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WEAK ASSOCIATED PRIMES OVER DIFFERENTIAL POLYNOMIAL RINGS

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ABSTRACT. In this paper we introduce the notion of weak associated primes of a ring R which is a generalization of associated primes of R, and study how the weak associated primes of a ring R behave under passage to the differential polynomial ring $R[x;\delta]$. If we impose δ-compatibility and reversibility assumptions on ring R, then we can describe all weak associated primes of the differential polynomial ring $R[x;\delta]$ in terms of the weak associated primes of R in a very straightforward way. Consequently, several well-known properties of associated primes in [**2, 3, 10**] are extended to a more general setting.

1. Introduction. Let $\delta : R \to R$ be a derivation on R, that is, δ is an additive map such that $\delta(ab) = \delta(a)b + a\delta(b)$ for $a, b \in R$. We denote by $R[x;\delta]$ the differential polynomial ring whose elements are the polynomials $\sum_{i=0}^{n} r_i x^i \in R[x;\delta]$, $r_i \in R$, where the addition is defined as usual and the multiplication by $xb = bx + \delta(b)$ for any $b \in R$. From this rule, an inductive argument can be made to calculate an expression for $x^j a$, for any positive integer j and $a \in R$.

One can show with routine computations that [**7**]:

$$x^j a = \sum_{i=0}^{j} \binom{j}{i} \delta^{j-i}(a) x^i.$$

Let δ be a derivation on a ring R. Following Hashemi and Moussavi [**4**], a ring R is said to be δ-compatible if, for all a, $b \in R$, $ab = 0 \Rightarrow$

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$a\delta(b) = 0$. Given a right R-module N_R, the right annihilator of N_R is denoted by $r_R(N_R) = \{a \in R \mid Na = 0\}$. We say that N_R is prime if $N \neq 0$, and $r_R(N_R) = r_R(N'_R)$ for every nonzero submodule $N' \subseteq N$. Let M_R be a right R-module; an ideal $\wp$ of R is called an associated prime of M if a prime submodule $N_R \subseteq M_R$ exists such that $\wp = r_R(N_R)$. The set of associated primes of M_R is denoted by $\mathrm{Ass}\,(M_R)$. In [**2**], Brewer and Heinzer used localization theory to prove that, over a commutative ring R, the associated primes of the polynomial ring $R[x]$ (viewed as a module over itself) are all extended: that is, every $\wp \in \mathrm{Ass}\,(R[x])$ may be expressed as $\wp = \wp_0[x]$, where $\wp_0 = \wp \cap R \in \mathrm{Ass}\,(R)$. Using the results of Shock in [**10**] on good polynomials, Faith has provided a new proof in [**3**] of the same result which does not rely upon localization or other tools from commutative algebra. In [**1**], Annin showed that Brewer and Heinzer's result still holds in the more general setting of a polynomial module $M[x]$ over a skew polynomial ring $R[x;\alpha]$.

Motivated by the results in [**1–3, 10**], in this paper, we continue the study of weak associated primes over differential polynomial rings. We first introduce the notions of weak annihilators and weak associated primes, which are generalizations of annihilators and associated primes. We next describe all weak associated primes of the differential polynomial ring $R[x;\delta]$ in terms of the weak associated primes of the ring R. So several earlier results in [**2, 3, 10**] are extended to a more general setting.

Throughout this paper, all rings R are associative with identity and $\mathrm{nil}\,(R)$ standing for the subset of all nilpotent elements of R. Let $U \subseteq R$ be a subset of R. We write $U[x;\delta] = \{a_0 + a_1x + \cdots + a_lx^l \in R[x;\delta] \mid a_i \in U\} \subseteq R[x;\delta]$, that is, for any $f(x) = a_0 + a_1x + \cdots + a_lx^l \in R[x;\delta]$, $f(x) \in U[x;\delta]$ if and only if $a_i \in U$ for all $0 \leq i \leq l$. It is easy to see that, if U is a right ideal of R, then $U[x;\delta]$ is a right ideal of $R[x;\delta]$. Let $f(x) = a_0 + a_1x + \cdots + a_nx^n \in R[x;\delta]$. We say that $f(x) \in \mathrm{nil}\,(R)[x;\delta]$ if and only if for all $0 \leq i \leq n$, $a_i \in \mathrm{nil}\,(R)$, and $f(x) \in \mathrm{nil}\,(R[x;\delta])$ if and only if $f(x)$ is a nilpotent element of $R[x;\delta]$. A ring R is said to be reversible if, for all $a, b \in R$, $ab = 0$ implies $ba = 0$, and a ring R is called *semi-commutative* if, for all a, $b \in R$, $ab = 0$ implies that $aRb = 0$. Clearly, reversible rings are semi-commutative rings, but the converse does not in general hold true [**9**].

2. δ-compatibility.

Definition 2.1 Let R be a ring. For a subset X of a ring R, we define $N_R(X) = \{a \in R \mid xa \in \mathrm{nil}\,(R) \text{ for all } x \in X\}$, which is called the weak annihilator of X in R. If X is singleton, say $X = \{r\}$, we use $N_R(r)$ in place of $N_R(\{r\})$.

Obviously, for any subset X of ring R, $N_R(X) = \{a \in R \mid xa \in \mathrm{nil}\,(R)$ for all $x \in X\} = \{b \in R \mid bx \in \mathrm{nil}\,(R)$ for all $x \in X\}$, and $r_R(X) \subseteq N_R(X)$ and $l_R(X) \subseteq N_R(X)$. If R is reduced (i.e., rings with nonzero nilpotent elements), then $r_R(X) = N_R(X) = l_R(X)$ for any subset X of R. It is easy to see that, for any subset $X \subseteq R$, $N_R(X)$ is an ideal of R in case $\mathrm{nil}\,(R)$ is an ideal.

Example 2.2. Let Z be the ring of integers and $T_2(Z)$ the 2×2 upper triangular matrix ring over Z. We consider the subset $X = \left\{\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\right\}$. Clearly, $r_{T_2(Z)}(X) = 0$, and $N_{T_2(Z)}(X) = \left\{\begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix}, \mid m \in Z\right\}$. Thus, $r_{T_2(Z)}(X) \neq N_{T_2(Z)}(X)$. Hence, a weak annihilator is not a trivial generalization of a annihilator.

Proposition 2.3. *Let X, Y be subsets of R. Then we have the following*:

(1) $X \subseteq Y$ *implies* $N_R(X) \supseteq N_R(Y)$.

(2) $X \subseteq N_R(N_R(X))$.

(3) $N_R(X) = N_R(N_R(N_R(X)))$.

Proof. (1) and (2) are really easy.

(3) Applying (2) to $N_R(X)$, we obtain $N_R(X) \subseteq N_R(N_R(N_R(X)))$. Since $X \subseteq N_R(N_R(X))$, we have $N_R(X) \supseteq N_R(N_R(N_R(X)))$ by (1). Therefore, we have $N_R(X) = N_R(N_R(N_R(X)))$.

Definition 2.4. Let I be a right ideal of a nonzero ring R. We say that I is an R-prime ideal if $I \not\subseteq \mathrm{nil}\,(R)$ and $N_R(I) = N_R(I')$ for every right ideal $I' \subseteq I$ and $I' \not\subseteq \mathrm{nil}\,(R)$.

Definition 2.5. Let $\operatorname{nil}(R)$ be an ideal of ring R. An ideal $\wp$ of R is called a weak associated prime of R if an R-prime ideal I exists such that $\wp = N_R(I)$. The set of weak associated primes of R is denoted by $N\operatorname{Ass}(R)$.

Example 2.6. Let R be a domain, and Let

$$R_n = \left\{ \begin{pmatrix} a & a_{12} & \cdots & a_{1n} \\ 0 & a & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & a \end{pmatrix} \mid a, a_{ij} \in R \right\}$$

be the subring of $n \times n$ upper triangular matrix ring. Then $\operatorname{nil}(R_n)$ is an ideal of R_n and

$$\operatorname{nil}(R_n) = \left\{ \begin{pmatrix} 0 & x_{12} & \cdots & x_{1n} \\ 0 & 0 & \cdots & x_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \mid x_{ij} \in R \right\}.$$

By routine computations, we know that each right ideal $I \nsubseteq \operatorname{nil}(R_n)$ is an R_n-prime ideal, and $N\operatorname{Ass}(R_n) = \{\operatorname{nil}(R_n)\}$.

Example 2.7. Let k be any field, and consider the ring $R = \left(\begin{smallmatrix} k & 0 \\ k & k \end{smallmatrix}\right)$ of 2×2 lower triangular matrices over k. One easily checks that $\left(\begin{smallmatrix} k & 0 \\ k & k \end{smallmatrix}\right) \supsetneq \left(\begin{smallmatrix} k & 0 \\ k & 0 \end{smallmatrix}\right) \supsetneq \left(\begin{smallmatrix} k & 0 \\ 0 & 0 \end{smallmatrix}\right) \supsetneq \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ is a composition series for R_R. In particular, R_R has finite length.

Next we shall determine the set $\operatorname{Ass}(R)$. By an easy ad hoc calculation, we can write down all of the proper nonzero ideals of R :

$$\left\{ m_1 = \begin{pmatrix} 0 & 0 \\ k & k \end{pmatrix},\ m_2 = \begin{pmatrix} k & 0 \\ k & 0 \end{pmatrix},\ \alpha = \begin{pmatrix} 0 & 0 \\ k & 0 \end{pmatrix} \right\}.$$

Now, since $\alpha^2 = 0$, 0 is not a prime ideal. Moreover, since $m_1 R m_2 \subseteq \alpha$, α is not a prime ideal. So the only candidates for the associated primes of R are the maximal ideals m_1 and m_2.

We claim that $m_2 \notin \operatorname{Ass}(R)$. Otherwise, a right ideal $I \supsetneq 0$ of R would exist with $m_2 = r_R(I)$. So $I \cdot m_2 = 0$. Now, given

$\begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \in I$, we have $0 = \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}$, so $a = b = 0$. Also, $0 = \begin{pmatrix} 0 & 0 \\ 0 & c \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix}$ implies that $c = 0$. Thus, $I = 0$, a contradiction. Hence, $m_2 \notin \operatorname{Ass}(R)$.

By virtue of R_R being Noetherian, we know that $\operatorname{Ass}(R) \neq 0$. Hence, $\operatorname{Ass}(R) = \{m_1\}$.

Finally, we should determine the set of $N\operatorname{Ass}(R)$. Clearly, $\operatorname{nil}(R) = \alpha$. Thus, $\operatorname{nil}(R)$ is an ideal. Now we show that $m_1 = N_R(m_2)$ and m_2 is a right R-prime ideal. Clearly, $m_1 \subseteq N_R(m_2)$ since $m_2 m_1 = 0$. Given $\begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \in N_R(m_2)$, we have $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a & 0 \\ b & c \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \in \operatorname{nil}(R)$. Then $a = 0$ and so $\begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \in m_1$. Hence, $m_1 = N_R(m_2)$. Next we see that m_2 is a right R-prime ideal. Let $n \not\subseteq \operatorname{nil}(R)$ and $n \subseteq m_2$. Since $N_R(n) \supseteq N_R(m_2)$ is clear, we now assume that $\begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \in N_R(n)$ and find $\begin{pmatrix} h & 0 \\ k & 0 \end{pmatrix} \in n$ with $h \neq 0$. Then we have $\begin{pmatrix} h & 0 \\ k & 0 \end{pmatrix}\begin{pmatrix} a & 0 \\ b & c \end{pmatrix} = \begin{pmatrix} ha & 0 \\ ka & 0 \end{pmatrix} \in \operatorname{nil}(R)$. Thus, $a = 0$ and so $\begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \in N_R(m_2)$. Hence, we obtain $N_R(n) = N_R(m_2)$ and so m_2 is a right R-prime ideal. Thus we obtain $m_1 \in N\operatorname{Ass}(R)$. Similarly, we have $m_2 \in N\operatorname{Ass}(R)$. Therefore, $N\operatorname{Ass}(R) = \{m_1, m_2\} \neq \operatorname{Ass}(R)$.

If R is reduced, then $\wp$ is a weak associated prime of R if and only if $\wp$ is an associated prime of R. So $N\operatorname{Ass}(R) = \operatorname{Ass}(R)$ in case R is reduced.

Lemma 2.8 [**4**]. *Let R be a δ-compatible ring. Then $ab = 0$ implies $a\delta^m(b) = 0$ for all a, $b \in R$ and any positive integer m.*

Proof. It is trivial. $\square$

Lemma 2.9 *Let R be reversible and δ-compatible. Then we have the following*:

(1) *If $ab \in \operatorname{nil}(R)$, then $a\delta^m(b) \in \operatorname{nil}(R)$, and $\delta^n(a)b \in \operatorname{nil}(R)$ for all positive integers m, n.*

(2) *If $abc \in \operatorname{nil}(R)$, then $ab\delta^s(c) \in \operatorname{nil}(R)$ and $a\delta^t(b)c \in \operatorname{nil}(R)$ for all positive integers s, t.*

(3) *If* $abc \in \mathrm{nil}(R)$, *then* $a\delta^s(b)\delta^t(c) \in \mathrm{nil}(R)$ *for all positive integers* s *and* t.

Proof. (1) Since $ab \in \mathrm{nil}(R)$, some positive integer k exists such that $(ab)^k = 0$. $(ab)^k = abab\cdots ab = 0 \Rightarrow abab\cdots aba\delta^m(b) = 0 \Rightarrow a\delta^m(b)abab\cdots ab = 0 \Rightarrow a\delta^m(b)abab\cdots aba\delta^m(b) = 0 \Rightarrow a\delta^m(b)a\delta^m(b)ab\cdots ab = 0 \Rightarrow \cdots \Rightarrow a\delta^m(b) \in \mathrm{nil}(R)$. $ab \in \mathrm{nil}(R)$ implies $ba \in \mathrm{nil}(R)$. Hence, we obtain $b\delta^n(a) \in \mathrm{nil}(R)$, and so $\delta^n(a)b \in \mathrm{nil}(R)$.

(2) Let $abc \in \mathrm{nil}(R)$. It is enough to show that $ab\delta(c) \in \mathrm{nil}(R)$ and $a\delta(b)c \in \mathrm{nil}(R)$. By (1), we have $ab\delta(c) \in \mathrm{nil}(R)$, and $a\delta(bc) = ab\delta(c) + a\delta(b)c \in \mathrm{nil}(R)$. Hence, $a\delta(b)c = a\delta(bc) - ab\delta(c) \in \mathrm{nil}(R)$.

(3) It is an immediate consequence of (1) and (2).

Lemma 2.10 [**8**]. *Let* R *be a semi-commutative ring. Then* $\mathrm{nil}(R)$ *is an ideal of* R.

Lemma 2.11. *Let* R *be a semicommutative ring and* $ab \in \mathrm{nil}(R)$ *for* $a, b \in R$. *Then* $xayb \in \mathrm{nil}(R)$ *for all* $x, y \in R$.

Proof. Suppose a, b, x, $y \in R$, and $ab \in \mathrm{nil}(R)$. Some positive integer n exists such that $(ab)^n = abab\cdots ab = 0$. Thus, $xaybxayb\cdots xayb = (xayb)^n = 0$ because R is a semi-commutative ring. Hence, $xayb \in \mathrm{nil}(R)$.

Lemma 2.12. *Let* R *be* $\delta-$*compatible and reversible. Then* $f(x) = a_0 + a_1x + \cdots + a_mx^m \in R[x;\delta]$ *is a nilpotent element of* $R[x;\delta]$ *if and only if* $a_i \in \mathrm{nil}(R)$ *for all* $0 \le i \le m$.

Proof. ($\Rightarrow$). Suppose $f(x) = a_0 + a_1x + \cdots + a_mx^m \in \mathrm{nil}(R[x;\delta])$. Some positive integer k exists such that $f(x)^k = (a_0 + a_1x + \cdots + a_mx^m)^k = 0$. Then

$$0 = f(x)^k = \text{“lower terms”} + a_m^k x^{mk}.$$

Hence, we obtain $a_m \in \mathrm{nil}(R)$. So $a_m = 1 \cdot a_m \in \mathrm{nil}(R)$ implies $1 \cdot \delta^s(a_m) = \delta^s(a_m) \in \mathrm{nil}(R)$ for any positive integer s by Lemma 2.9.

Thus, we obtain $(a_0 + a_1x + \cdots + a_{m-1}x^{m-1})^k =$ "lower terms" $+ a_{m-1}^k x^{(m-1)k} \in \mathrm{nil}\,(R)[x;\delta]$ because $\mathrm{nil}\,(R)$ of a reversible ring is an ideal. Hence, we obtain $a_{m-1} \in \mathrm{nil}\,(R)$. Using induction on m, we obtain $a_i \in \mathrm{nil}\,(R)$ for all $0 \le i \le m$.

($\Leftarrow$). Suppose that $a_i^{m_i} = 0$, $0 \le i \le m$. Let $k = \sum_{i=0}^m m_i + 1$. Then, from

$$\begin{aligned}
\Big(\sum_{i=0}^m a_ix^i\Big)^2 &= \Big(\sum_{i=0}^m a_ix^i\Big)a_0 + \Big(\sum_{i=0}^m a_ix^i\Big)a_1x + \cdots \\
&\quad + \Big(\sum_{i=0}^m a_ix^i\Big)a_mx^m \\
&= \sum_{i=0}^m \binom{i}{0} a_i\delta^i(a_0) + \Big(\sum_{i=1}^m \binom{i}{1} a_i\delta^{i-1}(a_0)\Big)x + \cdots \\
&\quad + \Big(\sum_{i=s}^m \binom{i}{s} a_i\delta^{i-s}(a_0)\Big)x^s + \cdots + a_ma_0x^m \\
&\quad + \Big(\sum_{i=0}^m \binom{i}{0} a_i\delta^i(a_1) + \Big(\sum_{i=1}^m \binom{i}{1} a_i\delta^{i-1}(a_1)\Big)x \\
&\qquad\qquad + \cdots + a_ma_1x^m\Big)x \\
&\quad + \cdots + \Big(\sum_{i=0}^m \binom{i}{0} a_i\delta^i(a_m) + \Big(\sum_{i=1}^m \binom{i}{1} a_i\delta^{i-1}(a_m)\Big)x \\
&\qquad\qquad + \cdots + a_ma_mx^m\Big)x^m \\
&= \sum_{i=0}^m \binom{i}{0} a_i\delta^i(a_0) \\
&\quad + \Big(\sum_{i=1}^m \binom{i}{1} a_i\delta^{i-1}(a_0) + \sum_{i=0}^m \binom{i}{0} a_i\delta^i(a_1)\Big)x + \cdots \\
&\quad + \Big(\sum_{s+t=k} \Big(\sum_{i=s}^m \binom{i}{s} a_i\delta^{i-s}(a_t)\Big)\Big)x^k + \cdots \\
&\quad + a_ma_mx^{m+m},
\end{aligned}$$

where $0 \le s \le m$ and $0 \le t \le m$, it is easy to check that the coefficients of $(\sum_{i=0}^{m} a_i x^i)^k$ can be written as sums of the monomial $p \cdot a_{i_1}\delta^{s_2}(a_{i_2})\delta^{s_3}(a_{i_3})\cdots\delta^{s_k}(a_{i_k})$ where p is a positive integer, $a_{i_j}(1 \le j \le k) \in \{a_0, a_1, \ldots, a_m\}$ and $s_v(2 \le v \le k)$ is a nonnegative integer. Considering each monomial $p \cdot a_{i_1}\delta^{s_2}(a_{i_2})\delta^{s_3}(a_{i_3})\cdots\delta^{s_k}(a_{i_k})$, we claim that $a_{i_1}\delta^{s_2}(a_{i_2})\delta^{s_3}(a_{i_3})\cdots\delta^{s_k}(a_{i_k}) = 0$. If the number of a_0 in $a_{i_1}\delta^{s_2}(a_{i_2})\delta^{s_3}(a_{i_3})\cdots\delta^{s_k}(a_{i_k})$ is greater than m_0, then we can write $a_{i_1}\delta^{s_2}(a_{i_2})\delta^{s_3}(a_{i_3})\cdots\delta^{s_k}(a_{i_k})$ as

$$b_0(\delta^{s_{01}}(a_0))^{j_1} b_1(\delta^{s_{02}}(a_0))^{j_2} \cdots b_{v-1}(\delta^{s_{0v}}(a_0))^{j_v} b_v,$$

where $j_1 + j_2 + \cdots j_v > m_0$, and $s_{0l}(1 \le l \le v) \in \{s_2, s_3, \ldots, s_t\}$ and b_q $(q = 0, 1, 2, \ldots, v)$ is a product of some elements chosen from $\{a_{i_1}, \delta^{s_2}(a_{i_2}), \ldots, \delta^{s_k}(a_{i_k})\}$ or is equal to 1. Since $a_0^{j_1+j_2+\cdots j_v} = 0$ and R are reversible and δ-compatible, we have

$$\begin{aligned}
a_0^{j_1+j_2+\cdots+j_v} &= \underbrace{a_0 a_0 \cdots a_0 = 0}_{j_1+j_2+\cdots+j_v}, \\
&\Longrightarrow a_0 a_0 \cdots \delta^{s_{01}}(a_0)) = 0, \\
&\Longrightarrow \delta^{s_{01}}(a_0)) a_0 \cdots a_0 = 0, \\
&\Longrightarrow \delta^{s_{01}}(a_0) a_0 \cdots a_0 \delta^{s_{01}}(a_0)) = 0, \\
&\Longrightarrow \ldots, \\
&\Longrightarrow (\delta^{s_{01}}(a_0))^{j_1} a_0 \cdots a_0 = 0, \\
&\Longrightarrow \ldots, \\
&\Longrightarrow (\delta^{s_{01}}(a_0))^{j_1} (\delta^{s_{02}}(a_0))^{j_2} \cdots (\delta^{s_{0v}}(a_0))^{j_v} = 0, \\
&\Longrightarrow b_0(\delta^{s_{01}}(a_0))^{j_1} b_1(\delta^{s_{02}}(a_0))^{j_2} \cdots b_{v-1}(\delta^{s_{0v}}(a_0))^{j_v} b_v = 0.
\end{aligned}$$

If the number of a_i in $a_{i_1}\delta^{s_2}(a_{i_2})\delta^{s_3}(a_{i_3})\cdots\delta^{s_k}(a_{i_k})$ is greater than m_i, then a similar discussion yields that $a_{i_1}\delta^{s_2}(a_{i_2})\delta^{s_3}(a_{i_3})\cdots\delta^{s_k}(a_{i_k}) = 0$. Thus, we obtain that each monomial $p \cdot a_{i_1}\delta^{s_2}(a_{i_2})\delta^{s_3}(a_{i_3})\cdots\delta^{s_k}(a_{i_k})$ equals 0. Therefore, $f(x) = a_0 + a_1 x + \cdots + a_m x^m \in R[x;\delta]$ is a nilpotent element of the ring $R[x;\delta]$.

Corollary 2.13. *Let R be a reversible and δ-compatible ring. Then we have* $\mathrm{nil}\,(R)[x;\delta] = \mathrm{nil}\,(R[x;\delta])$.

Proof. It is an immediate consequence of Lemma 2.12. $\square$

Lemma 2.14. *Let R be a reversible and δ-compatible ring, and let $f = \sum_{i=0}^{m} a_i x^i$, $g = \sum_{j=0}^{n} b_j x^j$, $h = \sum_{k=0}^{p} c_k x^k \in R[x;\delta]$ and $c \in R$. Then we have the following*:

(1) $fg \in \mathrm{nil}\,(R[x;\delta]) \Leftrightarrow a_i b_j \in \mathrm{nil}\,(R)$ *for all* $0 \le i \le m$, $0 \le j \le n$.

(2) $fgc \in \mathrm{nil}\,(R[x;\delta]) \Leftrightarrow a_i b_j c \in \mathrm{nil}\,(R)$ *for all* $0 \le i \le m$, $0 \le j \le n$.

(3) $fgh \in \mathrm{nil}\,(R[x;\delta]) \Leftrightarrow a_i b_j c_k \in \mathrm{nil}\,(R)$ *for all* $0 \le i \le m$, $0 \le j \le n$, $0 \le k \le p$.

Proof. (1) ($\Rightarrow$). Let $f = \sum_{i=0}^{m} a_i x^i$, $g = \sum_{j=0}^{n} b_j x^j \in R[x;\delta]$ be such that $fg \in \mathrm{nil}\,(R[x;\delta])$. Then

$$fg = \left(\sum_{i=0}^{m} a_i x^i\right)\left(\sum_{j=0}^{n} b_j x^j\right) = \sum_{k=0}^{m+n}\left(\sum_{s+t=k}\left(\sum_{i=s}^{m}\binom{i}{s} a_i \delta^{i-s}(b_t)\right)\right)x^k$$

where $0 \le s \le m$ and $0 \le t \le n$.

Then we have the following equations by Lemma 2.12:

$$\Delta_{m+n} = a_m b_n \in \mathrm{nil}\,(R), \tag{1}$$

(2)

$$\Delta_{m+n-1} = a_m b_{n-1} + \sum_{i=m-1}^{m}\binom{i}{m-1} a_i \delta^{i-(m-1)}(b_n) \in \mathrm{nil}\,(R)$$

$$\begin{aligned}\Delta_{m+n-2} = {} & \sum_{i=m-2}^{m}\binom{i}{m-2} a_i \delta^{i-(m-2)}(b_n) \\ & + \sum_{i=m-1}^{m}\binom{i}{m-1} a_i \delta^{i-(m-1)}(b_{n-1}) \\ & + a_m b_{n-2} \in \mathrm{nil}\,(R)\end{aligned} \tag{3}$$

$$\vdots$$

$$\Delta_k = \sum_{s+t=k}\left(\sum_{i=s}^{m}\binom{i}{s} a_i \delta^{i-s}(b_t)\right) \in \mathrm{nil}\,(R). \tag{4}$$

From equation (1), we get $a_m b_n \in \mathrm{nil}\,(R)$ and $b_n a_m \in \mathrm{nil}\,(R)$. Now we show that $a_i b_n \in \mathrm{nil}\,(R)$ for all $0 \le i \le m$. If we multiply equation

(2) on the left side by b_n, then

$$b_n a_{m-1} b_n = b_n \Delta_{m+n-1} - b_n a_m b_{n-1} - \binom{m}{m-1} b_n a_m \delta(b_n) \in \operatorname{nil}(R)$$

since $\operatorname{nil}(R)$ is an ideal of R. So $b_n a_{m-1} \in \operatorname{nil}(R)$ and $a_{m-1} b_n \in \operatorname{nil}(R)$. If we multiply equation (3) on the left side by b_n, then $b_n a_{m-2} b_n = b_n \Delta_{m+n-2} - \binom{m-1}{m-2} b_n a_{m-1} \delta(b_n) - \binom{m}{m-2} b_n a_m \delta^2(b_n) - b_n a_{m-1} b_{n-1} - \binom{m}{m-1} b_n a_m \delta(b_{n-1}) - b_n a_m b_{n-2} = b_n \Delta_{m+n-2} - \binom{m-1}{m-2} (b_n a_{m-1}) \delta(b_n) - \binom{m}{m-2} (b_n a_m) \delta^2(b_n) - (b_n a_{m-1}) b_{n-1} - \binom{m}{m-1} (b_n a_m) \delta(b_{n-1}) - (b_n a_m) b_{n-2} \in \operatorname{nil}(R)$ since $\operatorname{nil}(R)$ is an ideal of R. Hence, $b_n a_{m-2} \in \operatorname{nil}(R)$, and so $a_{m-2} b_n \in \operatorname{nil}(R)$. Continuing this process yields that $a_i b_n \in \operatorname{nil}(R)$ for $0 \le i \le m$. By Lemma 2.9 we have $a_i \delta^s(b_n) \in \operatorname{nil}(R)$ for $0 \le i \le m$ and any positive integer s. Thus, we obtain

$$\begin{aligned}
(a_0 + a_1 x + \cdots + a_m x^m)&(b_0 + b_1 x + \cdots + b_{n-1} x^{n-1}) \\
&= \sum_{k=0}^{m+n-1} \Big(\sum_{s+t=k} \Big(\sum_{i=s}^{m} \binom{i}{s} a_i \delta^{i-s}(b_t) \Big) \Big) x^k \\
&= \Delta_0 + \Delta_1 x + \cdots + \Delta_{m+n-1} x^{m+n-1} \in \operatorname{nil}(R)[x;\delta],
\end{aligned}$$

where $0 \le s \le m$, $0 \le t \le n-1$.

Applying the same method as above, we obtain $a_i b_{n-1} \in \operatorname{nil}(R)$ for all $0 \le i \le m$. Using induction on n, we obtain $a_i b_j \in \operatorname{nil}(R)$ for all $0 \le i \le m$ and $0 \le j \le n$.

($\Leftarrow$). If $a_i b_j \in \operatorname{nil}(R)$ for all $0 \le i \le m$ and $0 \le j \le n$, then $a_i \delta^t(b_j) \in \operatorname{nil}(R)$ for all $0 \le i \le m$, $0 \le j \le n$ and any positive integer t by Lemma 2.9. Since $\operatorname{nil}(R)$ of a reversible ring is an ideal, we obtain $\sum_{s+t=k} (\sum_{i=s}^{m} \binom{i}{s} a_i \delta^{i-s}(b_t)) \in \operatorname{nil}(R)$. Hence, $fg \in \operatorname{nil}(R[x;\delta])$ by Lemma 2.12.

(2) ($\Rightarrow$). We have

$$\begin{aligned} gc &= \left(\sum_{j=0}^{n} b_j x^j\right) c \\ &= \sum_{j=0}^{n} \binom{j}{0} b_j \delta^j(c) \\ &\quad + \left(\sum_{j=1}^{n} \binom{j}{1} b_j \delta^{j-1}(c)\right) x + \cdots \\ &\quad + \left(\sum_{j=s}^{n} \binom{j}{s} b_j \delta^{j-s}(c)\right) x^s + \cdots + b_n c x^n \\ &= \Delta_0 + \Delta_1 x + \cdots + \Delta_s x^s + \cdots + \Delta_n x^n \end{aligned}$$

where $\Delta_s = \sum_{j=s}^{n} \binom{j}{s} b_j \delta^{j-s}(c)$, $0 \le s \le n$.

By (1), we have $a_i \Delta_s = a_i \left(\sum_{j=s}^{n} \binom{j}{s} b_j \delta^{j-s}(c)\right) \in \mathrm{nil}\,(R)$ for $0 \le i \le m$ and $0 \le s \le n$.

For $s = n$, we have $a_i \Delta_n = a_i b_n c \in \mathrm{nil}\,(R)$ for all $0 \le i \le m$.

For $s = n-1$, we have $a_i \Delta_{n-1} = a_i(b_{n-1}c + \binom{n}{n-1} b_n \delta(c)) = a_i b_{n-1} c + \binom{n}{n-1} a_i b_n \delta(c) \in \mathrm{nil}\,(R)$ for all $0 \le i \le m$. Since $a_i b_n c \in \mathrm{nil}\,(R)$, by Lemma 2.9, we have $a_i b_n \delta(c) \in \mathrm{nil}\,(R)$. Hence, $a_i b_{n-1} c \in \mathrm{nil}\,(R)$ for all $0 \le i \le m$.

Now suppose that k is a positive integer such that for all $0 \le i \le m$ $a_i b_j c \in \mathrm{nil}\,(R)$ when $j > k$. We show that $a_i b_k c \in \mathrm{nil}\,(R)$ for all $0 \le i \le m$.

If $s = k$, for all $0 \le i \le m$, we have

$$a_i \Delta_k = a_i \left(\sum_{j=k}^{n} \binom{j}{k} b_j \delta^{j-k}(c)\right) \in \mathrm{nil}\,(R).$$

Since $a_i b_j c \in \mathrm{nil}\,(R)$ for $0 \le i \le m$ and $k < j \le n$, by Lemma 2.9, we have $a_i b_j \delta^{j-k}(c) \in \mathrm{nil}\,(R)$ for all $0 \le i \le m$ and $k < j \le n$. Then $a_i b_k c \in \mathrm{nil}\,(R)$ for all $0 \le i \le m$. Therefore, by induction, we obtain $a_i b_j c \in \mathrm{nil}\,(R)$ for all $0 \le i \le m$ and $0 \le j \le n$.

($\Leftarrow$). If $a_ib_jc \in \mathrm{nil}\,(R)$ for all $0 \le i \le m$, $0 \le j \le n$, then $a_ib_j\delta^t(c) \in \mathrm{nil}\,(R)$ and so $a_i(\sum_{j=s}^{n}\binom{j}{s}b_j\delta^{j-s}(c)) \in \mathrm{nil}\,(R)$. Therefore, $fgc \in \mathrm{nil}\,(R[x;\delta])$ by (1).

(3) It suffices to show ($\Rightarrow$). First we show that $fgh \in \mathrm{nil}\,(R[x;\delta]) \Rightarrow fgc_k \in \mathrm{nil}\,(R[x;\delta])$ with $k \in \{0,1,\dots p\}$. We have

$$\begin{aligned} fg &= \left(\sum_{i=0}^{m} a_ix^i\right)\left(\sum_{j=0}^{n} b_jx^j\right) \\ &= \sum_{l=0}^{m+n}\Bigg(\sum_{s+t=l}\Bigg(\sum_{i=s}^{m}\binom{i}{s}a_i\delta^{i-s}(b_t)))x^l \\ &= \sum_{l=0}^{m+n}\Delta_lx^l. \end{aligned}$$

For any $0 \le k \le p$, since $fgh \in \mathrm{nil}\,(R[x;\delta])$, by (1), we have

$$\Delta_lc_k = \Bigg(\sum_{s+t=l}\Bigg(\sum_{i=s}^{m}\binom{i}{s}a_i\delta^{i-s}(b_t)))c_k \in \mathrm{nil}\,(R),$$
$$0 \le l \le m+n,$$

and so $fgc_k \in \mathrm{nil}\,(R[x;\delta])$ with $k \in \{0,1,\dots p\}$. Now (2) implies $a_ib_jc_k \in \mathrm{nil}\,(R)$ for all $0 \le i \le m$, $0 \le j \le n$, $0 \le k \le p$.

Given a polynomial $f(x) \in R[x]$, if the polynomial $f(x)$ has the property that each nonzero coefficient has the same right annihilator in R, then we say that such a polynomial is a good polynomial. Shock showed in [**10**] that, given any nonzero polynomial $f(x) \in R[x]$, one can find $r \in R$ such that $f(x)r$ is good. In order to prove the main result of this paper, we will need a generalized version of Shock's result which applies in our differential polynomial setting.

Let $m(x) = m_0 + m_1x + \cdots + m_kx^k + \cdots + m_nx^n \notin \mathrm{nil}\,(R)[x;\delta]$. If $m_k \notin \mathrm{nil}\,(R)$, and $m_i \in \mathrm{nil}\,(R)$ for all $i > k$, then we say that the weak degree of $m(x)$ is k. To simplify notations, we write $N\deg\,(m(x))$ for the weak degree of $m(x)$.

Definition 2.15. Let $m(x) = m_0+m_1x+\cdots+m_kx^k+\cdots+m_nx^n \notin \mathrm{nil}\,(R)[x;\delta]$ and $N\deg\,(m(x)) = k$. If $N_R(m_k) \subseteq N_R(m_i)$ for all $i \le k$, then we say that $m(x)$ is a weak good polynomial.

Lemma 2.16. *Let R be a δ-compatible and reversible ring. For any $m(x) = m_0 + m_1x + \cdots + m_kx^k + \cdots + m_nx^n \notin \mathrm{nil}\,(R)[x;\delta]$, there exists an $r \in R$ such that $m(x)r$ is a weak good polynomial.*

Proof. Assume the result is false, and let $m(x) = m_0 + m_1x + \cdots + m_kx^k + \cdots + m_nx^n \notin \mathrm{nil}\,(R)[x;\delta]$ be a counterexample of minimal weak degree $N\mathrm{deg}\,(m(x)) = k \geq 1$. In particular, $m(x)$ is not a weak good polynomial. Hence, there exists an $i < k$ such that $N_R(m_k) \nsubseteq N_R(m_i)$. So we can find $b \in R$ with $m_ib \notin \mathrm{nil}\,(R)$, and $m_kb \in \mathrm{nil}\,(R)$. Note that $N\mathrm{deg}\,(m(x)) = k$, so $m_j \in \mathrm{nil}\,(R)$ for all $j > k$. Since $\mathrm{nil}\,(R)$ of a reversible ring is an ideal, $m_j\delta^t(b) \in \mathrm{nil}\,(R)$ for all $j > k$ and any positive integer t. So the degree k coefficient of $m(x)b$ is $\sum_{i=k}^{n} \binom{i}{k} m_i\delta^{i-k}(b) \in \mathrm{nil}\,(R)$. On the other hand, it is easy to see that, for all $s > k$, the degree s coefficient of $m(x)b$ is $\sum_{i=s}^{n} \binom{i}{s} m_i\delta^{i-s}(b) \in \mathrm{nil}\,(R)$. Hence, $m(x)b$ has weak degree at most $k-1$. Since $m_ib \notin \mathrm{nil}\,(R)$, by Corollary 2.13, we have $m(x)b \notin \mathrm{nil}\,(R)[x;\delta]$. By the minimality of k, we know that $c \in R$ exists with $m(x)bc$ weakly good. But this contradicts the fact that $m(x)$ is a counterexample to the statement.

3. Main results.

Theorem 3.1. *Let δ be a derivation on R. If R is a reversible and δ-compatible ring, then $N\mathrm{Ass}\,(R[x;\delta]) = \{\wp[x;\delta] \mid \wp \in N\mathrm{Ass}\,(R)\}$.*

Proof. We first prove $\supseteq$. Let $\wp \in N\mathrm{Ass}\,(R)$. By definition, a right ideal $I \nsubseteq \mathrm{nil}\,(R)$ exists with I an R-prime ideal and $\wp = N_R(I)$. It suffices to prove

$$\wp[x;\delta] = N_{R[x;\delta]}(I[x;\delta]) \tag{5}$$

and

$$I[x;\delta] = R[x;\delta]\text{-prime}. \tag{6}$$

For equation (5), let $f(x) = a_0 + a_1x + \cdots + a_lx^l \in \wp[x;\delta]$, and let $i(x) = i_0 + i_1x + \cdots + i_mx^m \in I[x;\delta]$. Since $i_ka_j \in \mathrm{nil}\,(R)$ for each

k, j, applying Lemma 2.14 yields that $i(x)f(x) \in \mathrm{nil}\,(R[x;\delta])$. Hence, $\wp[x;\delta] \subseteq N_{R[x;\delta]}(I[x;\delta])$.

Conversely, if $f(x) = a_0 + a_1x + \cdots + a_lx^l \in N_{R[x;\delta]}(I[x;\delta])$, then $i(x)f(x) \in \mathrm{nil}\,(R[x;\delta])$ for all $i(x) = i_0 + i_1x + \cdots + i_mx^m \in I[x;\delta]$. Using Lemma 2.14, we obtain that $i_ka_j \in \mathrm{nil}\,(R)$ for each k, j. Thus, for all $0 \leq j \leq l$, $a_j \in N_R(I) = \wp$, and so $f(x) \in \wp[x;\delta]$. Hence, $N_{R[x;\delta]}(I[x;\delta]) \subseteq \wp[x;\delta]$. Therefore, $\wp[x;\delta] = N_{R[x;\delta]}(I[x;\delta])$.

Note that the right ideal I is an R-prime ideal. Then we have $I \nsubseteq \mathrm{nil}\,(R)$. Thus,

$$I[x;\delta] \nsubseteq \mathrm{nil}\,(R)[x;\delta] = \mathrm{nil}\,(R[x;\delta]).$$

To see (6), we must show that if a right ideal $\mho \nsubseteq \mathrm{nil}\,(R[x;\delta])$ and $\mho \subseteq I[x;\delta]$, then

$$N_{R[x;\delta]}(\mho) = N_{R[x;\delta]}(I[x;\delta]).$$

To this end, let D be a subset of R consisting of all coefficients of elements of $\mho$. Then, let $\wp_0$ denote the right ideal of R generalized by D. Since $\mho \nsubseteq \mathrm{nil}\,(R[x;\delta]) = \mathrm{nil}\,(R)[x;\delta]$, $D \nsubseteq \mathrm{nil}\,(R)$, and hence $\wp_0 \subseteq I$, $\wp_0 \nsubseteq \mathrm{nil}\,(R)$. So we have $N_R(\wp_0) = N_R(I) = \wp$ because I is R-prime. Since $N_{R[x;\delta]}(\mho) \supseteq N_{R[x;\delta]}(I[x;\delta])$ is clear, we now assume that

$$h(x) = h_0 + h_1x + \cdots + h_ux^u \in N_{R[x;\delta]}(\mho),$$

and

$$s(x) = s_0 + s_1x + \cdots + s_vx^v \in \mho.$$

Then we have $s(x)h(x) \in \mathrm{nil}\,(R[x;\delta])$. By Lemma 2.14, we obtain

$$s_ih_j \in \mathrm{nil}\,(R) \quad \text{for all } 0 \leq i \leq v,\ 0 \leq j \leq u.$$

It follows from Lemma 2.11 that $s_iRh_j \in \mathrm{nil}\,(R)$. Thus we obtain

$$h_j \in N_R(\wp_0) = N_R(I) = \wp \quad \text{for all } 0 \leq j \leq u.$$

Let $i(x) = i_0 + i_1x + \cdots + i_px^p \in I[x;\delta]$. We have $i_mh_j \in \mathrm{nil}\,(R)$ for all $0 \leq m \leq p$, $0 \leq j \leq u$. Then $i(x)h(x) \in \mathrm{nil}\,(R[x;\delta])$ by

Lemma 2.14. Hence, $N_{R[x;\delta]}(\mho) \subseteq N_{R[x;\delta]}(I[x;\delta])$ is proved, and so is $\supseteq$ in Theorem 3.1.

Now we turn our attention to proving $\subseteq$ in Theorem 3.1. Let $I \in N\mathrm{Ass}\,(R[x;\delta])$. By definition, we have an $R[x;\delta]$-prime ideal $\pounds$ with $I = N_{R[x;\delta]}(\pounds)$. Pick any

$$m(x) = m_0 + m_1 x + \cdots + m_k x^k + \cdots + m_n x^n \notin \mathrm{nil}\,(R)[x;\delta] \text{ in } \pounds.$$

By $\pounds \not\subseteq \mathrm{nil}\,(R[x;\delta])$ and Lemma 2.16, we may assume that $m(x)$ is weak good, and $N\deg\,(m(x)) = k$. Set $\pounds_0 = m(x) \cdot R[x;\delta]$. Note that $m(x) \notin \mathrm{nil}\,(R)[x;\delta])$, so we get

$$\pounds_0 = m(x)R[x;\delta] \not\subseteq \mathrm{nil}\,(R)[x;\delta] = \mathrm{nil}\,(R[x;\delta]).$$

Then we have

$$N_{R[x;\delta]}(\pounds) = N_{R[x;\delta]}(\pounds_0) = N_{R[x;\delta]}(m(x) \cdot R[x;\delta]) = I$$

because $\pounds$ is $R[x;\delta]$-prime. Consider the right ideal $m_k R$, and assume that $U = N_R(m_k R)$. We wish to claim that $I = U[x;\delta]$. Let

$$g(x) = b_0 + b_1 x + \cdots + b_l x^l \in U[x;\delta].$$

Then

$$m_k R b_j \in \mathrm{nil}\,(R) \quad \text{for all } 0 \le j \le l.$$

Since $m(x)$ is weakly good, and $N\deg\,(m(x)) = k$, $m_i R b_j \in \mathrm{nil}\,(R)$ for all $0 \le i \le k$, $0 \le j \le l$. On the other hand, for all $i > k$, $m_i \in \mathrm{nil}\,(R)$. Thus, we have $m_i R b_j \in \mathrm{nil}\,(R)$ for all $0 \le i \le n$, $0 \le j \le l$. Choose any

$$h(x) = h_0 + h_1 x + \cdots + h_p x^p \in R[x;\delta].$$

From $m_i h_d b_j \in \mathrm{nil}\,(R)$ for all $0 \le i \le n$, $0 \le d \le p$ and $0 \le j \le l$ and Lemma 2.14, we obtain $m(x)h(x)g(x) \in \mathrm{nil}\,(R[x;\delta])$. Hence, $g(x) \in N_{R[x;\delta]}(m(x)R[x;\delta]) = I$, and so $U[x;\delta] \subseteq I$. Conversely, let $g(x) = b_0 + b_1 x + \cdots + b_l x^l \in I$. Then

$$m(x)h(x)g(x) \in \mathrm{nil}\,(R[x;\delta])$$

for all

$$h(x) = h_0 + h_1 x + \cdots + h_p x^p \in R[x;\delta].$$

By Lemma 2.14, we get $m_i h_d b_j \in \mathrm{nil}\,(R)$ for all $0 \le i \le n$, $0 \le d \le p$ and $0 \le j \le l$. Thus, $b_j \in N_R(m_k R)$ for all $0 \le j \le l$, and so $g(x) \in U[x;\delta]$. Hence, $I \subseteq U[x;\delta]$. Therefore, $I = U[x;\delta]$.

We are now ready to check that $m_k R$ is R-prime. Since $m_k \notin \mathrm{nil}\,(R)$, $m_k R \nsubseteq \mathrm{nil}\,(R)$. Assume that a right ideal $Q \subseteq m_k R$, and $Q \nsubseteq \mathrm{nil}\,(R)$. Then $N_R(Q) \supseteq N_R(m_k R)$ is clear. Now we show that

$$N_R(Q) \subseteq N_R(m_k R).$$

Set $W = \{m(x)r \mid r \in Q\}$, and let $WR[x;\delta]$ be the right ideal of $R[x;\delta]$ generated by W. It is obvious that $WR[x;\delta] \subseteq m(x)R[x;\delta]$. Since $Q \nsubseteq \mathrm{nil}\,(R)$, $a \in R$ exists such that $m_k a \in Q$ and $m_k a \notin \mathrm{nil}\,(R)$. If $m_k \cdot m_k a \in \mathrm{nil}\,(R)$, then we have $m_k a \in \mathrm{nil}\,(R)$ by Lemma 2.11. This contradicts the fact that $m_k a \notin \mathrm{nil}\,(R)$. Thus, $m_k \cdot m_k a \notin \mathrm{nil}\,(R)$, and so $m(x) \cdot m_k a \nsubseteq \mathrm{nil}\,(R[x;\delta])$ by Lemma 2.14, and this implies that $WR[x;\delta] \nsubseteq \mathrm{nil}\,(R[x;\delta])$. Since $\pounds$ is $R[x;\delta]$-prime, we obtain

$$N_{R[x;\delta]}(WR[x;\delta]) = N_{R[x;\delta]}(m(x)R[x;\delta]) = I.$$

Suppose $q \in N_R(Q)$. Then $rq \in \mathrm{nil}\,(R)$ for each $r \in Q$. For any $m(x)rf(x) \in WR[x;\delta]$ where $f(x) = a_0 + a_1 x + \cdots + a_l x^l \in R[x;\delta]$. The typical term of $m(x)rf(x)$ is $m_i x^i r a_j x^j$. From $rq \in \mathrm{nil}\,(R)$ and Lemma 2.11, we have $m_i r a_j q \in \mathrm{nil}\,(R)$. Thus, $m_i x^i r a_j x^j q \in \mathrm{nil}\,(R)[x;\delta]$ due to the δ-compatibility of R, and so

$$m(x)rf(x)q \in \mathrm{nil}\,(R)[x;\delta] = \mathrm{nil}\,(R[x;\delta]).$$

Thus, for any

$$\sum m(x) r_i f_i(x) \in WR[x;\alpha],$$

it is easy to see that

$$\left(\sum m(x) r_i f_i(x)\right) q \in \mathrm{nil}\,(R[x;\delta]).$$

Hence, $q \in N_{R[x;\delta]}(WR[x;\delta]) = I = U[x;\delta]$, and so $q \in U = N_R(m_k R)$. So $N_R(Q) \subseteq N_R(m_k R)$, and this implies that $N_R(Q) = N_R(m_k R)$. Thus, $m_k R$ is $R-$prime.

Assembling the above results, we finish the proof of Theorem 3.1.

Corollary 3.2. *Let R be a reversible ring. Then* $N\mathrm{Ass}\,(R[x]) = \{\wp[x] \mid \wp \in N\mathrm{Ass}\,(R)\}$.

Proof. Take $\delta = 0$ in Theorem 3.1. $\square$

Example 3.3. Let R be a domain, and let

$$T(R,3) = \left\{ \begin{pmatrix} a_1 & a_2 & a_3 \\ 0 & a_1 & a_2 \\ 0 & 0 & a_1 \end{pmatrix} \mid a_i \in R \right\}.$$

Then it is easy to verify that $T(R,3)$ is reversible. Let

$$m = \left\{ \left(\begin{pmatrix} 0 & u & v \\ 0 & 0 & u \\ 0 & 0 & 0 \end{pmatrix} \right) \mid u,\ v \in R \right\}.$$

By routine computations, we have $N\mathrm{Ass}\,(T(R,3)) = \{m\}$. For a derivation δ on R, the natural extension $\overline{\delta} : T(R,3) \to T(R,3)$ defined by $\overline{\delta}((a_i)) = (\delta(a_i))$ is a derivation on $T(R,3)$. Suppose that δ is compatible on R. We can show that $\overline{\delta}$ is compatible on $T(R,3)$ as well. Then, by Theorem 3.1, we have $N\mathrm{Ass}\,(T(R,3)[x;\overline{\delta}]) = \{m[x;\overline{\delta}]\}$.

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