ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 42, Number 5, 2012

SUPERADDITIVITY AND MONOTONICITY OF SOME FUNCTIONALS ASSOCIATED WITH THE HERMITE-HADAMARD INEQUALITY FOR CONVEX FUNCTIONS IN LINEAR SPACES

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ABSTRACT. The superadditivity and monotonicity properties of some functionals associated with convex functions and the Hermite-Hadamard inequality in the general setting of linear spaces are investigated. Applications for norms and convex functions of a real variable are given. Some inequalities for arithmetic, geometric, harmonic, logarithmic and identric means are improved.

1. Introduction. For any convex function we can consider the well-known inequality due to Hermite and Hadamard. It was first discovered by Ch. Hermite in 1881 in the journal *Mathesis* (see [**6**]). Hermite mentioned that the following inequality holds for any convex function f defined on $\mathbf{R}$

$$(1.1)\qquad (b-a)f\left(\frac{a+b}{2}\right) < \int_a^b f(x)\,dx < (b-a)\frac{f(a)+f(b)}{2},\quad a,b\in\mathbf{R}.$$

But this result was nowhere mentioned in the mathematical literature and was not widely known as Hermite's result [**8**]. Beckenbach, a leading expert on the history and the theory of convex functions, wrote that this inequality was proven by Hadamard in 1893 [**1**]. In 1974, Mitrinović found Hermite's note in *Mathesis* [**6**]. Since (1.1) was known as Hadamard's inequality, the inequality is now commonly referred to as the Hermite-Hadamard inequality [**8**].

Let X be a vector space, $x,y\in X$, $x\neq y$. Define the segment $[x,y]:=\{(1-t)x+ty,\ t\in[0,1]\}$. We consider the function $f:[x,y]\to\mathbf{R}$ and

2010 AMS *Mathematics subject classification.* Primary 26D15, 26D10.
Keywords and phrases. Convex functions, Hermite-Hadamard's inequality, norms, means.
Received by the editors on February 7, 2010.
DOI:10.1216/RMJ-2012-42-5-1447

the associated function

$$g(x,y):[0,1]\longrightarrow \mathbf{R},\qquad g(x,y)(t):=f[(1-t)x+ty],\quad t\in[0,1].$$

Note that f is convex on $[x,y]$ if and only if $g(x,y)$ is convex on $[0,1]$.

For any convex function defined on a segment $[x,y]\subset X$, we have the Hermite-Hadamard integral inequality (see [**2**, page 2], [**3**, page 2])

$$f\left(\frac{x+y}{2}\right)\le\int_0^1 f[(1-t)x+ty]\,dt\le\frac{f(x)+f(y)}{2}, \tag{1.2}$$

which can be derived from the classical Hermite-Hadamard inequality (1.1) for the convex function $g(x,y):[0,1]\to\mathbf{R}$.

Since $f(x)=\|x\|^p$ ($x\in X$ and $1\le p<\infty$) is a convex function, we have the following norm inequality from (1.2) (see [**7**, page 106])

$$\left\|\frac{x+y}{2}\right\|^p\le\int_0^1\|(1-t)x+ty\|^p dt\le\frac{\|x\|^p+\|y\|^p}{2}, \tag{1.3}$$

for any $x,y\in X$. Particularly, if $p=2$, then

$$\left\|\frac{x+y}{2}\right\|^2\le\int_0^1\|(1-t)x+ty\|^2 dt\le\frac{\|x\|^2+\|y\|^2}{2}, \tag{1.4}$$

holds for any $x,y\in X$. We also get the following refinement of the triangle inequality when $p=1$

$$\left\|\frac{x+y}{2}\right\|\le\int_0^1\|(1-t)x+ty\|\,dt\le\frac{\|x\|+\|y\|}{2}. \tag{1.5}$$

2. Some functional properties.

Consider a convex function $f:C\subset X\to\mathbf{R}$ defined on the convex subset C in the real linear space X and two distinct vectors $x,y\in C$. We denote by $[x,y]$ the closed segment defined by $\{(1-t)x+ty,t\in[0,1]\}$. We also define the functional

$$\Psi_f(x,y;t):=(1-t)f(x)+tf(y)-f((1-t)x+ty)\ge 0, \tag{2.1}$$

where $x,y\in C$ and $t\in[0,1]$.

Theorem 1. *Let $f : C \subset X \to \mathbf{R}$ be a convex function on the convex set C. Then, for each $x, y \in C$ and $z \in [x, y]$, we have*

$$(0 \leq)\, \Psi_f(x, z; t) + \Psi_f(z, y; t) \leq \Psi_f(x, y; t) \tag{2.2}$$

for each $t \in [0, 1]$, i.e., the functional $\Psi_f(\cdot, \cdot; t)$ is superadditive as a function of interval.

If $[z, u] \subset [x, y]$, then

$$(0 \leq)\, \Psi_f(z, u; t) \leq \Psi_f(x, y; t) \tag{2.3}$$

for each $t \in [0, 1]$, i.e., the functional $\Psi_f(\cdot, \cdot; t)$ is nondecreasing as a function of the interval.

Proof. Let $z = (1-s)x + sy$ with $s \in (0, 1)$. For $t \in (0, 1)$, we have

$$\begin{aligned}\Psi_f(z, y; t) &= (1-t) f((1-s)x + sy) \\ &\qquad + tf(y) - f((1-t)[(1-s)x + sy] + ty)\end{aligned}$$

and

$$\begin{aligned}\Psi_f(x, z; t) &= (1-t) f(x) + tf((1-s)x + sy) \\ &\qquad - f((1-t)x + t[(1-s)x + sy])\end{aligned}$$

giving that

$$\begin{aligned}&\Psi_f(x, z; t) + \Psi_f(z, y; t) - \Psi_f(x, y; t) \\ &\quad = f((1-s)x + sy) + f((1-t)x + ty) \\ &\qquad - f((1-t)(1-s)x + [(1-t)s + t]y) - f((1-ts)x + tsy).\end{aligned} \tag{2.4}$$

Now, for a convex function $\varphi : I \subset \mathbf{R} \to \mathbf{R}$, where I is an interval, and any real numbers t_1, t_2, s_1 and s_2 from I and with the properties that $t_1 \leq s_1$ and $t_2 \leq s_2$ we have that

$$\frac{\varphi(t_1) - \varphi(t_2)}{t_1 - t_2} \leq \frac{\varphi(s_1) - \varphi(s_2)}{s_1 - s_2}. \tag{2.5}$$

Indeed, since φ is convex on I then for any $a \in I$ the function $\psi : I \backslash \{a\} \to \mathbf{R}$

$$\psi(t) := \frac{\varphi(t) - \varphi(a)}{t - a}$$

is monotonic nondecreasing on $t \setminus \{a\}$. Utilizing this property repeatedly, we have

$$\frac{\varphi(t_1)-\varphi(t_2)}{t_1-t_2} \le \frac{\varphi(s_1)-\varphi(t_2)}{s_1-t_2} = \frac{\varphi(t_2)-\varphi(s_1)}{t_2-s_1}$$
$$\le \frac{\varphi(s_2)-\varphi(s_1)}{s_2-s_1} = \frac{\varphi(s_1)-\varphi(s_2)}{s_1-s_2},$$

which proves inequality (2.5).

Consider the function $\varphi : [0,1] \to \mathbf{R}$ given by $\varphi(t) := f((1-t)x+ty)$. Since f is convex on C, it follows that φ is convex on $[0,1]$. Now, if we consider for a given $t, s \in (0,1)$,

$$t_1 := ts < s =: s_1 \quad \text{and} \quad t_2 := t < t + (1-t)s =: s_2,$$

then we have

$$\varphi(t_1) = f((1-ts)x + tsy), \qquad \varphi(t_2) = f((1-t)x+ty)$$

giving that

$$\frac{\varphi(t_1)-\varphi(t_2)}{t_1-t_2} = \frac{f((1-ts)x+tsy) - f((1-t)x+ty)}{t(s-1)}.$$

Also,

$$\varphi(s_1) = f((1-s)x+sy),$$
$$\varphi(s_2) = f((1-t)(1-s)x + [(1-t)s+t]y),$$

giving that

$$\frac{\varphi(s_1)-\varphi(s_2)}{s_1-s_2} = \frac{f((1-s)x+sy) - f((1-t)(1-s)x+[(1-t)s+t]y)}{t(s-1)}.$$

Utilizing inequality (2.5) and multiplying with $t(s-1) < 0$, we deduce the inequality

$$\begin{aligned} &f((1-ts)x+tsy) - f((1-t)x+ty) \\ &\quad \ge f((1-s)x+sy) - f((1-t)(1-s)x+[(1-t)s+t]y). \end{aligned} \tag{2.6}$$

Finally, by (2.4) and (2.6), we get the desired result (2.2).

Applying repeatedly the superadditivity property, we have for $[z,u] \subset [x,y]$ that

$$\Psi_f(x,z;t) + \Psi_f(z,u;t) + \Psi_f(u,y;t) \le \Psi_f(x,y;t)$$

giving that

$$0 \le \Psi_f(x,z;t) + \Psi_f(u,y;t) \le \Psi_f(x,y;t) - \Psi_f(z,u;t)$$

which proves (2.3). $\square$

For $t = 1/2$, we consider the functional

$$\Psi_f(x,y) := \Psi_f\left(x,y;\frac{1}{2}\right) = \frac{f(x)+f(y)}{2} - f\left(\frac{x+y}{2}\right),$$

which obviously inherits the superadditivity and monotonicity properties of the functional $\Psi_f(\cdot,\cdot;t)$. We are then able to state the following.

Corollary 1. *Let $f : C \subset X \to \mathbf{R}$ be a convex function on the convex set C and $x, y \in C$. Then we have the bounds*

$$\inf_{z\in[x,y]}\left[f\left(\frac{x+z}{2}\right) + f\left(\frac{z+y}{2}\right) - f(z)\right] = f\left(\frac{x+y}{2}\right) \tag{2.7}$$

and

$$\sup_{z,u\in[x,y]}\left[\frac{f(z)+f(u)}{2} - f\left(\frac{z+u}{2}\right)\right] = \frac{f(x)+f(y)}{2} - f\left(\frac{x+y}{2}\right). \tag{2.8}$$

Proof. By the superadditivity of the functional $\Psi_f(\cdot,\cdot)$, we have for each $z \in [x,y]$ that

$$\frac{f(x)+f(y)}{2} - f\left(\frac{x+y}{2}\right) \ge \frac{f(x)+f(z)}{2} - f\left(\frac{x+z}{2}\right) + \frac{f(z)+f(y)}{2} - f\left(\frac{z+y}{2}\right)$$

which is equivalent to

$$f\left(\frac{x+z}{2}\right)+f\left(\frac{z+y}{2}\right)-f(z)\geq f\left(\frac{x+y}{2}\right). \tag{2.9}$$

Since the equality case in (2.9) is realized for either $z=x$ or $z=y$, we get the desired bound (2.7).

The bound (2.8) is obvious by the monotonicity of the functional $\Psi_f(\cdot,\cdot)$ as a function of the interval. □

Now consider the following functional

$$\Gamma_f(x,y;t):=f(x)+f(y)-f((1-t)x+ty)-f((1-t)y+tx),$$

where, as above, $f:C\subset X\to\mathbf{R}$ is a convex function on the convex set C and $x,y\in C$ while $t\in[0,1]$.

We notice that

$$\Gamma_f(x,y;t)=\Gamma_f(y,x;t)=\Gamma_f(x,y;1-t)$$

and

$$\Gamma_f(x,y;t)=\Psi_f(x,y;t)+\Psi_f(x,y;1-t)\geq 0$$

for any $x,y\in C$ and $t\in[0,1]$.

Therefore, we can state the following result as well:

Corollary 2. *Let $f:C\subset X\to\mathbf{R}$ be a convex function on the convex set C and $t\in[0,1]$. The functional $\Gamma_f(\cdot,\cdot;t)$ is superadditive and monotonic nondecreasing as a function of the interval.*

In particular, if $z\in[x,y]$, then we have the inequality

$$\begin{aligned}\frac{1}{2}&[f((1-t)x+ty)+f((1-t)y+tx)]\\&\leq\frac{1}{2}[f((1-t)x+tz)+f((1-t)z+tx)]\\&\quad+\frac{1}{2}[f((1-t)z+ty)+f((1-t)y+tz)]-f(z).\end{aligned} \tag{2.10}$$

Also, if $z, u \in [x, y]$, then we have the inequality

$$\begin{aligned} f(x) + f(y) - f((1-t)x + ty) - f((1-t)y + tx) \\ \geq f(z) + f(u) - f((1-t)z + tu) \\ - f((1-t)z + tu) \end{aligned} \tag{2.11}$$

for any $t \in [0, 1]$.

Perhaps the most interesting functional we can consider from the above is the following one:

$$\Theta_f(x, y) := \frac{f(x) + f(y)}{2} - \int_0^1 f((1-t)x + ty)\, dt \geq 0, \tag{2.12}$$

which is related to the second Hermite-Hadamard inequality.

We observe that

$$\Theta_f(x, y) = \int_0^1 \Psi_f(x, y; t)\, dt = \int_0^1 \Psi_f(x, y; 1-t)\, dt. \tag{2.13}$$

Utilizing this representation, we can state the following result as well:

Corollary 3. *Let $f : C \subset X \to \mathbf{R}$ be a convex function on the convex set C and $t \in [0, 1]$. The functional $\Theta_f(\cdot, \cdot)$ is superadditive and monotonically nondecreasing as a function of the interval. Moreover, we have the bounds*

$$\begin{aligned} \inf_{z \in [x,y]} \left[\int_0^1 [f((1-t)x + tz) + f((1-t)z + ty)]\, dt - f(z) \right] \\ = \int_0^1 f((1-t)x + ty)\, dt \end{aligned} \tag{2.14}$$

and

$$\begin{aligned} \sup_{z,u \in [x,y]} \left[\frac{f(z) + f(u)}{2} - \int_0^1 f((1-t)z + tu)\, dt \right] \\ = \frac{f(x) + f(y)}{2} - \int_0^1 f((1-t)x + ty)\, dt. \end{aligned} \tag{2.15}$$

For other functionals associated with the Hermite-Hadamard inequality for functions of a real variable, see the paper [**4**].

3. Applications for norms. Let $(X, \|\cdot\|)$ be a normed space and x, y two distinct vectors in X. Then, for any $p \geq 1$, the function $f : X \to [0, \infty)$, $f(x) = \|x\|^p$ is convex and, utilizing the results from the above section, we can state the following norm inequalities:

$$\inf_{z\in[x,y]} \left[\left\| \frac{x+z}{2} \right\|^p + \left\| \frac{z+y}{2} \right\|^p - \|z\|^p \right] = \left\| \frac{x+y}{2} \right\|^p, \tag{3.1}$$

and

$$\sup_{z,u\in[x,y]} \left[\frac{\|z\|^p + \|u\|^p}{2} - \left\| \frac{z+u}{2} \right\|^p \right] = \frac{\|x\|^p + \|y\|^p}{2} - \left\| \frac{x+y}{2} \right\|^p. \tag{3.2}$$

Moreover, we can state the following results as well:

$$\begin{aligned} \frac{1}{2} &[\|(1-t)\,x + ty\|^p + \|(1-t)\,y + tx\|^p] \\ &\leq \frac{1}{2} [\|(1-t)\,x + tz\|^p + \|(1-t)\,z + tx\|^p] \\ &\quad + \frac{1}{2} [\|(1-t)\,z + ty\|^p + \|(1-t)\,y + tz\|^p] - \|z\|^p \end{aligned} \tag{3.3}$$

for any $z \in [x, y]$ and $t \in [0, 1]$, and

$$\begin{aligned} \|x\|^p + \|y\|^p - \|(1-t)\,x + ty\|^p - \|(1-t)\,y + tx\|^p &\geq \|z\|^p + \|u\|^p \\ &\quad - \|(1-t)\,z + tu\|^p - \|(1-t)\,z + tu\|^p \end{aligned} \tag{3.4}$$

for any $z, u \in [x, y]$ and $t \in [0, 1]$.

In [**5**] Kikianty and Dragomir introduced the concept of the p-HH-norm as $\|\cdot\|_{p-\mathrm{HH}} : X \times X \to [0, \infty)$ with

$$\|(x, y)\|_{p-\mathrm{HH}} := \left(\int_0^1 \|(1-t)\,x + ty\|^p \, dt \right)^{1/p}, \quad p \geq 1$$

and studied its various properties.

From the integral inequalities established in the above section, we can deduce the following results for the p-HH-norm of two distinct vectors x, y in the normed linear space $(X, \|\cdot\|)$:

$$\inf_{z\in[x,y]}\left[\|(x,z)\|_{p-\mathrm{HH}}^{p}+\|(z,y)\|_{p-\mathrm{HH}}^{p}-\|(z,z)\|_{p-\mathrm{HH}}^{p}\right] = \|(x,y)\|_{p-\mathrm{HH}}^{p} \tag{3.5}$$

and

$$\sup_{z,u\in[x,y]}\left[\frac{\|(z,z)\|_{p-\mathrm{HH}}^{p}+\|(u,u)\|_{p-\mathrm{HH}}^{p}}{2}-\|(z,u)\|_{p-\mathrm{HH}}^{p}\right] = \frac{\|(x,x)\|_{p-\mathrm{HH}}^{p}+\|(y,y)\|_{p-\mathrm{HH}}^{p}}{2}-\|(x,y)\|_{p-\mathrm{HH}}^{p}. \tag{3.6}$$

4. Applications for convex functions of a real variable. Let $f : I \to \mathbf{R}$ be a convex function on the interval $I \subset \mathbf{R}$ and $x, y \in I$ with $x < y$. Due to the obvious fact that

$$\int_0^1 f\left((1-t)\,x+ty\right)\,dt=\frac{1}{y-x}\int_x^y f\left(s\right)\,ds,$$

the functional

$$\Theta_f\left(x,y\right):=\frac{f\left(x\right)+f\left(y\right)}{2}-\frac{1}{y-x}\int_x^y f\left(s\right)\,ds$$

is *superadditive* and *monotonic nondecreasing* as a function of interval. We also have the inequalities

$$\inf_{z\in[x,y]}\left[\frac{1}{z-x}\int_x^z f\left(s\right)\,ds+\frac{1}{y-z}\int_z^y f\left(s\right)\,ds-f\left(z\right)\right] = \frac{1}{y-x}\int_x^y f\left(s\right)\,ds \tag{4.1}$$

and

$$\sup_{z,u\in[x,y]}\left[\frac{f\left(z\right)+f\left(u\right)}{2}-\frac{1}{z-u}\int_u^z f\left(s\right)\,ds\right] = \frac{f\left(x\right)+f\left(y\right)}{2}-\frac{1}{y-x}\int_x^y f\left(s\right)\,ds. \tag{4.2}$$

The above inequalities may be used to obtain some interesting results for means.

For $0 < x \leq y < \infty$ and $t \in (0,1)$, consider the weighted arithmetic, geometric and harmonic means defined by

$$A_t(x,y) := (1-t)x + ty, \ G_t(x,y) := x^{1-t}y^t$$

and

$$H_t(x,y) := \frac{1}{(1-t)/x + t/y}.$$

For $t = 1/2$, we simply write $A(x,y), G(x,y)$ and $H(x,y)$.

It is well known that the following inequality holds:

$$A_t(x,y) \geq G_t(x,y) \geq H_t(x,y).$$

1. Consider the convex function $f : (0,\infty) \to (0,\infty)$, $f(s) = s^{-1}$. Then for $0 < x \leq y < \infty$ and $t \in (0,1)$, we have

$$\begin{aligned} \Psi_{(\cdot)^{-1}}(x,y;t) &= (1-t)x^{-1} + ty^{-1} - [(1-t)x + ty]^{-1} \\ &= H_t^{-1}(x,y) - A_t^{-1}(x,y) \\ &= \frac{A_t(x,y) - H_t(x,y)}{A_t(x,y)H_t(x,y)}. \end{aligned} \tag{4.3}$$

Making use of Theorem 1, we have for $0 < x \leq z \leq y < \infty$ and $t \in (0,1)$ that

$$\begin{aligned} (0 \leq) \frac{A_t(x,z) - H_t(x,z)}{A_t(x,z)H_t(x,z)} + \frac{A_t(z,y) - H_t(z,y)}{A_t(z,y)H_t(z,y)} \\ \leq \frac{A_t(x,y) - H_t(x,y)}{A_t(x,y)H_t(x,y)} \end{aligned} \tag{4.4}$$

and, in particular,

$$\begin{aligned} (0 \leq) \frac{A(x,z) - H(x,z)}{A(x,z)H(x,z)} + \frac{A(z,y) - H(z,y)}{A(z,y)H(z,y)} \\ \leq \frac{A(x,y) - H(x,y)}{A(x,y)H(x,y)}, \end{aligned} \tag{4.5}$$

and for $0 < x \le z \le u \le y < \infty$ and $t \in (0,1)$, that

$$(4.6) \qquad (0 \le) \frac{A_t(z,u) - H_t(z,u)}{A_t(z,u) H_t(z,u)} \le \frac{A_t(x,y) - H_t(x,y)}{A_t(x,y) H_t(x,y)}$$

and, in particular,

$$(4.7) \qquad (0 \le) \frac{A(z,u) - H(z,u)}{A(z,u) H(z,u)} \le \frac{A(x,y) - H(x,y)}{A(x,y) H(x,y)}.$$

Now, if we consider the *logarithmic mean* of two positive numbers x, y defined as

$$L(x,y) := \begin{cases} (y-x)/(\ln y - \ln x) & \text{if } x \neq y \\ x & \text{if } x = y \end{cases}$$

then

$$(4.8) \qquad \begin{aligned} \Theta_{(\cdot)^{-1}}(x,y) &:= \frac{x^{-1} + y^{-1}}{2} - \frac{1}{y-x} \int_x^y s^{-1} ds \\ &= H^{-1}(x,y) - L^{-1}(x,y) \\ &= \frac{L(x,y) - H(x,y)}{L(x,y) H(x,y)}. \end{aligned}$$

Making use of Corollary 3, we have for $0 < x \le z \le y < \infty$, that

$$(4.9) \qquad \begin{aligned} (0 \le) \frac{L(x,z) - H(x,z)}{L(x,z) H(x,z)} &+ \frac{L(z,y) - H(z,y)}{L(z,y) H(z,y)} \\ &\le \frac{L(x,y) - H(x,y)}{L(x,y) H(x,y)}, \end{aligned}$$

and for $0 < x \le z \le u \le y < \infty$, that

$$(4.10) \qquad (0 \le) \frac{L(z,u) - H(z,u)}{L(z,u) H(z,u)} \le \frac{L(x,y) - H(x,y)}{L(x,y) H(x,y)}.$$

2. Consider the convex function $f : (0,\infty) \to (0,\infty)$, $f(s) = -\ln s$. Then, for $0 < x \le y < \infty$ and $t \in (0,1)$, we have

$$\begin{aligned} \Psi_{-\ln}(x,y;t) &= \ln[(1-t)x + ty] - (1-t)\ln x - t \ln y \\ &= \ln \left[\frac{A_t(x,y)}{G_t(x,y)} \right]. \end{aligned}$$

Making use of Theorem 1, we have for $0 < x \leq z \leq y < \infty$ and $t \in (0,1)$ that

$$(1 \leq) \frac{A_t(x,z)}{G_t(x,z)} \cdot \frac{A_t(z,y)}{G_t(z,y)} \leq \frac{A_t(x,y)}{G_t(x,y)} \tag{4.11}$$

and, in particular,

$$(1 \leq) \frac{A(x,z)}{G(x,z)} \cdot \frac{A(z,y)}{G(z,y)} \leq \frac{A(x,y)}{G(x,y)}, \tag{4.12}$$

and for $0 < x \leq z \leq u \leq y < \infty$ and $t \in (0,1)$ that

$$(1 \leq) \frac{A_t(z,u)}{G_t(z,u)} \leq \frac{A_t(x,y)}{G_t(x,y)} \tag{4.13}$$

and, in particular,

$$(1 \leq) \frac{A_t(z,u)}{G_t(z,u)} \leq \frac{A_t(x,y)}{G_t(x,y)}. \tag{4.14}$$

Now, if we consider the *identric mean* of two positive numbers x, y defined as

$$I(x,y) := \begin{cases} (1/e) \cdot (y^y/x^x)^{1/(y-x)} & \text{if } x \neq y, \\ x & \text{if } x = y, \end{cases}$$

then

$$\Theta_{-\ln}(x,y) := \frac{1}{y-x} \int_x^y \ln s \, ds - \frac{\ln x + \ln y}{2} = \ln \left[\frac{I(x,y)}{G(x,y)} \right].$$

Making use of Corollary 3, we have for $0 < x \leq z \leq y < \infty$, that

$$(1 \leq) \frac{I(x,z)}{G(x,z)} \cdot \frac{I(z,y)}{G(z,y)} \leq \frac{I(x,y)}{G(x,y)}, \tag{4.15}$$

and for $0 < x \leq z \leq u \leq y < \infty$, that

$$(1 \leq) \frac{I(z,u)}{G(z,u)} \leq \frac{I(x,y)}{G(x,y)}. \tag{4.16}$$

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