ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 42, Number 5, 2012

ON THE EXISTENCE OF SOLUTIONS FOR SCHRÖDINGER-MAXWELL SYSTEMS IN $\mathbf{R}^3$

MINBO YANG, FUKUN ZHAO AND YANHENG DING

ABSTRACT. In this paper we discuss the existence of solutions for the following Schrödinger-Maxwell systems

$$\begin{cases} -\Delta\psi + \lambda\psi + b(x)\phi\psi = a(x)|\psi|^{p-1}\psi & \text{in } R^3, \\ -\Delta\phi = 4\pi b(x)\psi^2 & \text{in } R^3. \end{cases}$$

Under suitable assumptions on $a(x)$ and $b(x)$, we establish existence results by variational methods.

1. Introduction and main results. In recent years, the following Schrödinger-Maxwell system has been widely considered

$$\begin{cases} -\varepsilon^2\Delta\psi + V(x)\psi + b(x)\phi\psi = a(x)|\psi|^{p-1}\psi & \text{in } R^3, \\ -\Delta\phi = 4\pi b(x)\psi^2 & \text{in } R^3. \end{cases} \tag{1.1}$$

This system arises in quantum mechanics and can be used to describe the standing waves of a classical Schrödinger equation interacting with an unknown electromagnetic field [**10, 12, 13**].

In recent years, a number of papers have contributed to investigating the existence of solutions of (1.1) and the concentration phenomena of these solutions as $\varepsilon \to 0$. As we know, if $\varepsilon \to 0$, the laws of quantum mechanics must reduce to those of classical mechanics, and this describes the transition between quantum mechanics and classical mechanics. D'Aprile and Wei [**14**], Ruiz [**21**], Ianni [**16**], Ianni and Vaira [**17, 18**] studied the semiclassical limit and constructed a family of radial bound states concentrated around a sphere. For other results concerning the semiclassical limit for a single Schrödinger equation, we refer readers to [**1, 11, 15, 26**] and the references therein.

2010 AMS *Mathematics subject classification.* Primary 35J20, 35J60, 35Q55.
Keywords and phrases. Variational methods, Schrödinger-Maxwell systems, critical points.
The first author was supported by NSFC (11271331, 11101374), the second author by NSFC (11061040) and the third author by NSFC (10831005).
Received by the editors on October 27, 2009, and in revised form on February 22, 2010.
DOI:10.1216/RMJ-2012-42-5-1655

If $\varepsilon = V(x) = b(x) = 1$, Benci and Fortunato [**10**] obtained the existence of infinitely many solutions of an eigenvalue problem in a bounded domain with $a(x)|u|^{p-1}u$ replaced by 0. D'Aprile and Mugnai [**13**] studied the existence of radially symmetric solitary waves with $3 \leq p < 5$ and obtained the solutions as mountain-pass critical points for the associated energy functional. Ruiz [**20**] and Ambrosetti and Ruiz [**7**] considered

$$\begin{cases} -\Delta\psi + \psi + \mu\phi\psi = |\psi|^{p-1}\psi & \text{in } R^3, \\ -\Delta\phi = 4\pi\psi^2 & \text{in } R^3. \end{cases} \tag{1.2}$$

By working in the radial functions subspace of $H^1(\mathbf{R}^3)$, they were able to obtain the existence and multiplicity results of (1.2) depending upon the parameter p and $\mu > 0$.

If the potential $V(x)$ is not a constant, many authors considered the following case

$$\begin{cases} -\Delta\psi + V(x)\psi + \phi\psi = f(\psi) & \text{in } R^3, \\ -\Delta\phi = 4\pi\psi^2 & \text{in } R^3. \end{cases} \tag{1.3}$$

Since $V(x)$ might not be radial, authors cannot work in the radial function space. In [**27**], by assuming the non-radial potential $V(x)$ satisfying $V_\infty = \lim_{|x|\to\infty} V(x) \geq V(x)$ and bounded from below by a positive constant, Wang and Zhou considered problem (1.3) with asymptotically linear nonlinearities. Later, Azzollini and Pomponio [**8**] studied the existence of ground state solutions of (1.3) with $f(u) = |u|^{p-1}u$, $3 < p < 5$ and they also obtained existence results for the critical growth case by concentration compactness arguments.

Inspired by recent works [**2, 3, 9**], the aim of this paper is to discuss the existence of solutions for the following Schrödinger-Maxwell systems, $\lambda > 0$,

$$\begin{cases} -\Delta\psi + \lambda\psi + b(x)\phi\psi = a(x)|\psi|^{p-1}\psi & \text{in } R^3, \\ -\Delta\phi = 4\pi b(x)\psi^2 & \text{in } R^3. \end{cases} \tag{1.4}$$

As we know, it is quite difficult to verify the (P.S) condition when we want to apply critical point theory. Different from the methods used in [**8, 13, 20, 27**], we will not use the critical point theory to establish existence results. Instead, we will apply the perturbation method developed by Ambrosetti and Badiale in [**2, 3**] to show the existence of solutions of (1.4). In [**2, 3**], Ambrosetti and Badiale established an abstract theory to reduce a class of perturbation problems to a finite-dimensional one by some careful observations on the unperturbed problems and the Lyapunov-Schmit reduction procedure. This method has also been applied to many different problems, see [**2, 3, 4, 5, 9**] for example. Let $\lambda = \varepsilon^2$. The main results of the paper are the following two theorems.

Theorem 1.1. *Assume* $1 < p < 3$ *and a positive constant* A *exist such that* $a(x), b(x)$ *satisfy*:

(a_1) $a(x) - A$ *is continuous, bounded and* $a(x) - A \in L^1(\mathbf{R}^3)$ *with* $\int_{\mathbf{R}^3}(a(x) - A) \neq 0$;

(b_1) $b(x)$ *is continuous, bounded and* $b(x) \in L^{6/5}(\mathbf{R}^3)$.

Then for any ε *small, a solution* $(\phi_\varepsilon, \psi_\varepsilon)$ *exists in* $D^{1,2}(\mathbf{R}^3) \times H^1(\mathbf{R}^3)$ *for problem* (1.4). *Moreover, if* $1 < p < 7/3$, *then* $(\phi_\varepsilon, \psi_\varepsilon) \to 0$ *as* $\varepsilon \to 0$.

Theorem 1.2. *Assume that* $1 < p < 3$ *and* (b_1) *hold, instead of* (a_1). *We assume*

(a_2) $a - A$ *is continuous and* $L \neq 0$ *and* $0 < \gamma < 3$ *exist such that* $|x|^\gamma(a(x) - A) \to L$ *as* $|x| \to \infty$.

Then, for any ε *small, a solution* $(\phi_\varepsilon, \psi_\varepsilon)$ *exists in* $D^{1,2}(\mathbf{R}^3) \times H^1(\mathbf{R}^3)$ *for problem* (1.4). *Moreover, if* $1 < p < 7/3$, *then* $(\phi_\varepsilon, \psi_\varepsilon) \to 0$ *as* $\varepsilon \to 0$.

Remark 1.3. These two results are related to the problem of bifurcating from the essential spectrum of the Laplace operator in the whole space. Ambrosetti and Badiale have studied this kind of problem in [**3**] for a one-dimensional differential equation. Later, some of those results were generalized to higher dimensions by Badiale and Pomponio in [**9**] under the affect of local nonlinear terms.

Throughout this paper, we denote the norm of $H^1(\mathbf{R}^3)$ by

$$\|u\| = \left(\int_{\mathbf{R}^3} \left(|\nabla u|^2 + u^2\right) \, dx \right)^{1/2},$$

the norm of $D^{1,2}(\mathbf{R}^3)$ by

$$\|u\|_D = \left(\int_{\mathbf{R}^3} |\nabla u|^2 \, dx \right)^{1/2}$$

and by $|\cdot|_s$ we denote the usual L^s-norm; C and C_i stand for different positive constants.

The paper is organized as follows. In Section 1, we recall some recent results about Schrödinger-Maxwell systems. In Section 2, we outline the abstract critical point theory for perturbed functionals. In Section 3, we prove Theorems 1.1 and 1.2.

2. The abstract theorem. To prove the main results, we need the following known propositions.

Proposition 2.1 [**19**] (Hardy-Littlewood-Sobolev inequality). *Let $p, r > 1$ and $0 < \mu < 3$ with $1/p + \mu/3 + 1/r = 2$. Let $f \in L^p(\mathbf{R}^3)$ and $h \in L^r(\mathbf{R}^3)$. A sharp constant $C(p, r)$ exists, independent of f, h, such that*

$$\int_{\mathbf{R}^3 \times \mathbf{R}^3} \frac{f(x)h(y)}{|x-y|^\mu} \le C(p,r)|f|_p |h|_r. \tag{2.1}$$

Proposition 2.2. *For any positive constant A, consider the following problem*

$$\begin{cases} -\Delta u + u = Au^p, \\ u > 0 \qquad\qquad u \in H^{1,2}(\mathbf{R}^3). \end{cases}$$

There is a unique positive radial solution U, which satisfies the following decay property:

$$\lim_{r \to \infty} U(r) r e^r = C > 0, \qquad \lim_{r\to\infty} \frac{U'(r)}{U(r)} = -1, \quad r = |x|,$$

$C > 0$ is a constant. The function U is a critical point of the C^2 functional $I_0 : H^1(\mathbf{R}^3) \to \mathbf{R}$ defined by

$$I_0(u) = \frac{1}{2}\|u\|^2 - \frac{A}{p+1} \int_{\mathbf{R}^3} |u|^{p+1} dx.$$

Moreover, I_0 possesses a three-dimensional manifold of critical points

$$Z = \{z_\theta = U(x+\theta),\ \theta \in \mathbf{R}^3\}.$$

Set

$$Q(u) := I_0''(U)[u,u] = \int_{\mathbf{R}^3} [|\nabla u|^2 + u^2 - pAU^{p-1}u^2]\, dx$$

and denote $X = \operatorname{span}\{(\partial U/\partial x_i), 1 \le i \le 3\}$. We have

(1) $Q(U) = (1-p)A \int_{\mathbf{R}^3} U^{p+1} dx < 0$,

(2) $\operatorname{Ker} Q = X$,

(3) $Q(w) \ge C\|w\|^2$, *for all $w \in (\mathbf{R}U \oplus X)^\perp$.*

In the following we outline the abstract theorem of a variational method to study critical points of perturbed functionals. Let E be a real Hilbert space. We will consider the perturbed functional defined on it of the form

$$I_\varepsilon(u) = I_0(u) + G(\varepsilon, u) \tag{2.2}$$

where $I_0 : E \to \mathbf{R}$ and $G : \mathbf{R} \times E \to \mathbf{R}$. We need the following hypotheses and assume that

(1) I_0 and G are C^2 with respect to u;

(2) G is continuous in (ε, u) and $G(0,u) = 0$ for all u;

(3) $G'(\varepsilon, u)$ and $G''(\varepsilon, u)$ are continuous maps from $\mathbf{R} \times E \to E$ and $L(E,E)$ respectively, $L(E,E)$ is the space of linear continuous operators from E to E.

(4) There is a d-dimensional C^2 manifold Z, $d \ge 1$, consisting of critical points of I_0, such that a Z will be called a critical manifold of I_0.

(5) Let $T_\theta Z$ denote the tangent space to Z at z_θ; the manifold Z is non-degenerate in the following sense:

$$\operatorname{Ker}(I_0''(z)) = T_\theta Z \text{ and } I_0''(z_\theta) \text{ is an index-0 Fredholm operator for any } z_\theta \in Z.$$

(6) An $\alpha > 0$ and a continuous function $\Gamma : Z \to \mathbf{R}$ exist such that

$$\Gamma(z) = \lim_{\varepsilon \to 0} \frac{G(\varepsilon, z)}{\varepsilon^\alpha}$$

and

$$G'(\varepsilon, z) = o(\varepsilon^{\alpha/2}).$$

In [**2, 3**] the following abstract theorem is proved.

Theorem 2.3. *Suppose assumptions* (1)–(6) *are satisfied, and* $\delta > 0$ *and* $z^* \in Z$ *exist such that*

$$\text{either} \min_{\|z-z^*\|=\delta} \Gamma(z) > \Gamma(z^*) \text{ or } \max_{\|z-z^*\|=\delta} \Gamma(z) < \Gamma(z^*). \tag{2.3}$$

Then, for any ε *small,* u_ε *exists which is a critical point of* I_ε*.*

3. Proof of the main results. In this section we will apply the abstract tools of the previous section to prove Theorem 1.1.

Applying the Lax-Milgram theorem, we know that for every $\psi \in H^1(\mathbf{R}^3)$, a unique $\phi_\psi \in D^{1,2}(\mathbf{R}^3)$ exists such that

$$-\Delta \phi_\psi = 4\pi b(x)\psi^2 \tag{3.1}$$

and ϕ_u can be expressed by

$$\phi_\psi(x) = \int_{\mathbf{R}^3} \frac{b(y)\psi^2(y)}{|x-y|}.$$

Thus, problem (1.4) is equivalent to the following one:

$$-\Delta\psi + \lambda\psi + b(x)\bigg(\int \frac{b(y)|\psi|^2}{|x-y|}\bigg)\psi = a(x)|\psi|^{p-1}\psi. \tag{3.2}$$

Set

$$\begin{cases} \lambda = \varepsilon^2, \\ \psi(x) = \varepsilon^{2/(p-1)} u(\varepsilon x). \end{cases}$$

We have

$$-\Delta u + u + \varepsilon^{4(2-p)/p-1} b\Big(\frac{x}{\varepsilon}\Big)\bigg(\int \frac{b(y/\varepsilon)|u(y)|^2}{|x-y|}\bigg)u(x) = a\Big(\frac{x}{\varepsilon}\Big)|u|^{p-1}u. \tag{3.3}$$

It can be proved that $\psi(x) = \varepsilon^{2/(p-1)}u(\varepsilon x) \in H^1(\mathbf{R}^3)$ is a solution of system (1.4) if and only if $u \in H^1(\mathbf{R}^3)$ is a critical point of the functional $I_\varepsilon : H^1(\mathbf{R}^3) \to \mathbf{R}$ defined as

$$\begin{aligned} I_\varepsilon(u) = \frac{1}{2}\|u\|^2 + \frac{\varepsilon^{4(2-p)/p-1}}{4}\int_{\mathbf{R}^3\times\mathbf{R}^3} \frac{b(x/\varepsilon)u^2(x)b(y/\varepsilon)u^2(y)}{|x-y|} \\ - \frac{1}{p+1}\int_{\mathbf{R}^3} a\left(\frac{x}{\varepsilon}\right)|u|^{p+1}. \end{aligned}$$

Set

$$I_0(u) = \frac{1}{2}\|u\|^2 - \frac{A}{p+1}\int_{\mathbf{R}^3} |u|^{p+1}.$$

Then $I_\varepsilon(u)$ can be rewritten as

$$\begin{aligned} I_\varepsilon(u) = I_0(u) \\ + \frac{1}{4}\varepsilon^{4(2-p)/p-1}\int_{\mathbf{R}^3\times\mathbf{R}^3} \frac{b(x/\varepsilon)u^2(x)b(y/\varepsilon)u^2(y)}{|x-y|} \\ + \frac{1}{p+1}\int_{\mathbf{R}^3}\left(A - a\left(\frac{x}{\varepsilon}\right)\right)|u|^{p+1}. \end{aligned} \tag{3.4}$$

Define

$$\begin{aligned} \widetilde{G}(\varepsilon,u) &= \frac{1}{p+1}\int_{\mathbf{R}^3}\left[A - a\left(\frac{x}{\varepsilon}\right)\right]|u|^{p+1} \\ &\quad + \frac{1}{4}\varepsilon^{4(2-p)/p-1}\int_{\mathbf{R}^3\times\mathbf{R}^3} \frac{b(x/\varepsilon)u^2(x)b(y\varepsilon)u^2(y)}{|x-y|} \\ &= \widetilde{G}_1(\varepsilon,u) + \widetilde{G}_2(\varepsilon,u), \end{aligned}$$

and, for $i = 1, 2$,

$$G_i(\varepsilon,u) = \begin{cases} \widetilde{G_i}(\varepsilon,u) & \text{if } \varepsilon \neq 0, \\ 0 & \text{if } \varepsilon = 0. \end{cases}$$

3.1. Proof of Theorem 1.1.

Lemma 3.1. *Under assumptions (a_1) and (b_1), G is continuous in (ε, u).*

Proof. From the proof of the lemma in [**9**], we know G_1 is continuous in $(\varepsilon, u) \in \mathbf{R} \times H^1(\mathbf{R}^3)$, and hence we only need to prove that G_2 is continuous in (ε, u).

If $(\varepsilon, u) \to (\varepsilon_0, u_0)$, with $\varepsilon_0 \neq 0$, then we can verify that

$$
\begin{aligned}
&4|G_2(\varepsilon,u) - G_2(\varepsilon_0,u_0)| \\
&= \Bigg| \varepsilon^{4(2-p)/p-1} \int_{\mathbf{R}^3\times\mathbf{R}^3} \frac{b(x/\varepsilon)u^2(x)b(y/\varepsilon)u^2(y)}{|x-y|} \\
&\qquad\qquad - \varepsilon_0^{4(2-p)/p-1} \int_{\mathbf{R}^3\times\mathbf{R}^3} \frac{b(x/\varepsilon_0)u_0^2(x)b(y/\varepsilon_0)u_0^2(y)}{|x-y|} \Bigg| \\
&\leq |\varepsilon|^{4(2-p)/p-1} \\
&\quad \times \left| \int_{\mathbf{R}^3\times\mathbf{R}^3} \frac{b(x/\varepsilon)u^2(x)b(y/\varepsilon)u^2(y) - b(x/\varepsilon_0)u_0^2(x)b(y/\varepsilon_0)u_0^2(y)}{|x-y|} \right| \\
&\quad + |\varepsilon^{4(2-p)/p-1} - \varepsilon_0^{4(2-p)/p-1}| \left| \int_{\mathbf{R}^3\times\mathbf{R}^3} \frac{b(x/\varepsilon_0)u_0^2(x)b(y/\varepsilon_0)u_0^2(y)}{|x-y|} \right| \\
&:= |\varepsilon|^{4(2-p)/p-1} I_1 + I_2.
\end{aligned}
$$

It is obvious that $I_2 \to 0$, as $\varepsilon \to \varepsilon_0$. Also, we know that

$$
\begin{aligned}
I_1 \leq &\left| \int_{\mathbf{R}^3\times\mathbf{R}^3} \frac{[b(x/\varepsilon) - b(x/\varepsilon_0)]u^2(x)b(y/\varepsilon)u^2(y)}{|x-y|} \right| \\
&+ \left| \int_{\mathbf{R}^3\times\mathbf{R}^3} \frac{b(x/\varepsilon_0)(u^2(x) - u_0^2(x))b(y/\varepsilon)u^2(y)}{|x-y|} \right| \\
&+ \left| \int_{\mathbf{R}^3\times\mathbf{R}^3} \frac{b(x/\varepsilon_0)u_0^2(x)(b(y/\varepsilon) - b(y/\varepsilon_0))u^2(y)}{|x-y|} \right| \\
&+ \left| \int_{\mathbf{R}^3\times\mathbf{R}^3} \frac{b(x/\varepsilon_0)u_0^2(x)b(y/\varepsilon_0)(u^2(y) - u_0^2(y))}{|x-y|} \right| \\
:= &\, \Pi_1 + \Pi_2 + \Pi_3 + \Pi_4.
\end{aligned}
$$

Estimating the first term Π_1, by the Hardy-Littlewood-Sobolev inequality, we know that

$$
\begin{aligned}
\Pi_1 &= \left| \int_{\mathbf{R}^3\times\mathbf{R}^3} \frac{[b(x/\varepsilon) - b(x/\varepsilon_0)]u^2(x)b(y/\varepsilon)u^2(y)}{|x-y|} \right| \\
&\leq \left(\int_{\mathbf{R}^3} \left| b\left(\frac{x}{\varepsilon}\right) - b\left(\frac{x}{\varepsilon_0}\right) \right|^{6/5} u^{12/5} \right)^{5/6} \left(\int_{\mathbf{R}^3} \left| b\left(\frac{y}{\varepsilon}\right) \right|^{6/5} u^{12/5} \right)^{5/6}.
\end{aligned}
$$

Since $b(x)$ is bounded and continuous, $u \to u_0$, the dominated convergence theorem implies that

$$\Pi_1 \longrightarrow 0, \quad \text{as } \varepsilon \to \varepsilon_0.$$

Similarly, we can deduce that Π_2, Π_3, Π_4 vanish, as $\varepsilon \to \varepsilon_0$. Hence, $|G_2(\varepsilon, u) - G_2(\varepsilon_0, u_0)| \to 0$ as $(\varepsilon, u) \to (\varepsilon_0, u_0)$.

If $(\varepsilon, u) \to (0, u_0)$, by definition, $G_2(0, u) = 0$. Applying both the Hardy-Littlewood-Sobolev and Hölder inequalities, we get

$$\begin{aligned}
4|G_2(\varepsilon, u)| &\le |\varepsilon|^{4(2-p)/p-1} \int_{\mathbf{R}^3 \times \mathbf{R}^3} \left| \frac{b(x/\varepsilon) u^2(x) b(y/\varepsilon) u^2(y)}{|x-y|} \right| \\
&\le |\varepsilon|^{4(2-p)/p-1} \left(\int_{\mathbf{R}^3} \left| b\left(\frac{x}{\varepsilon}\right) u^2(x) \right|^{6/5} \right)^{5/3} \\
&\le |\varepsilon|^{4(2-p)/p-1} \left(\int_{\mathbf{R}^3} |b\left(\frac{x}{\varepsilon}\right)|^2 \right) \left(\int_{\mathbf{R}^3} |u|^6 \right)^{2/3} \\
&= |\varepsilon|^{5-p/p-1} \int_{\mathbf{R}^3} |b(x)|^2 \left(\int_{\mathbf{R}^3} |u|^6 \right)^{2/3}.
\end{aligned}$$

We changed the variables $y = x/\varepsilon$ in the last step, since $1 < p < 3$, $G_2(\varepsilon, u) \to 0$, as $(\varepsilon, u) \to (0, u)$. Hence, $G = G_1 + G_2$ is continuous, and the lemma is proved. $\square$

Lemma 3.2. *Under assumptions* (a_1) *and* (b_1), G' *and* G'' *are continuous in* (ε, u).

Proof. G_1' and G_1'' are continuous in (ε, u), see [**9**] for details. Here we only prove that G_2' and G_2'' are continuous in (ε, u).

If $(\varepsilon, u) \to (\varepsilon_0, u_0)$ with $\varepsilon_0 \neq 0$, then

$$
\begin{aligned}
&\|G_2'(\varepsilon,u) - G_2'(\varepsilon_0,u_0)\| \\
&= \sup_{\|v\|=1}\left\{\left|\varepsilon^{4(2-p)/p-1}\int_{\mathbf{R}^3\times\mathbf{R}^3}\frac{b(x/\varepsilon)u^2(x)b(y/\varepsilon)u(y)v(y)}{|x-y|}\right.\right.\\
&\quad\left.\left. - \varepsilon_0^{4(2-p)/p-1}\int_{\mathbf{R}^3\times\mathbf{R}^3}\frac{b(x/\varepsilon_0)u_0^2(x)b(y/\varepsilon_0)u_0(y)v(y)}{|x-y|}\right|\right\}\\
&\le \sup_{\|v\|=1}\left\{|\varepsilon|^{4(2-p)/p-1}\right.\\
&\quad\times\int_{\mathbf{R}_3\times\mathbf{R}^3}\frac{b(x/\varepsilon)u^2(x)b(y/\varepsilon)u(y)v(y)-b(x/\varepsilon_0)u_0^2(x)b(y/\varepsilon_0)u_0(y)v(y)}{|x-y|}\\
&\quad\left.+\left|\varepsilon^{4(2-p)/p-1}-\varepsilon_0^{4(2-p)/p-1}\right|\left|\int_{\mathbf{R}_3\times\mathbf{R}^3}\frac{b(x/\varepsilon_0)u_0^2(x)b(y/\varepsilon_0)u_0(y)v(y)}{|x-y|}\right|\right\}\\
&:= |\varepsilon|^{4(2-p)/p-1}\sup_{\|v\|=1} I_1 + \sup_{\|v\|=1} I_2.
\end{aligned}
$$

Estimating the second term, by the Hardy-Littlewood-Sobolev inequality, we know that

$$
\begin{aligned}
\sup_{\|v\|=1} I_2 \le& \left|\varepsilon^{4(2-p)/p-1} - \varepsilon_0^{4(2-p)/p-1}\right|\\
&\times\left(\int_{\mathbf{R}^3}\left|b\left(\frac{x}{\varepsilon_0}\right)u_0^2(x)\right|^{6/5}\right)^{5/6}\\
&\times\left(\int_{\mathbf{R}^3}\left|b\left(\frac{y}{\varepsilon_0}\right)u_0(y)v(y)\right|^{6/5}\right)^{5/6}\\
\le& \left|\varepsilon^{4(2-p)/p-1} - \varepsilon_0^{4(2-p)/p-1}\right|\\
&\left(\int_{\mathbf{R}^3}\left|b\left(\frac{x}{\varepsilon_0}\right)\right|^2\right)\left(\int_{\mathbf{R}^3}|u_0|^6\right)^{1/2}\left(\int|v|^6\right)^{1/6}\\
\le& C_0\left|\varepsilon^{4(2-p)/p-1} - \varepsilon_0^{4(2-p)/p-1}\right|.
\end{aligned}
$$

Thus, $\sup_{\|v\|=1} I_2 \to 0$ as $\varepsilon \to \varepsilon_0$. Estimating the first term I_1, we know that

$$
\begin{aligned}
&\sup_{\|v\|=1} I_1\\
&= \left\{\left|\int_{\mathbf{R}^3\times\mathbf{R}^3}\frac{b(x/\varepsilon)u^2(x)b(y/\varepsilon)u(y)v(y)-b(x/\varepsilon_0)u_0^2(x)b(y/\varepsilon_0)u(y)v(y)}{|x-y|}\right|\right\}\\
&\leq \left|\int_{\mathbf{R}^3\times\mathbf{R}^3}\frac{b(x/\varepsilon)-b(x/\varepsilon_0)u^2(x)b(y/\varepsilon)u(y)v(y)}{|x-y|}\right|\\
&\quad+\left|\int_{\mathbf{R}^3\times\mathbf{R}^3}\frac{b(x/\varepsilon_0)(u^2(x)-u_0^2(x))b(y/\varepsilon)u(y)v(y)}{|x-y|}\right|\\
&\quad+\left|\int_{\mathbf{R}^3\times\mathbf{R}^3}\frac{b(x/\varepsilon_0)u_0^2(x)(b(y/\varepsilon)-b(y/\varepsilon_0))u(y)v(y)}{|x-y|}\right|\\
&\quad+\left|\int_{\mathbf{R}^3\times\mathbf{R}^3}\frac{b(x/\varepsilon_0)u_0^2(x)b(y/\varepsilon_0)(u(y)v(y)-u_0(y)v(y))}{|x-y|}\right|\\
&:= A_1+A_2+A_3+A_4.
\end{aligned}
$$

As in Lemma 3.1, by the Hardy-Littlewood-Sobolev inequality, we can prove that $A_i\to 0$, as $\varepsilon\to\varepsilon_0$, $i=1,2,3,4$. Therefore, $\|G_2'(\varepsilon,u)-G_2'(\varepsilon_0,u_0)\|\to 0$ as $(\varepsilon,u)\to(\varepsilon_0,u_0)$.

If $\varepsilon_0=0$, by the definition of G_2, we know that $\|G_2'(0,u_0)\|=0$. Hence,

$$
\begin{aligned}
\|G_2'(\varepsilon,u)\| &= \sup_{\|v\|=1}\left|\varepsilon^{4(2-p)/p-1}\int_{\mathbf{R}^3\times\mathbf{R}^3}\frac{b(x/\varepsilon)u^2(x)b(y/\varepsilon)u(y)v(y)}{|x-y|}\right|\\
&\leq \sup_{\|v\|=1}\left\{|\varepsilon^{4(2-p)/p-1}|\left(\int_{\mathbf{R}^3}|b\left(\frac{x}{\varepsilon}\right)|^2\right)\right.\\
&\qquad\qquad\left.\times\left(\int_{R^3}|u|^6\right)^{1/2}\left(\int_{\mathbf{R}^3}|v|^6\right)^{1/6}\right\}.
\end{aligned}
$$

By a change of variables, we know that

$$
\|G_2'(\varepsilon,u)\|\leq C|\varepsilon|^{5-p/p-1}|b|_2^2|u|_6^3;
$$

thus, $||G_2'(\varepsilon,u)||\to 0$, as $\varepsilon\to 0$. From the above arguments, we know that $G'=G_1'+G_2'$ is continuous (ε,u).

In the following we prove that G'' is continuous in (ε, u). As we know

$$
\begin{aligned}
G_2''(\varepsilon,u)[w,v] = \varepsilon^{4(2-p)/p-1} & \\
& \times \int_{\mathbf{R}^3\times\mathbf{R}^3} \frac{b(x/\varepsilon)u^2(x)b(y/\varepsilon)w(y)v(y)}{|x-y|} \\
& \quad + \frac{2b(x/\varepsilon)u(x)w(x)b(y/\varepsilon)u(y)v(y)}{|x-y|}.
\end{aligned}
$$

If $(\varepsilon, u) \to (\varepsilon_0, u_0)$ with $\varepsilon_0 \neq 0$, then

$$
\begin{aligned}
&\|G_2''(\varepsilon,u) - G_2''(\varepsilon_0,u_0)\| \\
&\quad = \sup_{\|w\|=\|v\|=1} |G_2''(\varepsilon,u)(w,v) - G_2''(\varepsilon_0,u_0)(w,v)| \\
&\quad = \sup_{\|w\|=\|v\|=1} \Big| \varepsilon^{4(2-p)/p-1} \\
&\qquad \times \int_{\mathbf{R}^3\times\mathbf{R}^3} \frac{b(x/\varepsilon)u^2(x)b(y/\varepsilon)w(y)v(y)}{|x-y|} \\
&\qquad\quad + \frac{2b(x/\varepsilon)u(x)w(x)b(y/\varepsilon)u(y)v(y)}{|x-y|} \\
&\qquad - \varepsilon_0^{4(2-p)/p-1} \\
&\qquad \times \int_{\mathbf{R}^3\times\mathbf{R}^3} \frac{b(x/\varepsilon_0)u_0^2(x)b(y/\varepsilon_0)w(y)v(y)}{|x-y|} \\
&\qquad\quad + \frac{2b(x/\varepsilon_0)u_0(x)w(x)b(y/\varepsilon_0)u_0(y)v(y)}{|x-y|} \Big| \\
&\quad \le I_1 + I_2,
\end{aligned}
$$

where

$$
\begin{aligned}
I_1 := \Big| \varepsilon^{4(2-p)/p-1} \int_{\mathbf{R}^3\times\mathbf{R}^3} & \frac{b(x/\varepsilon)u^2(x)b(y/\varepsilon)\omega(y)v(y)}{|x-y|} \\
& - \varepsilon_0^{4(2-p)/p-1} \int_{\mathbf{R}^3\times\mathbf{R}^3} \frac{b(x/\varepsilon_0)u_0^2(x)b(y/\varepsilon_0)\omega(y)v(y)}{|x-y|} \Big|
\end{aligned}
$$

and

$$
\begin{aligned}
I_2 := \Big| \varepsilon^{4(2-p)/p-1} \int_{\mathbf{R}^3\times\mathbf{R}^3} & \frac{b(x/\varepsilon)u(x)w(x)b(y/\varepsilon)u(y)v(y)}{|x-y|} \\
& - \varepsilon_0^{4(2-p)/p-1} \int_{\mathbf{R}^3\times\mathbf{R}^3} \frac{b(x/\varepsilon_0)u_0(x)w(x)b(y/\varepsilon_0)u_0(y)v(y)}{|x-y|} \Big|.
\end{aligned}
$$

We only estimate I_1; I_2 can be estimated in a similar way. Indeed,

$$\begin{aligned} I_1 \le& \left|\varepsilon^{4(2-p)/p-1} - \varepsilon_0^{4(2-p)/p-1}\right| \left| \int_{\mathbf{R}^3\times\mathbf{R}^3} \frac{b(x/\varepsilon)u^2(x)b(y/\varepsilon)w(y)v(y)}{|x-y|} \right| \\ &+ |\varepsilon_0|^{4(2-p)/p-1} \\ &\times \left| \int_{\mathbf{R}^3\times\mathbf{R}^3} \frac{b(x/\varepsilon)u^2(x)b(y/\varepsilon)w(y)v(y)}{|x-y|} \right. \\ &\qquad \left. - \frac{b(x/\varepsilon_0)u_0^2(x)b(y/\varepsilon_0)w(y)v(y)}{|x-y|} \right|. \end{aligned}$$

Similar to the proof in Lemma 3.1, we know $I_1 \to 0$ as $\varepsilon \to \varepsilon_0$. Thus, we know $\|G_2''(\varepsilon,u) - G_2''(\varepsilon_0,u_0)\| \to 0$ as $(\varepsilon,u) \to (\varepsilon_0,u_0)$.

If $(\varepsilon,u) \to (0,u_0)$, then from the definition of G_2, we know that

$$\begin{aligned} \|G_2''(\varepsilon,u)\| =& \sup_{\|w\|=\|v\|=1} |G_2''(\varepsilon,u)[w,v]| \\ =& \sup_{\|w\|=\|v\|=1} \Bigg\{ |\varepsilon|^{4(2-p)/p-1} \\ &\qquad \times \left| \int_{\mathbf{R}^3\times\mathbf{R}^3} \frac{b(x/\varepsilon)b(y/\varepsilon)u^2(x)w(y)v(y)}{|x-y|} \right| \\ &\qquad + 2|\varepsilon|^{4(2-p)/p-1} \\ &\times \left| \int_{\mathbf{R}^3\times\mathbf{R}^3} \frac{2b(x/\varepsilon)u(x)w(x)b(y/\varepsilon)u(y)v(y)}{|x-y|} \right| \Bigg\} := I_1 + I_2. \end{aligned}$$

By the Hardy-Littlewood-Sobolev inequality again, we know that

$$\begin{aligned} I_1 \le& |\varepsilon|^{4(2-p)/p-1} \left(\int_{\mathbf{R}^3} |b\left(\frac{x}{\varepsilon}\right) w(x)v(x)|^{6/5} \right)^{5/6} \\ &\times \left(\int_{\mathbf{R}^3} |b\left(\frac{x}{\varepsilon}\right) u^2(x)|^{6/5} \right)^{5/6} \\ \le& |\varepsilon|^{5-p/p-1} |b|_2^2 |u|_6^2 \end{aligned}$$

and

$$\begin{aligned} I_2 \le& 2|\varepsilon|^{4(2-p)/p-1} \left(\int_{\mathbf{R}^3} |b(x/\varepsilon)u(x)w(x)|^{6/5} \right)^{5/6} \\ &\times \left(\int_{\mathbf{R}^3} |b\left(\frac{x}{\varepsilon}\right) u(x)v(x)|^{6/5} \right)^{5/6} \\ \le& 2|\varepsilon|^{5-p/p-1} |b|_2^2 |u|_6^2. \end{aligned}$$

Therefore,

$$\|G_2''(\varepsilon,u)\| \longrightarrow 0 \quad \text{as } \varepsilon \to 0.$$

From the above arguments, we know that G'' is continuous in (ε,u), and the proof is complete. □

Lemma 3.3. *Assume* (a_1) *and* (b_1) *are satisfied. Define*

$$\Gamma(\theta) = -\frac{1}{p+1}U^{p+1}(\theta)\int_{\mathbf{R}^3}\big(a(x)-A\big). \tag{3.5}$$

Then

$$\lim_{\varepsilon\to 0}\frac{G(\varepsilon,z_\theta)}{\varepsilon^3} = \Gamma(\theta)$$

and

$$G'(\varepsilon,z_\theta) = o(\varepsilon^{3/2}).$$

Proof. By a change of variables, we know that

$$\begin{aligned} G_1(\varepsilon,z_\theta) &= -\frac{1}{p+1}\int_{\mathbf{R}^3}\left(a\left(\frac{x}{\varepsilon}\right)-A\right)U^{p+1}(x+\theta)\\ &= -\frac{\varepsilon^3}{p+1}\int_{\mathbf{R}^3}(a(x)-A)U^{p+1}(\varepsilon x+\theta). \end{aligned}$$

Since $a(x)$ is continuous and bounded, the dominated convergence theorem implies that

$$\lim_{\varepsilon\to 0}\frac{G_1(\varepsilon,z_\theta)}{\varepsilon^3} = \Gamma(\theta).$$

On the other hand, by a change of variables we know that

$$\begin{aligned} \left|\frac{G_1(\varepsilon,z_\theta)}{\varepsilon^3}\right| &= \left|\frac{1}{4}\varepsilon^{[4(2-p)/p-1]-3}\int_{\mathbf{R}^3\times\mathbf{R}^3}\frac{b(x/\varepsilon)z_\theta^2(x)b(y/\varepsilon)z_\theta^2(y)}{|x-y|}\right|\\ &\le \frac{1}{4}|\varepsilon|^{[4(2-p)/p-1]-3}\left(\int_{\mathbf{R}^3}\left|b\left(\frac{x}{\varepsilon}\right)z_\theta^2\right|^{6/5}\right)^{5/3}\\ &\le C_0|\varepsilon|^{[6-2p/p-1]}\left(\int_{\mathbf{R}^3}|b(x)|^{6/5}\right)^{5/3} \longrightarrow 0 \quad \text{as } \varepsilon\to 0, \end{aligned}$$

since $1 < p < 3$, $b(x) \in L^{6/5}(R^3)$. Thus, we obtain

$$\lim_{\varepsilon \to 0} \frac{G(\varepsilon, z_\theta)}{\varepsilon^3} = \Gamma(\theta).$$

Now we are ready to prove

$$G'(\varepsilon, z_\theta) = o(\varepsilon^{3/2}).$$

From the proof of [**9**], we first know

$$G'_1(\varepsilon, z_\theta) = o(\varepsilon^{3/2}).$$

Also, since z is bounded, it is easy to check that

$$\begin{aligned}
\frac{\|G'_2(\varepsilon, z_\theta)\|}{|\varepsilon|^{3/2}} &= |\varepsilon|^{[4(2-p)/p-1]-3/2} \sup_{\|v\|=1} |(G'_2(\varepsilon, z_\theta), v)| \\
&= |\varepsilon|^{[4(2-p)/p-1]-3/2} \\
&\quad \times \sup_{\|v\|=1} \left| \int_{\mathbf{R}^3 \times \mathbf{R}^3} \frac{b(x/\varepsilon) z_\theta^2(x) b(y/\varepsilon) z_\theta(y) v(y)}{|x-y|} \right| \\
&\le |\varepsilon|^{[4(2-p)/p-1]-3/2} \sup_{\|v\|=1} \left| \left(\int_{\mathbf{R}^3} \left| b\left(\frac{x}{\varepsilon}\right) z_\theta^2 \right|^{6/5} \right)^{5/6} \right. \\
&\qquad \left. \times \left(\int_{\mathbf{R}^3} |b\left(\frac{x}{\varepsilon}\right) z_\theta v(x)|^{6/5} \right)^{5/6} \right| \\
&\le C_0 |\varepsilon|^{[4(2-p)/p-1]-3/2} \sup_{\|v\|=1} \left| \left(\int_{\mathbf{R}^3} \left| b\left(\frac{x}{\varepsilon}\right) \right|^{6/5} \right)^{5/6} \right. \\
&\qquad \left. \times \left(\int_{\mathbf{R}^3} \left| b\left(\frac{x}{\varepsilon}\right) v(x) \right|^{6/5} \right)^{5/6} \right|.
\end{aligned}$$

Moreover, recall that $b(x) \in L^{6/5}$ and is bounded, so it belongs to any q for $q \ge 6/5$. Using the Hölder inequality, we get

$$\begin{aligned}
\frac{\|G'_2(\varepsilon, z_\theta)\|}{|\varepsilon|^{3/2}} &\le C_0 |\varepsilon|^{[4(2-p)/p-1]-3/2+5/2+2} \\
&\quad \times \sup_{\|v\|=1} \left| \left(\int_{\mathbf{R}^3} |b(x)|^{6/5} \right)^{5/6} \left(\int_{\mathbf{R}^3} |b(x)|^{3/2} \right)^{2/3} \right. \\
&\qquad \left. \times \left(\int_{\mathbf{R}^3} |v(x)|^6 \right)^{1/6} \right| \\
&\le C_1 |\varepsilon|^{5-p/p-1},
\end{aligned}$$

since $1 < p < 3$. We have

$$\lim_{\varepsilon \to 0} \frac{G_2'(\varepsilon, z_\theta)}{\varepsilon^{3/2}} = 0.$$

From the above arguments, we know

$$\lim_{\varepsilon \to 0} \frac{G'(\varepsilon, z_\theta)}{\varepsilon^{3/2}} = 0,$$

and the proof is completed. □

Completion of the proof of Theorem 1.1.

Proof. By the exponential decay property of proposition U, it is easy to check that I_0'' is a compact perturbation of the identity map, and so it is an index-0 Fredholm operator. By Proposition 2.2, we know that Z is a non degenerate three-dimensional critical manifold. From Lemmas 3.1 to 3.3, we know all the assumptions of Theorem 2.3 are satisfied. U has a strict (global) maximum at $x = 0$. So Γ has a strict (global) maximum or minimum at $\theta = 0$ depending upon the sign of $\int_{\mathbf{R}^3}(a(x) - A)$. By the abstract theorem, we know the existence of a family of solutions $\{(\varepsilon, u_\varepsilon)\} \subset \mathbf{R} \times H^1(\mathbf{R}^3)$. If $1 < p < 7/3$, it is easy to check that $(\phi_\varepsilon, \psi_\varepsilon) \to 0$ as $\varepsilon \to 0$. □

Remark 3.4. The hypothesis $\int_{\mathbf{R}^3}(a(x) - A) \neq 0$ is used to apply Theorem 2.3 and has already been used in [**2, 9**]. If $\int_{\mathbf{R}^3}(a(x) - A)$ is identically zero, we cannot conclude that critical points of I_ε exist.

In the next part we prove Theorem 1.2.

3.2. Proof of Theorem 1.2.

Lemma 3.5. *Assume that* (a_2) *and* (b_1) *are satisfied. Then* G, G' *and* G'' *are continuous in* (ε, u).

Proof. Keeping the exponential decay property of U in mind, the continuity of G_1, G_1' and G_1'' in (ε, u) can be proved similarly as in [**9**].

We can also repeat the proof in Lemma 3.1 to know the continuity of G_2, etc. Thus, the lemma is concluded. $\square$

Lemma 3.6. *Assume* (a_2) *and* (b_1) *are satisfied. Define*

$$\Gamma(\theta) = -\frac{L}{p+1}\int_{\mathbf{R}^3} |x|^{-\gamma} U^{p+1}(x+\theta). \tag{3.6}$$

Then, for all $\theta \in R^N$, *we have*

$$\lim_{\varepsilon\to 0}\frac{G(\varepsilon, z_\theta)}{\varepsilon^\gamma} = \Gamma(\theta), \qquad \lim_{|\theta|\to\infty}\Gamma(\theta) = 0$$

and

$$\lim_{\varepsilon\to 0}\frac{G'(\varepsilon, z_\theta)}{\varepsilon}^{\gamma/2} = 0.$$

Proof. As we know,

$$G_1(\varepsilon, z_\theta) = -\frac{\varepsilon^\gamma}{p+1}\int_{\mathbf{R}^3}\left(a\left(\frac{x}{\varepsilon}\right) - A\right)\frac{|x|^\gamma}{\varepsilon^\gamma}\frac{U^{p+1}(x+\theta)}{|x|^\gamma}.$$

By assumption (a_2) and the decay property of U,

$$\lim_{\varepsilon\to 0}\frac{G_1(\varepsilon, z_\theta)}{\varepsilon^\gamma} = \Gamma(\theta).$$

Moreover, by the boundedness of z and the Hardy-Littlewood-Sobolev inequality, we know that

$$\begin{aligned} G_2(\varepsilon, z_\theta) &= \frac{1}{4}\varepsilon^{4(2-p)/p-1}\int_{\mathbf{R}^3\times\mathbf{R}^3}\frac{b(x/\varepsilon)z_\theta^2(x)b(y/\varepsilon)z_\theta^2(y)}{|x-y|} \\ &\le \frac{1}{4}\varepsilon^{4(2-p)/p-1}\left(\int_{\mathbf{R}^3}\left|b\left(\frac{x}{\varepsilon}\right)z_\theta^2\right|^{6/5}\right)^{5/3} \\ &\le C_0\varepsilon^{3+p/p-1}\left(\int_{\mathbf{R}^3}|b(x)|^{6/5}\right)^{5/3}. \end{aligned}$$

Since $3+p/p-1 > 3 > \gamma$, therefore

$$\lim_{\varepsilon\to 0}\frac{G_2(\varepsilon, z_\theta)}{\varepsilon^\gamma} = 0;$$

thus, we know

$$\lim_{\varepsilon\to 0}\frac{G(\varepsilon, z_\theta)}{\varepsilon^\gamma} = \Gamma(\theta).$$

To study the property of $G'(\varepsilon, z_\theta)$, since $\gamma < 3$ and U exponentially decays at infinity, from the proof in [**9**], we know that

$$\lim_{\varepsilon\to 0}\frac{G_1'(\varepsilon, z_\theta)}{\varepsilon^{\gamma/2}} = 0.$$

On the other hand, from the proof of Lemma 3.2, we have

$$\|G_2'(\varepsilon, z_\theta)\| \leq C_0|\varepsilon|^{[5-p/p-1]+3/2}.$$

Since $\gamma < 3$ and $1 < p < 3$, we get

$$\lim_{\varepsilon\to 0}\frac{G_2'(\varepsilon, z_\theta)}{\varepsilon^{\gamma/2}} = 0.$$

From the above arguments, we know that

$$\lim_{\varepsilon\to 0}\frac{G'(\varepsilon, z_\theta)}{\varepsilon^{\gamma/2}} = 0. \qquad \square$$

Completion of the proof of Theorem 1.2.

Proof. From Lemmas 3.5 and 3.6, we know that all of the assumptions of Theorem 2.3 are satisfied. Since $\lim_{|\theta|\to\infty}\Gamma(\theta) = 0$ and $\Gamma(0) \neq 0$, we know that there is an $R > 0$ such that either

$$\min_{|\theta|=R}\Gamma(\theta) > \Gamma(0) \quad \text{or} \quad \max_{|\theta|=R}\Gamma(\theta) < \Gamma(0).$$

By the abstract Theorem 2.3, we know the existence of the family of solutions $\{(\varepsilon, u_\varepsilon)\} \subset \mathbf{R} \times H^1(\mathbf{R}^3)$. If $1 < p < 7/3$, it is easy to check that $(\phi_\varepsilon, \psi_\varepsilon) \to 0$ as $\varepsilon \to 0$. $\square$

Acknowledgments. The authors would like to thank the unknown referee for his/her valuable suggestions which helped to improve the presentation of this paper.

REFERENCES

1. N. Ackermann, *On a periodic Schrödinger equation with nonlocal superlinear part*, Math. Z. **248** (2004), 423–443.

2. A. Ambrosetti and M. Badiale, *Homoclinics*: *Poincaré-Melnikov type results via a variational approach*, Ann. I.H.P.–Analyse Nonlin. **15** (1998), 233–252.

3. ———, *Variational perturbative methods and bifurcation of bound states from the essential spectrum*, Proc. Royal Soc. Edinburgh **128** (1998), 1131–1161.

4. A. Ambrosetti, M. Badiale and S. Cingolani, *Semiclassical states of nonlinear Schrödinger equations*, Arch. Rat. Mech. Anal. **140** (1997), 285–300.

5. A. Ambrosetti, A. Malchiodi and S. Secchi, *Multiplicity results for some nonlinear Schrödinger equations with potentials*, Arch. Rat. Mech. Anal. **159** (2001), 253–271.

6. A. Ambrosetti and P.H. Rabinowitz, *Dual variational methods in critical point theory and applications*, J. Funct. Anal. **14** (1973), 349–381.

7. ———, *Multiple bound states for the Schrödinger-Poisson equation*, Comm. Cont. Math. **10** (2008), 391–404.

8. A. Azzollini and A. Pomponio, *Ground state solutions for the nonlinear Schrödinger-Maxwell equations*, J. Math. Anal. Appl. **345** (2008), 90–108.

9. M. Badiale and A. Pomponio, *Bifurcation results for semilinear elliptic problems in* $\mathbf{R}^N$, Proc. Roy. Soc. Edinburgh **134** (2004), 11–32.

10. V. Benci and D. Fortunato, *An eigenvalue problem for the Schrödinger-Maxwell equations*, Topol. Meth. Nonlinear Anal. **11** (1998), 283–293.

11. S. Cingolani, *Semiclassical stationary states of nonlinear Schrödinger equation with external magnetic field*, J. Diff. Equations **188** (2003), 52–79.

12. G. Coclite and V. Georgiev, *Solitary waves for Maxwell-Schrödinger equations*, Elec. J. Differental Equations **94** (2004), 1–31.

13. T. D'Aprile and D. Mugnai, *Solitary waves for nonlinear Klein-Gordon-Maxwell and Schrödinger-Maxwell equations*, Proc. Roy. Soc. Edin. **134** (2004), 1–14.

14. T. D'Aprile and J. Wei, *On bound states concentrating on spheres for the Maxwell-Schrödinger equation*, SIAM J. Math. Anal. **37** (2005), 321–342.

15. M. del Pino and P. Felmer, *Semi-classical states of nonlinear Schrödinger equations*: *A variational reduction method*, Math. Ann. **324** (2002), 1–32.

16. I. Ianni, *Solutions of the Schródinger-Poisson system concentrating on spheres*, Part II: *Existence*, Math. Models Meth. Appl. Sci. **19** (2009), 877–910.

17. ———, *Solutions of the Schrödinger-Poisson system concentrating on spheres*, Part I: *Necessary conditions*, Math. Models Meth. Appl. Sci. **19** (2009), 707–720.

18. I. Ianni and G. Vaira, *On concentration of positive bound states for the Schrödinger-Poisson problem with potentials*, Adv. Nonlin. Stud. **8** (2008), 573–595.

19. E. Lieb and M. Loss, *Analysis*, Grad. Stud. Math., American Mathematical Society, Providence, Rhode island, 2001.

20. D. Ruiz, *The Schrödinger-Poisson equation under the effect of a nonlinear local term*, J. Funct. Anal. **237** (2006), 655–674.

21. ———, *Semiclassical states for coupled Schrödinger-Maxwell equations concentration around a sphere*, Math. Models Meth. Appl. Sci. **15** (2005), 141–164.

22. C. Stuart, *Bifurcation from the essential spectrum for some non-compact non-linearities*, Math. Meth. Appl. Sci. **11** (1989), 525–542.

23. ———, *Bifurcation of homoclinic orbits and bifurcation from the essential spectrum*, SIAM J. Math. Anal. **20** (1989), 1145–1171.

24. ———, *Bifurcation in* $L^p(\mathbb{R}^N)$ *for a semilinear elliptic equation*, Proc. Lond. Math. Soc. **57** (1988), 511–541.

25. ———, *A global branch of solutions to a semilinear elliptic equation on an unbounded interval*, Proc. Roy. Soc. Edinb. **101** (1985), 273–282.

26. M.B. Yang, Z.F. Shen and Y.H. Ding, *Multiple semiclassical solutions for the nonlinear Maxwell-Schrödinger system*, Nonl. Anal. TMA **71** (2009), 730–739.

27. Z.P. Wang and H.S. Zhou, *Positive solution for a nonlinear stationary Schrödinger-Poisson system in* $\mathbb{R}^3$, Discrete Continuous Dynamical Syst. **18** (2007), 809–816.

28. M. Willem, *Minimax theorems*, Birkhauser, Boston, 1996.

Department of Mathematics, Zhejiang Normal University, Jinhua, 321004, P.R. China
Email address: mbyang@zjnu.cn

Department of Mathematics, Yunnan Normal University, Kunming, 650092, P.R. China
Email address: fukunzhao@163.com

Institute of Mathematics, AMSS, Chinese Academy of Sciences, Beijing, 100080, P.R. China
Email address: dingyh@math.ac.cn