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POLYNOMIALS AND WEYL ALGEBRAS ON TIME SCALES

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ABSTRACT. We define the Weyl algebra on a time scale as the $\mathbf{C}$-algebra generated by the coordinate function t and the Hilger derivative operator δ. This leads to a rich variety of algebras that generalize the usual Weyl algebra on $\mathbf{R}$ and also the q-Weyl algebra. The main focus of the paper is on the computation of the Gelfand-Kirillov dimension of these algebras. Related to the Gelfand-Kirillov dimension, we also introduce a new invariant for arbitrary time scales (i.e., arbitrary closed subsets of $\mathbf{R}$) that is invariant under the group of affine linear transformations.

1. Introduction.

1.1. Time scales. A *time scale* $\mathbf{T}$ is any nonempty closed subset of the set of real numbers $\mathbf{R}$ with the induced (standard) metric and linear order. Analysis on time scales was introduced by Stefan Hilger in his Ph.D. thesis (see [**4**]) to unify continuous and discrete analysis of one variable. For example, Hilger defined the notion of derivative for functions on a time scale that coincides with the usual derivative if $\mathbf{T} = \mathbf{R}$ and with the difference operator if $\mathbf{T} = \mathbf{Z}$. Similarly, there is a generalized notion of integration that coincides with the usual (Riemann) integral for continuous functions if $\mathbf{T} = \mathbf{R}$ and with summation if $\mathbf{T} = \mathbf{Z}$. Since its introduction, analysis on time scales has become an active field of research in applied mathematics with an abundance of applications to dynamical systems that arise in engineering and control theory (see [**1**] and the references therein for an overview).

1.2. The q-Weyl algebra. Calculus on time scales also generalizes the calculus of q-analogs (or "quantum calculus," see [**6**]) that plays an increasingly important role in algebraic combinatorics and representation theory. Fix a real number $q > 1$, and consider the time scale

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$\mathbf{T} = q^{\mathbf{N}_0} = \{q^\nu \mid \nu \in \mathbf{N}_0\}$. Then the Hilger derivative of a function

$$D_q f(t) = \frac{f(qt) - f(t)}{qt - t}$$

for $t \in \mathbf{T}$. The operator D_q and the coordinate function t, identified with the operator on functions given by left multiplication by t, satisfy the q-Heisenberg relation $D_q \cdot t - qt \cdot D_q = 1$. The $\mathbf{C}$-algebra generated by t and D_q is called the q-Weyl algebra and is a deformation of the Weyl algebra which is obtained in the limit as $q \to 1$. All of these algebras are Ore extensions of the polynomial ring $\mathbf{C}[t]$ and have Gelfand-Kirillov dimension equal two.

1.3. Weyl algebras on time scales. The main purpose of this paper is to define and study Weyl algebras on time scales. Let $\mathbf{T}$ be a time scale, and assume that the space of infinitely Hilger differentiable $\mathbf{C}$-valued functions forms an algebra, i.e., is closed under multiplication. (It is easy to give examples of time scales for which the function t^2 is not Hilger differentiable, even though t is always differentiable with constant Hilger derivative 1.) Then we define the Weyl algebra on $\mathbf{T}$ as the $\mathbf{C}$-algebra generated by t and the Hilger derivative operator. For example, if $\mathbf{T} = q^{\mathbf{N}_0}$ with $q > 1$, then the Weyl algebra on $\mathbf{T}$ is the q-Weyl algebra described above. Our general construction leads to a rich variety of new $\mathbf{C}$-algebras with two generators. The main focus in this paper will be on the Gelfand-Kirillov dimension of these algebras. We conjecture that, if r is any integer ≥ 2, then there exists a time scale for which the corresponding Weyl algebra has Gelfand-Kirillov dimension equal to r. Related to the Gelfand-Kirillov dimension of Weyl algebras on time scales, we also introduce a new invariant for arbitrary time scales (i.e. arbitrary closed subsets of $\mathbf{R}$) that is invariant under the group of affine linear transformations.

2. Preliminaries. The reader should consult [**1**, Chapter 1] for more details and proofs.

2.1. Basics. Let $\mathbf{T}$ be a time scale, i.e., a nonempty closed subset of $\mathbf{R}$. The *forward-jump operator* $\sigma : \mathbf{T} \to \mathbf{T}$ is defined by

$$\sigma(t) = \inf\{s \in \mathbf{T} \mid s > t\}$$

for all $t \in \mathbf{T}$. Similarly, the *backward-jump operator* $\rho : \mathbf{T} \to \mathbf{T}$ is defined by $\rho(t) = \sup\{s \in \mathbf{T} \mid s < t\}$. A point $t \in \mathbf{T}$ is called *left-dense* if $\rho(t) = t$ and *left-scattered* if $\rho(t) < t$. Similarly, $t \in \mathbf{T}$ is called *right-dense* if $\sigma(t) = t$ and *right-scattered* if $\sigma(t) > t$. Points that are left-dense and right-dense are called *dense.* If $f : \mathbf{T} \to \mathbf{C}$ is a function, we define the function $f^\sigma : \mathbf{T} \to \mathbf{C}$ by

$$f^\sigma(t) = f(\sigma(t)) \tag{2.1.1}$$

for all $t \in \mathbf{T}$, i.e., $f^\sigma = \sigma^*(f)$ is the *pull-back* of f by σ. We will often slightly abuse notation and write t to denote the coordinate function on $\mathbf{R}$ restricted to $\mathbf{T}$. Thus, we would write t^σ to denote the function on $\mathbf{T}$ whose value at $t_0 \in \mathbf{T}$ is $\sigma(t_0)$.

2.2. The Hilger derivative. If $\mathbf{T}$ has a left-scattered maximum m, define $\mathbf{T}^\kappa = \mathbf{T} - \{m\}$; otherwise, define $\mathbf{T}^\kappa = \mathbf{T}$. The *delta* or (*Hilger*) *derivative* of a function $f : \mathbf{T} \to \mathbf{C}$ at $t \in \mathbf{T}^\kappa$ is the complex number $f^\Delta(t)$ (if it exists) with the property that, given any $\varepsilon > 0$, there is a neighborhood U (in the relative topology) of t such that

$$\left|[f(\sigma(t)) - f(s)] - f^\Delta(t)\,[\sigma(t) - s]\right| \le \varepsilon\,|\sigma(t) - s|$$

for all $s \in U$. It is an easy theorem that, if f is differentiable at $t \in \mathbf{T}^\kappa$, then f is continuous at t. Moreover, if f is continuous at $t \in \mathbf{T}^\kappa$ and t is right-scattered, then f is differentiable at t with

$$f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t},$$

and if t is right-dense, then f is differentiable at t if and only if the standard limit exists, i.e.,

$$f^\Delta(t) = \lim_{s\to t} \frac{f(s) - f(t)}{s - t},$$

where the limit is over $s \in \mathbf{T}$. It follows immediately from the definition that, if f and g are differentiable at $t \in \mathbf{T}^\kappa$ and $c \in \mathbf{C}$, then $f + g$ and cf are differentiable at t with $(f + g)^\Delta(t) = f^\Delta(t) + g^\Delta(t)$ and $(cf)^\Delta(t) = cf^\Delta(t)$.

2.3. The product rule. If f and g are differentiable at $t \in \mathbf{T}^\kappa$, then the product fg is differentiable at t with

$$(fg)^\Delta(t) = f^\Delta(t)g(t) + f(\sigma(t))g^\Delta(t). \tag{2.3.1}$$

If f and g are differentiable functions on $\mathbf{T}$, we will frequently use (2.1.1) to write the product rule in the form

$$(fg)^\Delta = f^\Delta g + f^\sigma g^\Delta. \tag{2.3.2}$$

Warning. In general, fg is not twice differentiable even if both f and g are twice differentiable. For instance, consider a time scale $\mathbf{T}$ which contains a point t_0 that is left-dense and right-scattered so that the forward-jump function σ is not continuous and hence not differentiable at t_0. Since $(t^2)^\Delta = t + t^\sigma = t + \sigma$ by (2.3.2), it follows that t^2 is not twice differentiable at t_0 even though t is twice differentiable everywhere with $t^{\Delta\Delta} \equiv 0$.

2.4. Anti-derivatives. A function $f : \mathbf{T} \to \mathbf{C}$ is called *rd-continuous* if it is continuous at right-dense points in $\mathbf{T}$ and its left-sided limits exist at left-dense points in $\mathbf{T}$. Clearly, every continuous function is rd-continuous. A function $F : \mathbf{T} \to \mathbf{C}$ is called an *anti-derivative* of $f : \mathbf{T} \to \mathbf{C}$ if $F^\Delta(t) = f(t)$ for all $t \in \mathbf{T}^\kappa$. It is a theorem that every rd-continuous function has an anti-derivative that is unique up to an additive constant. If $f : \mathbf{T} \to \mathbf{C}$ is rd-continuous and $F : \mathbf{T} \to \mathbf{C}$ is an anti-derivative of f, we define

$$\int_a^b f(t)\,\Delta t = F(b) - F(a)$$

for any $a, b \in \mathbf{T}$.

2.5. Taylor's formula. For a fixed $s \in \mathbf{T}$, define $h_k(\_, s)$ recursively by

$$h_0(t,s) \equiv 1, \qquad h_{k+1}(t,s) = \int_s^t h_k(\tau, s)\,\Delta\tau. \tag{2.5.1}$$

If f is n-times delta differentiable on $\mathbf{T}^{\kappa^n}$ and $a \in \mathbf{T}^{\kappa^{n-1}}$, then for any $t \in \mathbf{T}$,

$$(2.5.2)\quad f(t) = \sum_{k=0}^{n-1} f^{\Delta^k}(a) h_k(t,a) + \int_a^{\rho^{n-1}(t)} f^{\Delta^n}(\tau) h_{n-1}(t,\sigma(\tau))\,\Delta\tau.$$

3. Polynomials.

3.1. Restrictions of polynomial functions. Let $\mathbf{C}[t]$ be the polynomial ring in the variable t over $\mathbf{C}$. We will interpret the elements of $\mathbf{C}[t]$ as $\mathbf{C}$-valued functions on $\mathbf{R}$ and, by restriction, on any time scale $\mathbf{T}$.

In the following we will assume that $|T| = \infty$.

Then a polynomial $p(t) \in \mathbf{C}[t]$ is identically zero as a function on $\mathbf{T}$ if and only if $p(t) = 0$ as a polynomial, and hence we may view $\mathbf{C}[t]$ as a subalgebra of the algebra of $\mathbf{C}$-valued functions on $\mathbf{T}$, where multiplication is given by $(fg)(t) = f(t)g(t)$, $t \in \mathbf{T}$. In general, the algebra $\mathbf{C}[t]$ is not invariant under the delta derivative.

Lemma 1. *The polynomial ring $\mathbf{C}[t]$ viewed as an algebra of functions on a time scale $\mathbf{T}$ is invariant under the delta derivative if and only if $t^\sigma = p(t)$ for some $p(t) \in \mathbf{C}[t]$.*

Proof. Suppose that $\mathbf{C}[t]$ is invariant under the delta derivative. Since $(t^2)^\Delta = t + t^\sigma$, it follows that $t^\sigma = p(t)$ for some $p(t) \in \mathbf{C}[t]$. Conversely, suppose $t^\sigma = p(t)$ for some $p(t) \in \mathbf{C}[t]$. Then $(t^n)^\Delta = t^{n-1} + t^\sigma (t^{n-1})^\Delta = t^{n-1} + p(t)(t^{n-1})^\Delta$, and hence it follows by induction that $(t^n)^\Delta \in \mathbf{C}[t]$ for all $n \in \mathbf{N}_0$. $\square$

3.2. Another notion of polynomial functions. In the case when $\mathbf{T} = \mathbf{R}$, the polynomial functions are the precisely the functions with the property that some higher derivative vanishes identically. For a general time scale $\mathbf{T}$, define

$$\mathcal{P}(\mathbf{T}) = \{f : \mathbf{T} \to \mathbf{C} \mid f^{\Delta^n} \equiv 0 \text{ on } T^{\kappa^n} \text{ for some } n \in N\}.$$

Clearly, $\mathcal{P}(\mathbf{T})$ is a $\mathbf{C}$-vector space. As a vector space, $\mathcal{P}(\mathbf{T})$ shares many properties with the polynomial ring $\mathbf{C}[t]$. For instance, one has the following result.

Lemma 2. *For $n \in \mathbf{N}_0$, let $\mathcal{P}(\mathbf{T})_n = \{f : \mathbf{T} \to \mathbf{C} \mid f^{\Delta^{n+1}} \equiv 0$ on $\mathbf{T}^{\kappa^{n+1}}\}$. Then $\mathcal{P}(\mathbf{T})_n$ is a $\mathbf{C}$-vector space of dimension $n+1$. More precisely, for any fixed $s \in \mathbf{T}$, the functions $h_k(t,s)$, $0 \le k \le n$, form a basis of $\mathcal{P}(\mathbf{T})_n$.*

Proof. Let $f \in \mathcal{P}(\mathbf{T})_n$ and $s \in \mathbf{T}^{\kappa^{n+1}}$. Then, by Taylor's formula (2.5.2), f is a linear combination of the functions $h_k(t,s)$, $0 \le k \le n$. $\square$

Since $(t)^{\Delta} = 1$ and 1^{Δ}, one has $t \in \mathcal{P}(\mathbf{T})$ for every time scale $\mathbf{T}$. Recall, however, that t^2 is not twice differentiable, and hence $t^2 \notin \mathcal{P}(\mathbf{T})$ in general. Thus, $\mathcal{P}(\mathbf{T})$ is not an algebra in general.

Proposition 3. *If $\mathcal{P}(\mathbf{T}) = \mathbf{C}[t]$, then $t^{\sigma} = at + b$ for some $a, b \in \mathbf{R}$.*

Proof. Since $t^2 \in \mathbf{C}[t] = \mathcal{P}(\mathbf{T})$ and since $(t^2)^{\Delta} = t + t^{\sigma} = t + \sigma$, it follows that σ is n-times Δ-differentiable with $\sigma^{\Delta^n} = 0$ on $\mathbf{T}^{\kappa^n}$ for some $n \in \mathbf{N}$. In particular, $\sigma \in \mathcal{P}(\mathbf{T})$, so there exists a $p(t) \in \mathbf{C}[t]$ so that $t^{\sigma} = p(t)$.

By way of contradiction, suppose $d = \deg p(t) \ge 2$. We claim that there exists a sequence $\{t_0, t_1, t_2, \dots\} \subseteq \mathbf{T}$ with $t_\nu = \sigma(t_{\nu-1}) = p^{\circ\nu}(t_0)$ and $t_{\nu-1} < t_\nu$ for $\nu \in \mathbf{N}$. To see this, first suppose $\mathbf{T}$ possesses right-dense elements. Since $a \in \mathbf{T}$ is right-dense if and only if $a = \sigma(a) = p(a)$ and since $p(t) - t$ has only a finite number of zeros, it follows that there is a maximal right-dense element, $M \in \mathbf{T}$. By definition, there exists a $t_0 \in \mathbf{T}$ with $M < t_0$. Now define $t_\nu = \sigma(t_{\nu-1}) = p(t_{\nu-1}) = p^{\circ\nu}(t_0)$. Since $p(t) = \sigma(t) \ge t$ for $t \in \mathbf{T}$, it follows that $t_{\nu-1} \le t_\nu$. Since M was maximal, it also follows that $t_\nu < t_{\nu+1}$, and we are done. On the other hand, if $\mathbf{T}$ has no right-dense points, choose any $t_0 \in \mathbf{T}$ to start and set $t_\nu = \sigma(t_{\nu-1})$ as before.

We now claim that, for any $m \in \mathbf{N}$, $p^{\Delta^m}(t)$ is the restriction of a rational function $P_m(t)/Q_m(t)$ on $\{t_0, t_1, t_2, \dots\}$, where $P_m(t), Q_m(t) \in$

$\mathbf{C}[t]$ are polynomials with $Q_m(t)$ not vanishing on $\{t_0, t_1, t_2, \dots\}$ and

$$\deg P_m(t) - \deg Q_m(t) = \frac{d^{m+1}(d-2)+d}{d-1}.$$

Since $d = \deg p(t) \geq 2$, we have $[d^{m+1}(d-2)+d]/(d-1) > 0$, and hence $p^{\Delta^n}(t)$ cannot vanish identically on $\{t_0, t_1, t_2, \dots\}$, which will give us our desired contradiction and finish the proof.

Thus, it remains to verify the claim. First note that

$$p^{\Delta}(t) = \frac{p(p(t)) - p(t)}{p(t) - t}$$

on $\{t_0, t_1, t_2, \dots\}$. Set $P_1(t) = p(p(t)) - p(t)$ and $Q_1(t) = p(t) - t$. Since $\deg p^{\circ m}(t) = d^m$, it follows that $\deg P_1(t) = d^2$ and $\deg Q_1(t) = d$. Furthermore, $Q_1(t) > 0$ on $\{t_0, t_1, t_2, \dots\}$ by construction. Since $\deg P_1(t) - \deg Q_1(t) = d^2 - d$ and since $[d^2(d-2)+d]/(d-1) = d^2 - d$, we are done with the $m = 1$ case. For the inductive step, note that

$$\begin{aligned} p^{\Delta^m}(t) &= \frac{p^{\Delta^{m-1}}(p(t)) - p^{\Delta^{m-1}}(t)}{p(t) - t} \\ &= \frac{[P_{m-1}(p(t))]/[Q_{m-1}(p(t))] - [P_{m-1}(t)]/[Q_{m-1}(t)]}{p(t) - t} \\ &= \frac{P_{m-1}(p(t))Q_{m-1}(t) - P_{m-1}(t)Q_{m-1}(p(t))}{Q_{m-1}(p(t))Q_{m-1}(t)\,(p(t) - t)} \end{aligned}$$

on $\{t_0, t_1, t_2, \dots\}$. Set $P_m(t) = P_{m-1}(p(t))Q_{m-1}(t) - P_{m-1}(t)Q_{m-1}(p(t))$ and $Q_m(t) = Q_{m-1}(p(t))Q_{m-1}(t)(p(t) - t)$. Clearly, we have $\deg P_m(t) = d\deg P_{m-1}(t) + \deg Q_{m-1}(t)$ and $\deg Q_m(t) = d\deg Q_{m-1}(t) + d$. Therefore, using the induction hypothesis, $\deg P_m(t) - \deg Q_m(t) = d(\deg P_{m-1}(t) - \deg Q_{m-1}(t) - 1) = d((d^m(d-2) + d/d-1) - 1) = [d^{m+1}(d-2)+d]/(d-1)$. This completes the proof, and the lemma follows as noted above. ∎

Corollary 4. $\mathcal{P}(\mathbf{T}) = \mathbf{C}[t]$ *if and only if* $\mathbf{T}$ *is one of the following*:

(a) *a bounded or unbounded closed interval,*

(b) *an arithmetic sequence* (*possibly bounded below*), *or*

(c) *a geometric sequence or the closure of a geometric sequence.*

Proof. By the previous proposition, $t^\sigma = at + b$ whenever $\mathcal{P}(\mathbf{T}) = \mathbf{C}[t]$. Furthermore, $\mathcal{P}(\mathbf{T}) = \mathbf{C}[t]$ is equivalent to $\mathcal{P}(\mathbf{T}') = \mathbf{C}[t]$ if $\mathbf{T}'$ is obtained from $\mathbf{T}$ by an orientation preserving affine linear transformation of $\mathbf{R}$. Thus, without loss of generality, we can assume that either (a) $t^\sigma = t$, (b) $t^\sigma = t+1$, or (c) $t^\sigma = qt$ with $1 \neq q > 0$. The three normal forms correspond to the three cases of the proposition. Conversely, if t^σ is one of the three normal forms above, it is easy to verify that $\mathcal{P}(\mathbf{T}) = \mathbf{C}[t]$. For example, if $t^\sigma = qt$ with $q > 1$, we may assume (after another linear change of coordinates) that $\mathbf{T} = q^{\mathbf{N}_0}$ or $\mathbf{T} = q^{\mathbf{Z}} \cup \{0\}$. For these time scales (see [**1**, (1.19)]),

$$h_k(t,s) = \prod_{i=0}^{k-1} \frac{t - q^i s}{\sum_{j=0}^{i} q^j}.$$

Since the $h_k(t,s)$ for fixed $s \in \mathbf{T}$ form a basis for both $\mathcal{P}(\mathbf{T})$ and $\mathbf{C}[t]$, we conclude that $\mathcal{P}(\mathbf{T}) = \mathbf{C}[t]$ in this case. □

4. Weyl algebras and Ore extensions.

4.1. Differential operators. In the following, we will assume that

$$\mathbf{T} = \mathbf{T}^\kappa \quad \text{and} \quad C^\infty(\mathbf{T}) \text{ is closed under multiplication.}$$

Thus, the space $C^\infty(\mathbf{T})$ of infinitely delta differentiable functions on $\mathbf{T}$ is a $\mathbf{C}$-algebra with the product given (as usual) by pointwise multiplication: $(fg)(t) = f(t)g(t)$ for all $t \in \mathbf{T}$. Let $\operatorname{End} C^\infty(\mathbf{T})$ denote the algebra of $\mathbf{C}$-linear endomorphisms of $C^\infty(\mathbf{T})$. We have an injective $\mathbf{C}$-algebra homomorphism $C^\infty(\mathbf{T}) \hookrightarrow \operatorname{End} C^\infty(\mathbf{T})$ given by $f \mapsto (g \mapsto fg)$. Furthermore, the delta derivative defines a linear differential operator $\delta : C^\infty(\mathbf{T}) \to C^\infty(\mathbf{T})$ given by $\delta(g) = g^\Delta$. By the product rule (2.3.2), for $f \in C^\infty(\mathbf{T})$, we have the identity

$$\delta \cdot f = f^\sigma \delta + f^\Delta \tag{4.1.2}$$

in $\operatorname{End} C^\infty(\mathbf{T})$. Note that $\delta \cdot f$ denotes the operator given by $(\delta \cdot f)(g) = \delta(fg) = (fg)^\Delta$ which is not to be confused with $\delta(f) = f^\Delta$.

4.2. Weyl algebras on a time scale. Keeping our assumption (4.1.1), we define the *Weyl algebra* on $\mathbf{T}$ as the subalgebra $\mathcal{W}(\mathbf{T})$ of

$\operatorname{End} C^\infty(\mathbf{T})$ that is generated by the operators t and δ:

$$\mathcal{W}(\mathbf{T}) = \mathbf{C}\langle t, \delta\rangle.$$

If $\mathbf{T} = \mathbf{R}$ then $\mathcal{W}(\mathbf{T})$ is the usual Weyl algebra, i.e., the algebra of linear differential operators on $\mathbf{R}$ with polynomial coefficients. The generating relation in this case is the Heisenberg relation $\delta \cdot t = t\delta + 1$. If $\mathbf{T} = q^{\mathbf{N}_0}$ with $q > 1$, then $\mathcal{W}(\mathbf{T})$ is the quantum Weyl algebra with the generating relation $\delta \cdot t = qt\delta + 1$.

Lemma 5. *The time scale analogue of the Heisenberg relation is the identity*

$$[\delta, t] = \sigma^*.$$

Proof. By (4.1.2), $\delta \cdot t = t^\sigma \delta + 1$, and hence $[\delta, t] = (t^\sigma - t)\delta + 1$. If $t_0 \in \mathbf{T}$ and $\sigma(t_0) > t_0$, then

$$\begin{aligned}([\delta, t] - 1)(f)|_{t=t_0} &= (\sigma(t_0) - t_0)f^\Delta(t_0)\\ &= f(\sigma(t_0)) - f(t_0) = (\sigma^* - 1)(f)|_{t=t_0},\end{aligned}$$

and hence $[\delta, t](f)|_{t=t_0} = \sigma^*(f)|_{t=t_0}$. The argument in the case when $\sigma(t_0) = t_0$ is similar and left to the reader. $\square$

Remark. One can show that the total degree (Bernstein degree) of σ^* is two unless $\mathbf{T}$ is a closed interval (in which case $\sigma^* = 1$ is of total degree zero).

4.3. Ore extensions. Let A be a $\mathbf{C}$-algebra and $\alpha : A \to A$ an algebra endomorphism. A $\mathbf{C}$-linear map $\delta : A \to A$ is called an *α-derivation* if $\delta(ab) = \delta(a)b + \alpha(a)\delta(b)$ for all $a, b \in A$. If x is an indeterminate, we define a (non-commutative) multiplication on the free left A-module $\bigoplus_{i=0}^\infty Ax^i$ by the rule

$$x \cdot a = \alpha(a)x + \delta(a)$$

for $a \in A$. The resulting $\mathbf{C}$-algebra is denoted $A[x; \alpha, \delta]$ and is called a *skew-polynomial ring* over A or an *Ore extension* of A. If A is left

Noetherian (e.g., if A is a finitely generated commutative algebra) and α is an automorphism, then $A[x;\alpha,\delta]$ is also left Noetherian.

4.4. Ore extensions on time scales. Let A be a subalgebra of $C^\infty(\mathbf{T})$ that is invariant under the delta derivative and pull-pack by σ. Define $\sigma^* : A \to A$ by $\sigma^*(f) = f^\sigma$. Then σ^* is an algebra homomorphism and, by the product rule (2.3.2), the differential operator $\delta : A \to A$ given by $\delta(f) = f^\Delta$ is a σ^*-derivation. Thus, we can form an Ore extension $A[x;\sigma^*,\delta]$. In light of the identity, we obtain an algebra homomorphism $A[x;\sigma^*,\delta] \to A[\delta] \subseteq \operatorname{End} C^\infty(\mathbf{T})$ by mapping x to δ and $f \in A$ to the multiplication operator corresponding to f.

Lemma 6. *The homomorphism $A[x;\sigma^*,\delta] \to A[\delta] \subseteq \operatorname{End} C^\infty(\mathbf{T})$ is injective.*

Proof. Let $P = \sum_{i=0}^m a_i\delta^i$ be such that $P(f) = 0$ for all $f \in C^\infty(\mathbf{T})$. Consider the functions $f_k \in C^\infty(\mathbf{T})$ given by $f_k(t) = h_k(t,s)$, where the $h_k(t,s)$ are defined as in (2.5.1). Then $\delta^i(f_k) = 1$ if $i = k$ and $= 0$ if $i > k$. It follows that $a_i = 0$ for all i. $\square$

Corollary 7. *Let $\mathbf{T}$ be a time scale satisfying* (4.1.1)*. If $A \subseteq C^\infty(\mathbf{T})$ is a subalgebra containing $\mathbf{C}[t]$ that is stable under δ and σ^*, then the Weyl algebra $\mathcal{W}(\mathbf{T})$ is isomorphic to a subalgebra of the Ore extension $A[x;\sigma^*,\delta]$.*

4.5. Polynomial time scales. In general, the differential operators in $\mathcal{W}(\mathbf{T})$ do not have polynomial coefficients. More precisely, we have the result.

Proposition 8. *Let $\mathbf{T}$ be a time scale satisfying* (4.1.1)*. Then $\mathcal{W}(\mathbf{T}) = \operatorname{span}_{\mathbf{C}}\{t^i\delta^j \mid i,j \in \mathbf{N}_0\}$ if and only if $t^\sigma = p(t)$ for some $p(t) \in \mathbf{C}[t]$. In that case, the generating relation is $\delta \cdot t = p(t)\delta + 1$.*

Proof. First suppose that $t^\sigma = p(t) \in \mathbf{C}[t]$. Clearly $B = \operatorname{span}\{t^k\delta^j \mid k,j \in \mathbf{N}_0\} \subseteq \mathcal{W}(\mathbf{T})$, so it remains to see the reverse inclusion. For this, we show $V^n \subseteq B$ for each $n \in \mathbf{N}_0$, where V^n is the span of all words of length at most n in t and δ. To this end, we induct on n. Since the case of $n = 0$ is clear, assume $n \geq 1$. By construction, V^n is the span of V^{n-1}, tV^{n-1} and δV^{n-1}. By the inductive hypothesis, $V^{n-1} \subseteq B$ and hence $tV^{n-1} \subseteq B$ as well. It remains to examine δV^{n-1}. By linearity, it suffices to show that $\delta \cdot t^j \in B$ for $j \in \mathbf{N}_0$. For this, we proceed by induction on j. Since the case of $j = 0$ is clear, assume $j \geq 1$ and calculate

$$\delta \cdot t^j = \delta \cdot t \cdot t^{j-1} = (1 + t^\sigma \delta) \cdot t^{j-1} = t^{j-1} + p(t)\delta \cdot t^{j-1}.$$

By the induction hypothesis, $\delta \cdot t^j \in B$, and we are done.

For the converse, observe that $\delta \cdot t = 1 + t^\sigma \delta$, and hence $t^\sigma \delta \in \mathcal{W}(\mathbf{T}) = \operatorname{span}_{\mathbf{C}}\{t^i \delta^j \mid i, j \in \mathbf{N}_0\}$. Write $t^\sigma \delta = \sum_{k=0}^m p_k(t)\delta^k$ with $p_k(t) \in \mathbf{C}[t]$. Applying both sides to the function 1 gives $0 = p_0(t)$, and then applying both sides to the function t gives $t^\sigma = p_1(t)$. Since $t^\sigma = \sigma(t)$, we are done. $\square$

Corollary 9. *Let $\mathbf{T}$ be a time scale with $t^\sigma = p(t)$ for some $p(t) \in \mathbf{C}[t]$. Then the Weyl algebra $\mathcal{W}(\mathbf{T})$ is an Ore extension of $\mathbf{C}[t]$, namely, $\mathcal{W}(\mathbf{T}) = \mathbf{C}[t][x; \sigma^*, \delta]$.*

5. Gelfand-Kirillov dimension.

5.1. Basics. (For more details the reader should consult [**7**].) The *Gelfand-Kirillov dimension* of a $\mathbf{C}$-algebra A, denoted $\operatorname{GKdim} A$, is defined as

$$\operatorname{GKdim} A = \sup_V \limsup_{n \to \infty} \frac{\log(\dim V^n)}{\log n},$$

where V runs through all finite-dimensional subspaces of A with $1 \in V$. Here V^n denotes the subspace of A that is spanned by "monomials of degree n in elements of V," i.e., $V^n = \operatorname{span}_{\mathbf{C}}\{v_1 v_2 \cdots v_n \mid v_i \in V\}$. Note that $1 \in V$ implies that $V^1 \subseteq V^2 \subseteq V^3 \subseteq \cdots$. It is a theorem that, if A is a finitely generated $\mathbf{C}$-algebra, then the supremum is attained for any V containing a system of generators. If B is a subalgebra of A, then $\operatorname{GKdim} B \leq \operatorname{GKdim} A$. The Gelfand-Kirillov dimension of an Ore extension satisfies the inequality

$$\operatorname{GKdim} A[x; \alpha, \delta] \geq \operatorname{GKdim} A + 1,$$

where equality is attained if every finite-dimensional subspace U of A is contained in a finite-dimensional subspace V that is α-stable and satisfies $\delta(V) \subseteq V^m$ for some $m \geq 1$ (see [**5**]).

5.2. Gelfand-Kirillov of Weyl algebras. Assume that $\mathbf{T}$ is a time scale satisfying (4.1.1), and let $\mathcal{W}(\mathbf{T})$ be the Weyl algebra on $\mathbf{T}$. As we will see, computing $\operatorname{GKdim} \mathcal{W}(\mathbf{T})$ is a difficult problem in general. First, let us note the following elementary lower bound.

Lemma 10. $\operatorname{GKdim} \mathcal{W}(\mathbf{T}) \geq 2$.

Proof. Let $V = \operatorname{span}_{\mathbf{C}}\{1, t, \delta\}$. For $0 \leq i + j \leq n$, the operators $t^i\delta^j$ are contained in V^n. By the proof of Lemma 6, the $t^i\delta^j$'s are linearly independent. Thus, $\dim V^n \geq \binom{n+1}{2}$, and hence $\operatorname{GKdim} \mathcal{W}(\mathbf{T}) \geq 2$, as desired. □

5.3. Polynomial time scales. If $\mathbf{T} = \mathbf{R}$, $\mathbf{Z}$ or $q^{\mathbf{N}_0}$ with $q > 1$, then $\operatorname{GKdim} \mathcal{W}(\mathbf{T}) = 2$. The following result says that these examples are essentially the only examples (see the proof of Corollary 4) of polynomial time scales having a two-dimensional Weyl algebra.

Proposition 11. *Let* $\mathbf{T}$ *be a time scale such that* $t^\sigma = p(t)$ *for some* $p(t) \in \mathbf{C}[t]$. *Then* $\operatorname{GKdim} \mathcal{W}(\mathbf{T}) = 2$ *if and only if* $\deg p(t) \leq 1$.

Proof. By Corollary 9, $\mathcal{W}(\mathbf{T}) = \mathbf{C}[t][x, \sigma^*, \delta]$ is an Ore extension. By a theorem of Zhang [**10**], $\operatorname{GKdim} \mathbf{C}[t][x; \sigma^*, \delta] = \operatorname{GKdim} \mathbf{C}[t] + 1 = 2$ if and only if every finite-dimensional subspace of $\mathbf{C}[t]$ is contained in a σ^* stable finite-dimensional subspace. It is elementary to verify that σ^* in our context satisfies this property if and only if the degree of $p(t)$ is ≤ 1. □

5.4. Periodic time scales. Let $\mathbf{T} = \{t_\nu \mid \nu \in \mathbf{Z}\}$ with $t_{\nu+1} > t_\nu$ for all $\nu \in \mathbf{Z}$. We say that $\mathbf{T}$ is *periodic with period* k if $\sigma(t_{\nu+k}) - t_{\nu+k} = \sigma(t_\nu) - t_\nu$ for all $\nu \in \mathbf{Z}$. We will see below that the Weyl algebra of a discrete periodic time scale is two dimensional. First, we prove the following general lemma.

Lemma 12. *Let* $\mathbf{T}$ *be a time scale satisfying* (4.1.1), *and let* $A \subseteq C^\infty(\mathbf{T})$ *be a subalgebra that is* δ- *and* σ^*-*stable. If* $A = \mathbf{C}[t][e_0, \dots, e_m]$, *where the* e_i*'s are idempotent, then* $\operatorname{GKdim} A = 1$. *If, furthermore, every finite-dimensional subspace of* A *is contained in a finite-dimensional subspace that is* δ- *and* σ-*stable, then* $\operatorname{GKdim} A[x;\sigma^*,\delta] = 2$.

Proof. Let $U = \operatorname{span}_{\mathbf{C}}\{1, t, e_0, \dots, e_m\}$. Since $e_i^2 = e_i$ and $e_i e_j = 0$ for $i \neq j$, it follows that $U^n = \operatorname{span}_{\mathbf{C}}\{t^n, t^{n-1}e_1, \dots, t^{n-1}e_m\} + U^{n-1}$, and hence $n+1 \leq \dim U^n \leq n(m+1)+1$ for all $n \in \mathbf{N}$. This proves that $\operatorname{GKdim} A = 1$. The last statement follows from our remarks at the end of subsection 5.1. $\square$

Proposition 13. *Let* $\mathbf{T}$ *be a discrete periodic time scale. Then* $\operatorname{GKdim} \mathcal{W}(\mathbf{T}) = 2$.

Proof. Let $\mathbf{T} = \{t_\nu \mid \nu \in \mathbf{Z}\}$ with $t_{\nu+1} > t_\nu$ be a periodic time scale of period k. For $0 \leq i < k$, define the idempotent e_i by

$$e_i(t_\nu) = \begin{cases} 1 & \text{if } \nu \equiv i \bmod k \\ 0 & \text{if } \nu \not\equiv i \bmod k, \end{cases}$$

and let $A = \mathbf{C}[t][e_0, \dots, e_{k-1}]$. It is easy to check that A is δ- and σ^*-invariant. Furthermore, every finite-dimensional subspace of A is contained in a finite-dimensional subspace that is δ- and σ^*-stable. By the previous lemma, $\operatorname{GKdim} A[x;\sigma^*,\delta] = 2$. By Corollary 6, $\mathcal{W}(\mathbf{T})$ is a subalgebra of $A[x;\sigma^*,\delta]$, and hence $\operatorname{GKdim} \mathcal{W}(\mathbf{T}) \leq \operatorname{GKdim} A[x;\sigma^*,\delta] = 2$. $\square$

5.5. The time scale $\mathbf{T} = \mathbf{Z} \setminus \{0\}$. We have the following general conjecture.

Conjecture 14. *Let* $\mathbf{T}$ *and* $\mathbf{T}'$ *be two (infinite) discrete time scales such that their symmetric difference* $\mathbf{T}\Delta\mathbf{T}' = (\mathbf{T} \setminus \mathbf{T}') \cup (\mathbf{T}' \setminus \mathbf{T})$ *is a finite set. Then* $\operatorname{GKdim} \mathcal{W}(\mathbf{T}) = \operatorname{GKdim} \mathcal{W}(\mathbf{T}')$.

In the following, we will consider the simplest non-trivial special case of the conjecture. Let $\mathbf{T} = \mathbf{Z} \setminus \{0\}$ and $\mathbf{T}' = \mathbf{Z}$, i.e., $\mathbf{T}\Delta\mathbf{T}' = \{0\}$. For

$i \in \mathbf{N}_0$, define the idempotent function e_{-i} on $\mathbf{T}$ by

$$e_{-i}(t) = \begin{cases} 1 & \text{if } t = -i; \\ 0 & \text{otherwise.} \end{cases}$$

Note that $\sigma(t) = t + 1 + e_{-1}(t)$ for every $t \in \mathbf{T}$. We also note that the Weyl algebra $\mathcal{W}(\mathbf{T})$ can be embedded as a subalgebra in $A[x; \sigma^*, \delta]$ with $A = \mathbf{C}[t][e_{-i} \mid i \in \mathbf{N}_0]$. It is possible to show that $\operatorname{GKdim} A = 1$, but we are unable to conclude that $\operatorname{GKdim} A[x; \sigma^*, \delta] = 2$ since not every subspace of A is contained in a finite dimensional σ^*-stable subspace.

Proposition 15. *Let* $\mathbf{T} = \mathbf{Z} \setminus \{0\}$. *Then* $\operatorname{GKdim} \mathcal{W}(\mathbf{T}) = 2$.

Proof. Let $V = \operatorname{span}_{\mathbf{C}}\{1, t, \delta\}$. We claim that V^n is obtained from V^{n-1} by adding the span of

$$t^i \cdot \delta^{n-i}, \quad 0 \le i \le n,$$

and

$$e_{-i} \cdot (\delta + 1)^{i-1} \cdot \delta^{n-i}, \quad 1 \le i \le n - 1.$$

Clearly, the claim implies that $\dim V^n \le \dim V^{n-1} + 2n$ and hence $\dim V^n \le n^2 + n + 1$, which gives $\operatorname{GKdim} \mathcal{W}(\mathbf{T}) \le 2$. Thus, it remains to prove the claim. We proceed by induction on n. Since V^1 is spanned by $V^0 = \mathbf{C}$ and $\{t, \delta\}$, the case of $n = 1$ is complete. For the inductive step, note that V^n is obtained from V^{n-1} by adding the span of

$$\begin{aligned}
&t \cdot t^i \cdot \delta^{n-1-i} = t^{i+1} \cdot \delta^{n-1-i}, \quad 0 \le i \le n-1, \\
&\delta \cdot t^i \cdot \delta^{n-1-i}, \quad 0 \le i \le n-1, \\
&t \cdot e_{-i} \cdot (\delta+1)^{i-1}\delta^{n-1-i} = -ie_{-i} \\
&\qquad \cdot (\delta+1)^{i-1} \cdot \delta^{n-1-i}, \quad 1 \le i \le n-2, \\
&\delta \cdot e_{-i} \cdot (\delta+1)^{i-1} \cdot \delta^{n-1-i}, \quad 1 \le i \le n-2.
\end{aligned}$$

The $t^{i+1} \cdot \delta^{k-1-i}$ terms account for the $t^i \cdot \delta^{n-i}$ terms for $1 \le i \le n$, the term $\delta \cdot t^i \cdot \delta^{n-1-i}$ for $i = 0$ gives δ^n, and the $-ie_{-i} \cdot (\delta+1)^{i-1} \cdot \delta^{n-1-i}$ terms are in V^{n-1}. Therefore, we only need to examine the $\delta \cdot t^i \cdot \delta^{n-1-i}$

terms for $1 \le i \le n-1$ and the $\delta \cdot e_{-i} \cdot (\delta+1)^{i-1}\delta^{n-1-i}$ terms for $1 \le i \le n-2$. For the first case with $1 \le i \le n-1$, calculate that

$$\begin{aligned}\delta \cdot t^i \cdot \delta^{n-1-i} &= \left(((t+1)^i - t^i) + (t+1)^i\delta + \frac{3+(-1)^i}{2}\, e_{-1}\delta \right)\delta^{n-1-i} \\ &= ((t+1)^i - t^i)\delta^{n-1-i} + (t+1)^i\delta^{k-i} + \frac{3+(-1)^i}{2}\, e_{-1}\delta^{k-i}.\end{aligned}$$

Since $((t+1)^i - t^i)\delta^{n-1-i} \in V^{n-1}$ since we already have $t^i\delta^{n-i} \in V^n$ for $0 \le i \le k$, this term is equivalent to $[3+(-1)^i]/2\, e_{-1}\delta^{n-i}$. We also have $e_{-1}\delta^{n-i} \in V^{n-j+1} \subseteq V^{n-1}$ when $n-j+1 \le n-1$, i.e., when $j \ge 2$. Thus, it only remains to look at $j=1$. In that case we get $e_{-1}\delta^{n-1}$. We now turn to the second case. We must show that the given terms $\delta e_{-i} \cdot (\delta+1)^{i-1} \cdot \delta^{n-1-i}$ for $1 \le i \le n-2$ give rise to the desired terms $e_{-i} \cdot (\delta+1)^{i-1} \cdot \delta^{n-i}$, $2 \le i \le n-1$. To that end, we calculate (letting $c_1 = -1/2$ and $c_j = -1$ when $j \ge 2$) that

$$\begin{aligned}&\delta \cdot e_{-i} \cdot (\delta+1)^{i-1}\delta^{n-1-i} \\ &\qquad = (e_{-i-1} - c_j e_{-i} + e_{-i-1} \cdot \delta)\,(\delta+1)^{i-1}\delta^{n-1-i} \\ &\qquad = -c_j e_{-i} \cdot (\delta+1)^{i-1} \cdot \delta^{n-1-i} + e_{-i-1} \cdot (\delta+1)^i \cdot \delta^{n-1-i}.\end{aligned}$$

Since $e_{-i} \cdot (\delta+1)^{i-1} \cdot \delta^{n-1-i} \in V^{n-1}$, this is equivalent to $e_{-i-1} \cdot (\delta + 1)^i \cdot \delta^{n-1-i}$, which finishes the proof. □

6. Lower bounds for the Gelfand-Kirillov dimension.

6.1. Leading terms. In the following, let $\mathbf{T}$ be a time scale satisfying (4.1.1). For $k, i \in \mathbf{N}_0$, define

$$d_{k,i} = \dim \left\{\text{polynomials in } t,\, t^\sigma,\, t^{\sigma\sigma}, \dots, t^{\sigma^k} \text{ of degree} \le i\right\}. \tag{6.1.1}$$

For example, $d_{0,i} = i+1$ for all i, $d_{1,2} = \dim(\mathrm{span}_{\mathbf{C}}\{1, t, t^\sigma, t^2, t \cdot t^\sigma, (t^\sigma)^2\})$, and $d_{2,1} = \dim(\mathrm{span}_{\mathbf{C}}\{1, t, t^\sigma, t^{\sigma\sigma}\})$.

Lemma 16. *Let* $V = \mathrm{span}_{\mathbf{C}}\{1, t, \delta\} \subseteq \mathcal{W}(\mathbf{T})$. *Then* $\dim(V^n) \ge \sum_{k=0}^n d_{k,n-k}$.

Proof. We say that a differential operator $P = \sum_{i=0}^{k} a_i \delta^i \in \mathcal{W}(\mathbf{T})$ with $a_i \in C^\infty(\mathbf{T})$ has order k if $a_k \neq 0$; the term $a_k\delta^i$ is called the leading term of P. For $n, k \in \mathbf{N}_0$ such that $0 \leq k \leq n$, define

$$L_{n,k} = \operatorname{span}_{\mathbf{C}}\bigl\{\text{leading terms of elements in } V^n \text{ of order } k\bigr\}$$

and $L_n = \sum_{k=0}^{n} L_{n,k}$. By the proof of Lemma 6, this sum is direct. Since, for any $i, j \in \mathbf{N}_0$,

$$\delta^i \cdot t^j = \left(t^{\sigma^i}\right)^j \delta^i + \text{lower order terms},$$

it is straightforward to see that the coefficients of the leading terms of the monomials in t and δ of degree $\leq n$ are precisely the monomials in t, $t^\sigma, t^{\sigma\sigma}, \ldots, t^{\sigma^k}$ of degree $\leq n-k$. This shows that $\dim(L_{n,k}) = d_{k,n-k}$, and hence $\dim(L_n) = \sum_{k=0}^{n} d_{k,n-k}$. Since $L_n \subseteq V^n$, it follows that $\sum_{k=0}^{n} d_{k,n-k} \leq \dim(V^n)$, and the proof of the lemma is complete. □

Proposition 17.

$$\operatorname{GKdim} \mathcal{W}(\mathbf{T}) \geq \limsup_{n\to\infty} \frac{\log\left(\sum_{k=0}^{n} d_{k,n-k}\right)}{\log n}.$$

Proof. The proposition follows immediately from the previous lemma and the definition of $\operatorname{GKdim} \mathcal{W}(\mathbf{T})$. □

Conjecture 18. *The inequality in Proposition* 17 *is in fact an equality.*

6.2. A family of examples. As a first application of Proposition 17, we prove that there exist time scales $\mathbf{T}$ for which $\operatorname{GKdim} \mathcal{W}(\mathbf{T})$ is arbitrarily large.

Lemma 19. *Let $p_1, \ldots, p_m$ be the first m primes, and let $\mathbf{T} = \{\sum_{j=1}^{m} p_j^\nu \mid \nu \in \mathbf{N}_0\}$. Then the functions $t, t^\sigma, \ldots, t^{\sigma^{m-1}}$ on $\mathbf{T}$ are algebraically independent.*

Proof. For $\nu \in \mathbf{N}_0$, set $t_\nu = \sum_{j=1}^m p_j^\nu$. We claim that there exists an invertible $m \times m$-matrix $A = (a_{ij})$ with rational coefficients such that

$$p_j^\nu = \sum_{i=1}^{m-1} a_{ij} t_{\nu+i-1} \quad \text{for all } \nu \in N_0. \tag{6.2.1}$$

In fact, A is explicitly given as

$$A = \begin{pmatrix} 1 & p_1 & p_1^2 & \cdots & p_1^{m-1} \\ 1 & p_2 & p_2^2 & \cdots & p_2^{m-1} \\ 1 & p_m & p_m^2 & \cdots & p_m^{m-1} \end{pmatrix}^{-1}.$$

Equation (6.2.1) follows since

$$A = \begin{pmatrix} p_1^\nu & p_1^{\nu+1} & \cdots & p_1^{\nu+m-1} \\ p_2^\nu & p_2^{\nu+1} & \cdots & p_2^{\nu+m-1} \\ p_m^\nu & p_m^{\nu+1} & \cdots & p_m^{\nu+m-1} \end{pmatrix}^{-1} \begin{pmatrix} p_1^\nu & & & \\ & p_2^\nu & & \\ & & \ddots & \\ & & & p_m^\nu \end{pmatrix}$$

for all $\nu \in \mathbf{N}_0$. Clearly, showing that the functions $t, t^\sigma, \dots, t^{\sigma^{m-1}}$ are algebraically independent is equivalent to showing that the functions $f_j = \sum_{i=1}^{m-1} a_{ij} t^{\sigma^{i-1}}$, $1 \le j \le m$, are algebraically independent. So suppose that there is a relation

$$\sum_{i_1,\dots,i_m} c_{i_1,\dots,i_m} f_1^{i_1} \cdots f_m^{i_m} = 0, \tag{6.2.2}$$

with some coefficient $c_{i_1,\dots,i_m} \neq 0$. By (6.2.1), $f_j(t_\nu) = p_j^\nu$ for all $\nu \in \mathbf{N}_0$, and hence

$$\sum_{i_1,\dots,i_m} c_{i_1,\dots,i_m} (p_1^{i_1} \cdots p_m^{i_m})^\nu = 0 \tag{6.2.3}$$

with some coefficient $c_{i_1,\dots,i_m} \neq 0$. Let $M = \max\{p_1^{i_1} \cdots p_m^{i_m} \mid c_{i_1,\dots,i_m} \neq 0\}$. Multiplying equation (6.2.3) by $M^{-\nu}$, and taking the limit as $\nu \to \infty$, shows that

$$\sum_{p_1^{i_1} \cdots p_m^{i_m} = M} c_{i_1,\dots,i_m} = 0. \tag{6.2.4}$$

By the uniqueness of the prime decomposition of M, there is only one coefficient $c_{i_1,\dots,i_m}$ with $p_1^{i_1}\cdots p_m^{i_m} = M$. By (6.2.4), this coefficient is zero, and we have a contradiction. $\square$

Proposition 20. *Let $p_1,\dots,p_m$ be the first m primes, and let $\mathbf{T} = \{\sum_{j=1}^m p_j^\nu \mid \nu \in \mathbf{N}_0\}$. Then* $\operatorname{GKdim} \mathcal{W}(\mathbf{T}) \geq m+1$.

Proof. For $\nu \in \mathbf{N}_0$, let $t_\nu = \sum_{j=1}^m p_j^\nu$. For $1 \leq l \leq m$, define a_l by

$$\prod_{j=1}^m (1 - p_j x) = 1 - \sum_{l=1}^m a_l x^l.$$

Then, by the general theory of linear recursions, the t_ν's satisfy the relation

$$t_\nu = \sum_{l=1}^m a_l t_{\nu-l} \quad \text{for } \nu \geq m.$$

Thus, $t^{\sigma^k} = \sum_{l=1}^k a_l t^{\sigma^{k-l}}$, and hence

$$d_{k,i} = d_{m-1,i} \quad \text{for } k \geq m. \tag{6.2.5}$$

Furthermore, by Lemma 19,

$$d_{k,i} = \binom{k+i-1}{k+1} \quad \text{for } k \leq m-1. \tag{6.2.6}$$

From (6.2.5) and (6.2.6), one obtains

$$\sum_{k=0}^n d_{k,n-k} = \sum_{i=1}^{m+1} \binom{n+1}{i},$$

and hence $\lim_{n\to\infty} \log(\sum_{k=0}^n d_{k,n-k})/\log n = m+1$. The result now follows from Proposition 17. $\square$

Conjecture 21. *The inequality in Proposition* 20 *is in fact an equality.*

Remark. If Conjecture 21 is true, then for every integer $r \geq 2$, there exists a discrete time scale $\mathbf{T} = \{t_\nu \mid \nu \in \mathbf{N}_0\}$ such that $\operatorname{GKdim} \mathcal{W}(\mathbf{T}) = r$. We note that, in general, the Gelfand-Kirillov dimension of an algebra does not need to be an integer. In fact, Borho and Kraft proved in [**1**] that, for every real number $r \geq 2$, there exists a $\mathbf{C}$-algebra A generated by two generators such that $\operatorname{GKdim} A = r$. Thus, it is conceivable (and in fact we expect) that for every real number $r \geq 2$, there exists a discrete time scale $\mathbf{T} = \{t_\nu \mid \nu \in \mathbf{N}_0\}$ such that $\operatorname{GKdim} \mathcal{W}(\mathbf{T}) = r$.

6.3. An infinite-dimensional Weyl algebra. As a second application of Proposition 17, we prove that there exist time scales for which $\operatorname{GKdim} \mathcal{W}(\mathbf{T}) = \infty$.

Proposition 22. *Let $\mathbf{T}$ be a time scale with $t^\sigma = t^2$. Then $\operatorname{GKdim} \mathcal{W}(\mathbf{T}) = \infty$.*

Proof. If $t^\sigma = t^2$, then $t^{\sigma^k} = t^{2^k}$ for all k. Thus,

$$\begin{aligned}
d_{k,i} &= 1 + \dim \left(\operatorname{span}\{t^{\sum_{l=1}^{j} 2^{n_l}} \mid 1 \leq j \leq i,\ 0 \leq n_l \leq k\}\right) \\
&= 1 + \dim \left(\operatorname{span}\left\{t^{\sum_{m=0}^{k} c_m 2^m} \mid 1 \leq \sum_{m=0}^{k} c_m \leq i\right\}\right) \\
&= \dim \left(\operatorname{span}\left\{t^{\sum_{m=0}^{k} c_m 2^m} \mid 0 \leq \sum_{m=0}^{k} c_m \leq i\right\}\right) \\
&= \left|\left\{ \sum_{m=0}^{k} c_m 2^m \mid \sum_{m=0}^{k} c_m \leq i\right\}\right|.
\end{aligned}$$

Here, and in the following, the coefficients c_m are nonnegative integers. From this it follows (multiply by 2) that

$$\begin{aligned}
d_{k,i} &= \left|\left\{ \sum_{m=0}^{k} c_m 2^m \mid \sum_{m=0}^{k} c_m \leq i\right\}\right| \\
&= \left|\left\{ \sum_{m=0}^{k} c_m 2^{m+1} \mid \sum_{m=0}^{k} c_m \leq i\right\}\right|
\end{aligned}$$

$$= \left| \left\{ \sum_{m=0}^{k+1} c_m 2^m \mid \sum_{m=0}^{k+1} c_m \leq i \text{ with } c_0 = 0 \right\} \right|$$

and (multiply by 2 and add 1) that

$$\begin{aligned} d_{k,i-1} &= \left| \left\{ \sum_{m=0}^{k} c_m 2^m \mid \sum_{m=0}^{k} c_m \leq i-1 \right\} \right| \\ &= \left| \left\{ 1 + \sum_{m=0}^{k} c_m 2^{m+1} \mid \sum_{m=0}^{k} c_m \leq i-1 \right\} \right| \\ &= \left| \left\{ \sum_{m=0}^{k+1} c_m 2^m \mid \sum_{m=0}^{k+1} c_m \leq i \text{ with } c_0 = 1 \right\} \right|. \end{aligned}$$

By parity, the sets $\{\sum_{m=0}^{k+1} c_m 2^m \mid \sum_{m=0}^{k+1} c_m \leq i \text{ with } c_0 = j\}$ for $j = 0, 1$ are disjoint and, since both sets are contained in $\{\sum_{m=0}^{k+1} c_m 2^m \mid \sum_{m=0}^{k+1} c_m \leq i\}$, it follows that

$$d_{k+1,i} \geq d_{k,i} + d_{k,i-1}. \tag{6.3.1}$$

For $k, i \in \mathbf{N}_0$, define $s_{0,i} = d_{0,i} = i+1$, $s_{k,0} = d_{k,0} = 1$, and, recursively, for $i \geq 1$,

$$s_{k+1,i} = s_{k,i} + s_{k,i-1}.$$

By (6.3.1), and by construction, $d_{k,i} \geq s_{k,i}$ for all $k, i \in \mathbf{N}_0$. Let $S_n = \sum_{k=0}^{n} s_{k,n-k}$ so that $\sum_{k=0}^{n} d_{k,n-k} \geq S_n$. Then

$$\begin{aligned} S_{n-1} + S_n &= \sum_{k=0}^{n-1} s_{k,n-k-1} + \sum_{k=0}^{n} s_{k,n-k} \\ &= \sum_{k=0}^{n-1} (s_{k,n-k} + s_{k,n-k-1}) + s_{n,0} \\ &= \sum_{k=0}^{n-1} s_{k+1,n-k} + 1 \\ &= \sum_{k=1}^{n} s_{k,n+1-k} + s_{n+1,0} + s_{0,n+1} - (n+2) \\ &= S_{n+1} - (n+2), \end{aligned}$$

and hence $S_{n+1} = S_n + S_{n-1} + (n+2)$. Using that $S_0 = 1$ and $S_1 = 1+2 = 3$, it is straightforward to prove by induction that, for all $n \in \mathbf{N}_0$,

$$S_n = F_{n+6} - n - 4 = \frac{\varphi^{n+6} - (1-\varphi)^{n+6}}{\sqrt{5}} - n - 4, \tag{6.3.2}$$

where F_{n+6} is the $(n+6)$-th Fibonacci number and $\varphi = (1+\sqrt{5})/2$ is the golden ratio. By (6.3.2), and since $\sum_{k=0}^{n} d_{k,n-k} \geq S_n$, we obtain

$$\lim_{n\to\infty} \frac{\log\left(\sum_{k=0}^{n} d_{k,n-k}\right)}{\log n} = \infty.$$

Thus, by Proposition 17, $\operatorname{GKdim} \mathcal{W}(\mathbf{T}) = \infty$. $\square$

Remark. The above proof can easily be generalized to $t^\sigma = t^k$ for any $k \geq 2$. This, in turn, can be used to prove that, if $t^\sigma = p(t)$ is any polynomial of degree ≥ 2, then $\operatorname{GKdim} \mathcal{W}(\mathbf{T}) = \infty$.

7. More examples.

7.1. A table of examples. Table 1 shows the value of

$$\lim_{n\to\infty} \frac{\log(\sum_{k=0}^{n} d_{k,n-k})}{\log n}$$

for a sample of discrete time scales of the form $\mathbf{T} = \{t_\nu \mid \nu \in \mathbf{N}\}$ with $t_{\nu+1} > t_\nu$ for all $\nu \in \mathbf{N}$. A question mark indicates a conjectured value. In the following subsections, we provide more details for the time scales that were not discussed in previous sections.

TABLE 1. $\lim_{n\to\infty} \frac{\log\left(\sum_{k=0}^{n} d_{k,n-k}\right)}{\log n}$ for some discrete time scales.

t_ν	$\sum_{k=0}^{n} d_{k,n-k}$	$\lim\limits_{n\to\infty} \frac{\log(\sum_{k=0}^{n} d_{k,n-k})}{\log n}$
$\nu h \quad (h>0)$	$\binom{n+2}{2}$	2
$q^\nu \quad (q>1)$	$\binom{n+2}{2}$	2
$F_\nu = \nu$-th Fibonacci	$\sum_{i=1}^{3} \binom{n+1}{i} - \binom{n-2}{3}$	2
$2^\nu + \nu$	$\sum_{i=1}^{3} \binom{n+1}{i}$	3
$\nu!$	$\sum_{i=1}^{4} \binom{n+1}{i}$	4 (?)
$\sum_{j=1}^{\nu} \frac{1}{j}$	$\sum_{i=1}^{4} \binom{n+1}{i}$	4 (?)
$\sum_{j=1}^{m} p_j^\nu$	$\sum_{i=1}^{m+1} \binom{n+1}{i}$	$m+1$
$t_\nu = t_{\nu-1} + t_{\nu-2}t_{\nu-3}$	$\sum_{i=1}^{5} \binom{n+1}{i}$	5 (?)
$t_\nu = t_{\nu-1}^2$	$\geq F_{n+6} - n - 4$	∞
ν-th prime	$2^{n+1} - 1$	∞ (?)

7.2. The Fibonacci sequence. Let $\mathbf{T} = \{t_\nu \mid \nu \in \mathbf{N}\}$, where $t_\nu = F_\nu$ is the ν-th Fibonacci number. Since $t_\nu = t_{\nu-1} + t_{\nu-2}$ for $\nu \geq 3$,

$$t^{\sigma^k} = t^{\sigma^{k-1}} + t^{\sigma^{k-2}} \quad \text{for } k \geq 2. \tag{7.2.1}$$

Thus, every monomial in $t, t^\sigma, t^{\sigma\sigma}, \ldots, t^{\sigma^k}$ of degree $\leq i$ is a polynomial in t and t^σ of degree $\leq i$, which implies that $d_{k,i} = d_{1,i}$ for $k \geq 1$. Furthermore, $d_{1,i} \leq \binom{i+2}{2}$ for all $i \in \mathbf{N}_0$. If the functions t and t^σ were algebraically independent, then this last inequality would be an equality. However, t and t^σ satisfy the relation

$$(t^2 + t\,t^\sigma - (t^\sigma)^2 + 1)(t^2 + t\,t^\sigma - (t^\sigma)^2 - 1) \equiv 0, \tag{7.2.2}$$

which is a consequence of Cassini's identity $F_\nu^2 + F_\nu F_{\nu+1} - F_{\nu+1}^2 = (-1)^\nu$. Relation (7.2.2) of degree 4 implies that

$$d_{1,i} \leq \binom{i+2}{2} - \binom{i-2}{2} \tag{7.2.3}$$

for all i. Using that $d_{k,i} = d_{1,i}$ for $k \geq 1$ and $d_{0,i} = i + 1$, one obtains $\sum_{k=0}^{n} d_{k,n-k} \leq \sum_{i=1}^{3} \binom{n+1}{i} - \binom{n-2}{3}$, which gives $\lim_{n\to\infty} \log\left(\sum_{k=0}^{n} d_{k,n-k}\right)/\log n \leq 2$.

7.3. The time scale $\mathbf{T} = \{2^\nu + \nu \mid \nu \in \mathbf{N}_0\}$. Let $\mathbf{T} = \{t_\nu \mid \nu \in \mathbf{N}_0\}$, where $t_\nu = 2^\nu + \nu$. Note that $t_\nu = 3t_{\nu-1} - 2t_{\nu-2} - 1$ for $\nu \geq 2$, and hence

$$t^{\sigma^k} = 3t^{\sigma^{k-1}} - 2t^{\sigma^{k-2}} - 1 \quad \text{for } k \geq 2.$$

Thus, as for the Fibonacci sequence, every monomial in $t, t^\sigma, t^{\sigma\sigma}, \dots, t^{\sigma^k}$ of degree $\leq i$ is a polynomial in t and t^σ of degree $\leq i$. This implies that $d_{k,i} = d_{1,i}$ for all $k \geq 1$. An argument similar to the one given in the proof of Lemma 19 shows that the functions t and t^σ on $\mathbf{T}$ are algebraically independent. Thus,

$$d_{1,i} = \binom{i+2}{2}.$$

Using that $d_{k,i} = d_{1,i}$ for $k \geq 1$ and $d_{0,i} = i + 1$, one obtains $\sum_{k=0}^n d_{k,n-k} = \binom{n+1}{3} + \binom{n+1}{2} + \binom{n+1}{1}$, which gives $\lim_{n\to\infty} \log(\sum_{k=0}^n d_{k,n-k})/\log n = 3$.

7.4. The time scales $\mathbf{T} = \{\nu! \mid \nu \in \mathbf{N}\}$ and $\{\sum_{j=1}^{\nu}(1/j) \mid \nu \in \mathbf{N}\}$. For the time scales $\mathbf{T} = \{\nu! \mid \nu \in \mathbf{N}\}$ and $\{\sum_{j=1}^{\nu}(1/j) \mid \nu \in \mathbf{N}\}$, we used Mathematica to compute the $d_{k,i}$ for $k + i \leq 10$. The data suggests that the $d_{k,i}$ are polygonal numbers:

$$d_{k,i} = \frac{(1+i)(2+ki)}{2} \tag{7.4.1}$$

for all k, i. From (7.4.1), one obtains $\sum_{k=0}^n d_{k,n-k} = \sum_{i=1}^4 \binom{n+1}{i}$, and hence the conjectured value $\lim_{n\to\infty} \log(\sum_{k=0}^n d_{k,n-k})/\log n = 4$.

7.5. A time scale satisfying $t_\nu = t_{\nu-1} + t_{\nu-2}t_{\nu-3}$. Let $\mathbf{T} = \{t_\nu | \nu \in \mathbf{N}_0\}$ with $t_0 = 1$, $t_1 = 2$, $t_2 = 3$ and $t_\nu = t_{\nu-1} + t_{\nu-2}t_{\nu-3}$ for $\nu \geq 3$. Again, we used Mathematica to compute the $d_{k,i}$ for $k + i \leq 10$. Based on the data, we conjecture that

$$d_{k,i} = (k-1)\binom{i+2}{3} + \binom{i+2}{2} \quad \text{for } k \geq 1 \text{ and all } i.$$

From (7.5), one obtains $\sum_{k=0}^n d_{k,n-k} = \sum_{i=1}^5 \binom{n+1}{i}$, and hence the conjectured value $\lim_{n\to\infty} \log(\sum_{k=0}^n d_{k,n-k})/\log n = 5$.

7.6. Primes. For generic time scales, we expect that the functions t^{σ^k} and $k \in \mathbf{N}_0$ are algebraically independent, and hence

$$d_{k,i} = \binom{k+i+1}{k+1} \tag{7.6.1}$$

for all k, i. (In fact, for time scales satisfying (4.1.1), this is equivalent to $\mathcal{W}(\mathbf{T})$ being the free algebra in two generators.) An example of a time scale for which we conjecture (7.6.1) to hold is $\mathbf{T} = \{t_\nu | \nu \in \mathbf{N}\}$, where t_ν is the ν-th prime. We remark that (7.6.1) is related to some open problems in number theory. For example, (7.6.1) for $k = 1$ is equivalent to the statement that there exists no nonzero polynomial $P(x, y) \in \mathbf{C}[x, y]$ such that $P(t_\nu, t_{\nu+1}) = 0$ for all $\nu \in \mathbf{N}$.

8. Algebraic dimension of a time scale.

8.1. The Gelfand-Kirillov dimension $\operatorname{GKdim} \mathcal{W}(\mathbf{T})$ is an interesting invariant of a time scale. Unfortunately, in order to define the Weyl algebra $\mathcal{W}(\mathbf{T})$, it was necessary to assume (4.1.1). However, the right hand side of the inequality in Proposition 17 can be defined for general time scales as we will now show.

8.2. Generalized polynomials on arbitrary time scales. Let $\mathbf{T}$ be an arbitrary (possibly finite) time scale. Note that $\mathbf{T} \supseteq \mathbf{T}^\kappa \supseteq \mathbf{T}^{\kappa^2} \supseteq \cdots$, and hence for $m \in \mathbf{N}_0$, we can view polynomial expressions in $t, t^\sigma, t^{\sigma\sigma}, \ldots, t^{\sigma^m}$ as $\mathbf{C}$-valued functions (via restriction) on $\mathbf{T}^{\kappa^m}$. Thus, for $k, i \in \mathbf{N}_0$, it makes sense to define

$$d_{k,i} = \max_{0 \le m \le k} \dim\bigl\{\text{polynomials in } t,\, t^\sigma, \ldots, t^{\sigma^m} \text{ on } T^{\kappa^m} \text{ of degree} \le i\bigr\}.$$

(By convention, if $\mathbf{T}^{\kappa^m} = \varnothing$, any space of $\mathbf{C}$-valued functions on $\mathbf{T}^{\kappa^m}$ has dimension 0.) If $\mathbf{T}^\kappa = \mathbf{T}$, then the definition of the $d_{k,i}$ coincides with the definition (6.1.1); also, $\mathbf{T} \neq \varnothing$ implies that we always have $d_{k,i} \ge 1$ for all k, i since the constant functions are polynomials in t of degree 0.

8.3. The algebraic dimension of a time scale. We define the *algebraic dimension* of $\mathbf{T}$ as the number

$$\mathfrak{d}(\mathbf{T}) = \limsup_{n\to\infty} \frac{\log(\sum_{k=0}^n d_{k,n-k})}{\log n} - 1. \tag{8.3.1}$$

We note that it is not clear whether $\mathfrak{d}(\mathbf{T})$ is always an integer. However, the following proposition shows that $\mathfrak{d}(\mathbf{T})$ does not take any values between 0 and 1.

Proposition 23. *If* $|\mathbf{T}| < \infty$, *then* $\mathfrak{d}(\mathbf{T}) = 0$, *and if* $|\mathbf{T}| = \infty$, *then* $\mathfrak{d}(\mathbf{T}) \geq 1$.

Proof. If $|\mathbf{T}| < \infty$, then $1 \leq d_{k,i} \leq |\mathbf{T}|$ for all k, i. (The second inequality holds since the dimension of the space of all $\mathbf{C}$-valued functions on $\mathbf{T}$ equals $|\mathbf{T}|$.) Thus, $n+1 \leq \sum_{k=0}^{n} d_{k,n-k} \leq (n+1)|\mathbf{T}|$, which implies $\mathfrak{d}(\mathbf{T}) = 0$. If $|\mathbf{T}| = \infty$, then $d_{k,i} \geq i+1$ for all k, i since $\{1, t, t^2, \ldots, t^i\}$ is a linearly independent set of functions for all i. Thus, if $|\mathbf{T}| = \infty$, then $\sum_{k=0}^{n} d_{k,n-k} \geq \binom{n+2}{2}$, which implies $\mathfrak{d}(\mathbf{T}) \geq 1$. $\square$

8.4. Invariance under affine linear transformations. The following results says that our notion of algebraic dimensions is invariant under affine linear transformations.

Proposition 24. *Let* $\mathbf{T}$ *be a time scale.*

(a) *If* $\mathbf{T}' = a\mathbf{T} + b$ *for some* $a, b \in \mathbf{R}$ *with* $a > 0$, *then* $\mathfrak{d}(\mathbf{T}') = \mathfrak{d}(\mathbf{T})$.

(b) *If* $\mathbf{T}^\kappa = \mathbf{T}$ *and* $(-\mathbf{T})^\kappa = -\mathbf{T}$, *then* $\mathfrak{d}(-\mathbf{T}) = \mathfrak{d}(\mathbf{T})$.

Proof. Statement (a) follows immediately from the definition of the $d_{k,i}$. For (b), observe that the conditions $\mathbf{T}^\kappa = \mathbf{T}$ and $(-\mathbf{T})^\kappa = -\mathbf{T}$ imply that σ^* is an automorphism of the $\mathbf{C}$-algebra $\mathcal{F}(\mathbf{T})$ of all $\mathbf{C}$-valued functions $f : \mathbf{T} \to \mathbf{C}$ with the inverse given by ρ^*, the pullback by the backward-jump operator ρ. Since $(\rho^*)^{\circ k}$ defines a bijection between monomials in t, t^σ, $t^{\sigma\sigma}, \ldots, t^{\sigma^k}$ of degree $\leq i$ and monomials in t, t^ρ, $t^{\rho\rho}, \ldots, t^{\rho^k}$ of degree $\leq i$, the $d_{k,i}$ could have been defined by using ρ instead of σ. If $\overline{\sigma}$ denotes the forward jump operator of the time scale $-\mathbf{T}$, then for $t \in \mathbf{T}$, $\overline{\sigma}(-t) = -\rho(t)$. Together with the remarks above, this implies that the $d_{k,i}$ for $-\mathbf{T}$ are the same as for $\mathbf{T}$, and hence $\mathfrak{d}(-\mathbf{T}) = \mathfrak{d}(\mathbf{T})$. $\square$

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