition (13) reads as $\limsup_{t\to\infty} r^{1-q}(t)/\int_a^t r^{1-q}(s)\,ds < e^{(\alpha-1)(q-1)}$.

(iii) In Section 5 we give an example showing that the constant z_0 in (13) is somehow optimal.

(iv) In [**10, 11**] we gave two different proofs of the first part of Theorem 1 (where condition (12) is assumed), based upon the monotone properties of $\mathcal{S}$ and the ideas which were also used in the proof of Lemma 2, namely, the Riccati technique combined with the Schauder fixed point theorem and the function sequence technique, respectively.

4. The case $\int^\infty r^{1-q}(s)\,\Delta s < \infty$. Next we deal with the complementary case to the previous one: We assume the condition $\int^\infty r^{1-q}(s)\,ds < \infty$.

Note that, due to the lack of a transformation similar to that in the linear case, we have to distinguish the cases $\int^\infty r^{1-q}(s)\,\Delta s = \infty$ and $\int^\infty r^{1-q}(s)\,\Delta s < \infty$, and each of them have to be handled separately, using different approaches.

Theorem 2. *Let $\int^\infty r^{1-q}(s)\,\Delta s < \infty$ and $c(t) \geq 0$ for large t. Assume that*

$$\frac{r^{1-q}(t)}{R_C(t)} \leq 1 \quad \textit{for large } t, \tag{14}$$

where R_C is defined by (6). *Condition* (14) *may be dropped, provided $\mu(t) \geq 1$ eventually. If $\alpha \leq p$ and* (1) *is nonoscillatory, then* (11) *is nonoscillatory.*

Assume that $\mu(t) \leq h < 1$. Then (14) *may be replaced by the weaker condition*

$$\frac{r^{1-q}(t)}{R_C(t)} \leq \widetilde{z}_0^{(\alpha-1)(q-1)} \quad \textit{for large } t, \tag{15}$$

where $\widetilde{z}_0$ is the positive root of $\widetilde{f}_h(z) = 0$, $\widetilde{f}_h$ being defined in Lemma 4. *In addition, we have to assume that $h\widetilde{z}_0 < 1$. See also subsequent Remark* 5 *for important comments on additional conditions* (14) *and* (15).

Proof. Let y be a positive solution of (1). As in the proof of Lemma 3, we get that y is eventually monotone.

First assume that $y^\Delta(t) < 0$ for $t \geq t_0$. Then $w = r\Phi_p(y^\Delta/y)$ is negative and satisfies the Riccati type equation (2) with (3) for $t \geq t_0$, by Lemma 1. From Lemma 3, $|w(t)| \leq R_C^{1-p}(t)$, and so $|w(t)|/r(t) \leq (r^{1-q}(t)/R_C(t))^{p-1} \leq 1$ for large t assuming (14). Hence,

$\widetilde{f}_\mu(z) \geq 0$, and $\mathcal{S}(w,r,\alpha)(t) \leq \mathcal{S}(w,r,p)(t)$ for large t by Lemma 4. Consequently, w satisfies the inequality $w^\Delta(t)+c(t)+\mathcal{S}(w,r,\alpha)(t) \leq 0$. Moreover, $1-\mu(t)(|w(t)|/r(t))^{1/(\alpha-1)} \geq 1-\mu(t)(|w(t)|/r(t))^{1/(p-1)}$; thus, $\Phi^{-1}{}_\alpha(r(t))+\mu(t)\Phi^{-1}{}_\alpha(w(t)) > 0$. Equation (11) is nonoscillatory by Lemma 1. If (14) is relaxed to (15), then the statement follows from the previous arguments, taking into account Remark 2, the estimates $|w(t)|/r(t) \leq R_C^{1-p}(t)/r(t) \leq \widetilde{z}_0^{\alpha-1} \leq \widetilde{z}_0^{x-1}$, $x \geq \alpha$, and the fact that the inequality $\Phi^{-1}{}_\alpha(r(t)) + \mu(t)\Phi^{-1}{}_\alpha(w(t)) > 0$ is implied by $\mu(t)(|w(t)|/r(t))^{1/(\alpha-1)} \leq \mu(t)\widetilde{z}_0 \leq h\widetilde{z}_0 < 1$.

Now we assume that $y^\Delta(t) > 0$ for $t \geq a$. Then $w = r\Phi_p(y^\Delta/y)$ is positive, and hence $\lim_{t\to\infty} R_C^{p-1}(t)w(t) = 0$ by (7). Consequently, in view of (14), $w(r)/r(t) = w(t)R_C^{p-1}(t)(r^{1-q}(t)/R_C(t))^{p-1} \to 0$ as $t \to \infty$. Hence, $\mathcal{S}(w,r,\alpha)(t) \leq \mathcal{S}(w,r,p)(t)$ for large t, and the rest of the proof is the similar to that in the previous part. $\square$

Remark 5. (i) Observe that, for $\mathbf{T} = \mathbf{Z}$ (and also when $\mu(t) \geq 1$ eventually), function F from Lemma 4 always has the desired monotone properties, and hence (14) can be omitted.

(ii) If $\mathbf{T} = \mathbf{R}$, then (15) reads as $r^{1-q}(t)/\int_t^\infty r^{1-q}(s)\,ds \leq e^{(\alpha-1)(q-1)}$ for large t.

(iii) In Section 5 we give an example showing that the constant z_0 in (13) is somehow optimal.

(iv) A closer examination of the second part of the proof shows that if (1) has an eventually positive increasing solution, then (14) can be relaxed to the condition $r^{1-q}(t)/R_C(t)$ is bounded.

Next we present a different approach to the case $\int^\infty r^{1-q}(s)\,\Delta s < \infty$, based on the reciprocity principle. The Riccati type transformation is used as well. Notice that the "key" condition $\alpha \geq p$ is the opposite in comparison with that in Theorem 2. However, the resulting equation is different from (11).

Theorem 3. *Let $\mu(t) \equiv h \geq 0$ and $c(t) > 0$ for large t. Assume that $\int^\infty r^{1-q}(s)\,\Delta s < \infty$ and $\int^\infty c(s)\,\Delta s = \infty$. If $\alpha \geq p$ and (1) is nonoscillatory, then the equation*

$$\left((r^\sigma(t))^{(1-q)(1-\alpha)}\Phi_\alpha(x^\Delta)\right)^\Delta + c^\sigma(t)\Phi_\alpha(x^\sigma) = 0 \tag{16}$$

is nonoscillatory.

Proof. If (1) is nonoscillatory, then (10) is nonoscillatory as well by Lemma 5 since $\mu(t) \equiv h$. Moreover, $\int^\infty (c^{1-q}(s))^{1-p}\Delta s = \int^\infty c(s)\,\Delta s = \infty$, and $\int^\infty r^{1-q}(s)\Delta s < \infty$ implies $\int^\infty (r^\sigma(s))^{1-q}\Delta s < \infty$ thanks to $\mu(t) \equiv h$. Hence, by Lemma 2, there is a positive function v, which satisfies the generalized Riccati type equation

$$v(t) = \int_t^\infty (r^\sigma(s))^{1-q}\Delta s + \int_t^\infty \mathcal{S}(v, c^{1-q}, q)(s)\,\Delta s \tag{17}$$

for large t. In fact, v is given by $v = c^{1-q}\Phi_q(u^\Delta/u)$, where u is a positive increasing solution of (10), which indeed exists. It is easy to see that $\mathcal{S}(v, c^{1-q}, q) = \lim_{\lambda\to\mu} v[1-(1+\lambda c v^{1/(q-1)})^{1-q}]/\lambda$. Note that, if $\mu = 0$, then L'Hospital's rule yields $\mathcal{S}(v, c^{1-q}, q) = (q-1)cv^p$. Since $\lim_{t\to\infty} v(t) = 0$ and $\alpha \geq p$ (i.e., $\beta \leq q$), we get

$$\mathcal{S}(v, c^{1-q}, q)(t) \geq \mathcal{S}(v, c^{1-\beta}, \beta)(t). \tag{18}$$

Hence, v satisfies $v(t) \geq \int_t^\infty (r^\sigma(s))^{1-q}\Delta s + \int_t^\infty \mathcal{S}(w, c^{1-\beta}, \beta)(s)\,\Delta s$ for large t. Moreover, $\int^\infty (c^{1-\beta}(s))^{1-\alpha}\Delta s = \int^\infty c(s)\,\Delta s = \infty$ and $\int^\infty (r^\sigma(s))^{1-q}\Delta s < \infty$. Consequently, $(c^{1-\beta}(t)\Phi_\beta(z^\Delta))^\Delta + (r^\sigma(t))^{1-q}$ $\Phi_\beta(z^\sigma) = 0$ is nonoscillatory by Lemma 2. The statement now follows by using the reciprocity principle, Lemma 5. $\square$

Remark 6. (i) Observe that, in contrast to Theorem 2, in order to show monotonicity in the sense of (18), we do not need any condition of type (14) or (15), no matter how small the graininess, since c becomes "independent" of the power of nonlinearity.

(ii) Similarly, as in the standard cases $\mathbf{T} = \mathbf{R}$ and $\mathbf{T} = \mathbf{Z}$, where using, e.g., the Schauder fixed point theorem, it can be shown that conditions $\int^\infty r^{1-q}(s)\,\Delta s < \infty$, $c(t) > 0$ and $\int^\infty c(s)\,\Delta s < \infty$ imply the existence of a nonoscillatory solution of (1). Observe that, for equation (16), $\int^\infty ((r^\sigma(s))^{(1-q)(1-\alpha)})^{1-\beta}\Delta s = \int^\infty (r^\sigma(s))^{1-q}\Delta s$. If, as in Theorem 3, $\int^\infty r^{1-q}(s)\,\Delta s < \infty$, then $\int^\infty (r^\sigma(s))^{1-q}\Delta s < \infty$ (assuming $\mu(t) \equiv h$). Moreover, if the condition $\int^\infty c(s)\,\Delta s = \infty$ fails to hold, then, necessarily, $\int^\infty c(s)\,\Delta s < \infty$. Hence, $\int^\infty c^\sigma(s)\,\Delta s < \infty$. These observations show that the assumption $\int^\infty c(s)\,\Delta s = \infty$ in Theorem 3 is quite natural and means no restriction.

5. Special cases: Improvements, comparisons, peculiarities. In this section we discuss, in particular, how, in some special cases, the above results can be refined and also how the character of the parallel results can be changed when considering them on different time scales. We stress that the observations are new even in the well studied $\mathbf{T} = \mathbf{R}$ and $\mathbf{T} = \mathbf{Z}$ cases.

We start by showing that, in the continuous case, a sign condition on coefficient c in Theorem 1 can be dropped.

Theorem 4. *Let* $\mathbf{T} = \mathbf{R}$. *Assume that* $\int^{\infty} r^{1-q}(s)\,ds = \infty$, $\liminf_{t\to\infty} r(t) > 0$, *and* $\int^{\infty} c(s)\,ds$ *converges. If* $\alpha \le p$ *and* (1) *is nonoscillatory, then* (11) *is nonoscillatory.*

Proof. The proof is similar to that of Theorem 1, now using Remarks 1 (i) and 2 (i). □

We have already seen that, e.g., for the case $\mathbf{T} = \mathbf{R}$, in contrast, e.g., to the case $\mathbf{T} = \mathbf{Z}$, an additional condition is needed (see (12), (13), (14) or (15)). One can easily observe that, roughly speaking, a "bigger" graininess is "more favorable" for our needs. A natural question arises, whether such a condition (in particular, (13) and (15)) can be omitted or somehow relaxed. As the following theorem for case $\mathbf{T} = \mathbf{R}$ shows, this condition not only cannot be omitted, but even the constants z_0 and $\widetilde{z}_0$ in (13) and (15), respectively, are optimal (cannot be increased).

Theorem 5. *Let* $\mathbf{T} = \mathbf{R}$. *Suppose either*:

(i) $\int^{\infty} r^{1-q}(s)\,ds = \infty$, $c(t) \ge 0$, $\int^{\infty} c(s)\,ds < \infty$, *and*

$$\limsup_{t\to\infty} \frac{r^{1-q}(t)}{R_D(t,a)} < e^{(\alpha-1)(q-1)}, \tag{19}$$

or

(ii) $\int^{\infty} r^{1-q}(s)\,ds < \infty$, $c(t) \ge 0$, *and*

$$\frac{r^{1-q}(t)}{R_C(t)} \le e^{(\alpha-1)(q-1)} \quad \text{for large } t. \tag{20}$$

If $\alpha \le p$ *and* (1) *is nonoscillatory, then* (11) *is nonoscillatory. Moreover, the Euler number* e *in* (19) *and* (20) *is the best possible.*

Proof. We just prove the part concerning the best possible constant. Other parts are special cases of Theorems 1 and 2. Consider the equation

$$\left(e^{bt}\Phi_p(y')\right)' + \left(\frac{-b}{p}\right)^p e^{bt}\Phi_p(y) = 0, \tag{21}$$

where $b < 0$ and $p > 1$. Clearly, $c(t) > 0$, $\int^\infty c(s)\,ds < \infty$ and $\int^\infty r^{1-q}(s)\,ds = \infty$ We claim that (21) is nonoscillatory. Indeed, we have

$$\begin{aligned}\left(\int_a^t r^{1-q}(s)\,ds\right)^{p-1} \int_t^\infty c(s)\,ds &\le \frac{(-b)^p e^{tb(1-q)(p-1)}}{p^p(-b)^{p-1}} \cdot \frac{e^{bt}}{-b}\\ &= \frac{1}{p}\left(\frac{p-1}{p}\right)^{p-1}.\end{aligned}$$

Hence, (21) is nonoscillatory by the Hille-Nehari type criterion, [**6**, Theorem 5.5.8]. Further, we have

$$\lim_{t\to\infty} \frac{r^{1-q}(t)}{\int_a^t r^{1-q}(s)\,ds} = b(1-q) =: M.$$

Assume that $M = (e+\varepsilon)^{(\alpha-1)(q-1)}$, where $\varepsilon > 0$. Now consider the equation

$$\left(e^{bt}\Phi_\alpha(y')\right)' + \left(\frac{-b}{p}\right)^p e^{bt}\Phi_\alpha(y) = 0, \tag{22}$$

where $\alpha > 1$; the values of α and p will be specified later. Since $b = M(1-p) = (e+\varepsilon)^{(\alpha-1)(q-1)}(1-p)$, for equation (22) we have

$$\begin{aligned}&\lim_{t\to\infty}\left(\int_a^t r^{1-\beta}(s)\,ds\right)^{\alpha-1}\int_t^\infty c(s)\,ds\\ &\qquad = \frac{(-b)^{p-\alpha}}{p^p(\beta-1)^{\alpha-1}}\\ &\qquad = \frac{(e+\varepsilon)^{(\alpha-1)(q-1)(p-\alpha)}(p-1)^{p-\alpha}(\alpha-1)^{\alpha-1}}{p^p}.\end{aligned}$$

Denote the expression on the right-hand side by ω. We want to show that p, α exist with $p > \alpha$ such that $\omega > (\alpha-1)^{\alpha-1}/\alpha^\alpha$, which then guarantees oscillation of (22) by [**6**, Theorem 3.1.1]. Set $p = \gamma + \alpha$, where $\gamma > 0$. Then $\alpha < p$ and

$$\begin{aligned}\frac{\omega\alpha^\alpha}{(\alpha-1)^{\alpha-1}} &= (e+\varepsilon)^{(\alpha-1)(q-1)(p-\alpha)}\left(\frac{p-1}{p}\right)^p\left(\frac{\alpha}{p-1}\right)^\alpha\\ &= (e+\varepsilon)^{(\alpha-1)\gamma/(\alpha+\gamma-1)}\left(\frac{\alpha+\gamma-1}{\alpha+\gamma}\right)^{\alpha+\gamma}\\ &\longrightarrow (e+\varepsilon)^\gamma e^{-1}e^{1-\gamma} \quad (\text{as } \alpha\to\infty)\\ &= \left(\frac{e+\varepsilon}{e}\right)^\gamma > 1.\end{aligned}$$

Hence, for any $\varepsilon > 0$, we can find α and p with $\alpha < p$ and such that $\omega\alpha^\alpha/(\alpha-1)^{\alpha-1} > 1$. This implies oscillation of (22) in spite of (21) is nonoscillatory. Note that condition (19) fails to hold.

To show that the constant e is the best possible also in (20), we use arguments similar to those of the previous part. We again consider equation (21) where we take $b > 0$. To detect (non)oscillation of such an equation, we can use the criteria from [**6**, Theorem 3.1.6]. □

Remark 7. (i) We conjecture that a similar optimality result can be shown, e.g., also on $\mathbf{T} = h\mathbf{Z}$ with $h \in (0,1)$ where recently derived Hille-Nehari type criteria (see [**13**]) could find an application.

(ii) This remark concerns the case $\mathbf{T} = \mathbf{R}$. If $\int^\infty r^{1-q}(s)\,ds = \infty$, then differential equation (1) can be transformed by means of the transformation of the independent variable into the equation of the same form, but with the coefficient in the differential term being identically equal to 1. Hence, additional conditions, like (12) or (13) (i.e., (19)) are trivially fulfilled. But then we have to get over the following undesired property: the coefficient in the second term of the resulting equation becomes dependent on p. Nevertheless, also in such a case, we cannot exclude the possibility that a statement equivalent to our one might be obtained, by using a different method.

In the above results we can see that if the graininess is sufficiently large, then additional conditions on r can be omitted. In the next statement we show that, in such a case, all other conditions on the coefficients do not need to be assumed. In particular, there is no sign or integral condition on $c(t)$.

Theorem 6. *Let $\mu(t) \geq 1$ for large t. If $\alpha \leq p$ and* (1) *is nonoscillatory, then* (11) *is nonoscillatory.*

Proof. If (1) is nonoscillatory, then there is a w satisfying the Riccati type equation (2) with (3) for large t. We have no information on whether w is small or positive or negative, but in fact it is not necessary. In accordance with the notation of Lemma 4, we have $z = (|w|/r)^{1/(x-1)}$, where $x \geq 1$. If $w > 0$, then the numerator of $f_\mu(z)$ takes the form

$$(\mu z + 1)\ln(1+\mu z) - \mu z \ln z = \ln(1+\mu z) + \mu z \ln \frac{1+\mu z}{z}.$$

Since $\mu \geq 1$, we have $1 + \mu z \geq z$, and hence this numerator is nonnegative for all $z > 0$. Consequently, $x \mapsto \mathcal{S}(w, r, x)$ is increasing for $x > 1$. For the case $w < 0$, first note that the condition ${\Phi^{-1}}_p(r) + \mu {\Phi^{-1}}_p(w) > 0$ is equivalent to $1 - \mu(|w|/r)^{1/(p-1)} > 0$. Hence, $(|w|/r)^{1/(p-1)} < 1/\mu \leq 1$, or $|w|/r < 1$. This implies

$$z = \left(\frac{|w|}{r}\right)^{1/(x-1)} \leq \left(\frac{|w|}{r}\right)^{1/(p-1)} < \frac{1}{\mu} \leq 1 \tag{23}$$

for $1 < x \leq p$. Now it is easy to see that, with such z's for the numerator of $\tilde{f}_\mu$, we have $(\mu z - 1)\ln(1 - \mu z) - \mu z \ln z \geq 0$. Hence, $\mathcal{S}(w, r, p) \geq \mathcal{S}(w, r, \alpha)$ also when $w < 0$. Moreover, in this case, $(|w|/r)^{1/(\alpha-1)} < 1/\mu$, or ${\Phi^{-1}}_\alpha(r) + \mu(t){\Phi^{-1}}_\alpha(w) > 0$. Altogether, no matter if w is positive or negative, we obtain $w^\Delta(t) + c(t) + \mathcal{S}(w, r, \alpha)(t) \leq w^\Delta(t) + c(t) + \mathcal{S}(w, r, p)(t) = 0$ and ${\Phi^{-1}}_\alpha(r(t)) + \mu(t){\Phi^{-1}}_\alpha(w(t)) > 0$ for large t, which implies nonoscillation of (11) by Lemma 1. $\square$

Remark 8. In particular, the above theorem says that any nonoscillatory half-linear difference equation $\Delta(r(t)\Phi_p(\Delta y(t))) + c(t)\Phi_p(y(t+1)) = 0$ with $r(t) > 0$ or any nonoscillatory q-difference equation $D_q(r(t)\Phi_p(D_q y(t))) + c(t)\Phi_p(y(qt)) = 0$ with $r(t) > 0$ and $q > 1$ remains nonoscillatory provided p is decreased, no matter what the behavior of the coefficients is. Note that the number q in D_q has nothing to do with the conjugate number to p, which was also denoted by q in the above text. We just want to keep the usual notation of q-calculus.

6. One extension of the Sturm-Picone comparison theorem. Using an idea similar to that from the proof of Theorem 3, we may derive the following integral comparison theorem, which generalizes the Sturm-Picone type comparison theorem (see, e.g., [**9**]), in a certain sense. This time we do not compare nonlinearities, but the coefficients in the differential term. Along with (1), we consider the equation

$$\left(\widetilde{r}^{\sigma}(t)\Phi_p(x^{\Delta})\right)^{\Delta} + c^{\sigma}(t)\Phi_p(x^{\sigma}) = 0, \tag{24}$$

where $1/\widetilde{r}(t) > 0$ is an rd-continuous function on $[a,\infty)$. A similar observation as the one from Remark 6 (ii), concerning the condition $\int^{\infty} c(s)\,\Delta s = \infty$, applies as well for the next statement.

Theorem 7. *Let $\mu(t) \equiv h \geq 0$ and $c(t) > 0$ for large t. Assume that $\int^{\infty} c(s)\,\Delta s = \infty$ and*

$$\int_t^{\infty} \widetilde{r}^{1-q}(s)\,\Delta s \leq \int_t^{\infty} r^{1-q}(s)\,\Delta s < \infty. \tag{25}$$

If (1) *is nonoscillatory, then* (24) *is nonoscillatory.*

Proof. As in the previous proof, there is a $v > 0$ satisfying (17) for large t. Since $\mu(t) \equiv h$, condition (25) implies $\int_t^{\infty}(\widetilde{r}^{\sigma}(s))^{1-q}\Delta s \leq \int_t^{\infty}(r^{\sigma}(s))^{1-q}\Delta s < \infty$. Hence, v satisfies $v(t) \geq \int_t^{\infty}(\widetilde{r}^{\sigma}(s))^{1-q}\Delta s + \int_t^{\infty}\mathcal{S}(v, c^{1-q}, q)(s)\,\Delta s$ for large t. Consequently, $(c^{1-q}(t)\Phi_q(z^{\Delta}))^{\Delta} + (\widetilde{r}^{\sigma}(t))^{1-q}\Phi_q(z^{\sigma}) = 0$ is nonoscillatory by Lemma 2. Thus, by the reciprocity principle, (24) is nonoscillatory. ◻

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