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# RECONSTRUCTION OF THE STURM-LIOUVILLE OPERATOR ON A $p$-STAR GRAPH WITH NODAL DATA

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ABSTRACT. In this paper, we deal with the inverse problem of reconstructing the Sturm-Liouville operator defined on a $p$-star graph with a set of nodal data plus $p$ constant. Our reconstruct formulas are direct and explicit. We also derive the eigenvalue asymptotics in this paper.

**1. Introduction.** Recently there has been a lot of interest in the study of Sturm-Liouville on graphs (see, e.g., [**1, 4–6, 9, 14, 15**]). On one hand, the problem is a natural extension of the classical Sturm-Liouville problem on an interval. On the other hand, it has a number of applications in networks, spider webs, interlocking springs and even nanostructures. Kuchment called this Sturm-Liouville problem defined on *quantum graphs* [**6–8**]. The study of quantum graphs has a lot of potential applications. In [**9**], Kuchment and Post studied the spectral properties of periodic boundary value problems for the carbon atom in graphite.

In two papers [**14, 15**], Pivovarchik proved an inverse spectral problem with Dirichlet boundary conditions for $p$-star graphs. In short, he showed that, under certain conditions on the asymptotics of $p+1$ sequences of eigenvalues (in fact, one set for each of the $p$ edges and an extra set for the overall eigenvalue problem) are associated with an overall potential function $\vec{q} = (q_1, q_2, \dots, q_p)$. In the course he gave the asymptotic expansion of eigenvalues and showed that there are $p$ sequences of eigenvalues where one sequence is simple while the others might not be.

The inverse nodal problem is that of understanding the potential function using information of the nodal data. On the finite interval,

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issues of uniqueness ([**13, 16**]), reconstruction formula ([**10, 12, 16**]) and stability ([**3, 11**]) are well understood. The following theorem is the reconstruction formula derived by Law, Shen and Yang in 1999 [**10**].

**Theorem 1.1.** *Consider the Sturm-Liouville equation defined on the interval* $[0,1]$ *with separated boundary conditions*

$$\begin{aligned} &y'' + q(x)y = \lambda y, \\ &y(0)\cos\alpha + y'(0)\sin\alpha = 0, \\ &y(1)\cos\beta + y'(1)\sin\beta = 0, \end{aligned}$$

*where* $q \in L^1(0,1)$ *and* $\alpha, \beta \in [0,\pi)$*. Let* $\lambda_n$ *be the nth eigenvalue and* $0 = x_0^n < x_1^n < x_2^n < \cdots < x_{n-1}^n < x_n^n = 1$ *the nodal points of the nth eigenfunction. Then, letting* $l_k^n = x_{k+1}^n - x_k^n$ *and* $j = j_n(x) = \max\{k : x \in [x_k^n, x_{k+1}^n]\}$,

1. *the reconstruction formula*

$$q(x) - \int_0^1 q = \lim_{n\to\infty} 2\lambda_n\left(\frac{\sqrt{\lambda_n}\, l_j^n}{\pi} - 1\right),$$

*converges pointwisely almost everywhere in* $(0,1)$ *and in* $L^1$.

2. *Either* $\alpha = 0$ *or, with* $k \in \mathbf{N}$ *fixed,*

$$\cot\alpha = \lim_{n\to\infty} \sqrt{\lambda_n}\left(\left(k - \frac{1}{2}\right)\pi - \sqrt{\lambda_n}\, x_k^n\right);$$

*either* $\beta = 0$ *or, with* $k \in \mathbf{N}$ *fixed,*

$$\cot\beta = \lim_{n\to\infty} \sqrt{\lambda_n}\left(\sqrt{\lambda_n}(1 - x_{n-k}^n) - \left(k - \frac{1}{2}\right)\pi\right).$$

Although the eigenvalue $\lambda_n$ appears in the formula, it can be substituted by eigenvalue asymptotics, and hence this formula depends only upon the nodal data.

Consider the Sturm-Liouville problem on a $p$-star graph, with each edge of length 1, defined as the following:

$$\begin{aligned}
&-y_i'' + q_i(x)y_i = \lambda y_i, \quad i = 1, 2, \dots, p, \\
&y_i(0,\lambda)\cos\alpha_i + y_i'(0,\lambda)\sin\alpha_i = 0, \quad i = 1, 2, \dots, p, \\
&y_1(1,\lambda) = y_2(1,\lambda) = \cdots = y_p(1,\lambda), \\
&\sum_{i=1}^{p} y_i'(1,\lambda) = 0,
\end{aligned}$$

where $\alpha_i \in [0,\pi)$ and $q_i$ are real functions in $L^1(0,1)$, $i = 1, 2, \dots, p$. We shall reconstruct the potentials $q$'s and the boundary conditions $\alpha_i's$ using only nodal data. Our reconstruction formula is direct, and it automatically implies the uniqueness of this inverse problem. We remark that Currie and Watson [**4**] also studied the inverse nodal problems on general graphs. They showed that, for $q_i \in L^\infty$, a set of eigenvalues and nodal positions is sufficient to reconstruct the potentials $q_i$'s. In this paper, we shall only use the nodal positions for the reconstruction. The difficulty of this study is to have a detailed asymptotic expansion of eigenvalues and nodal points. The situation gets more complex as the multiplicity of each eigenvalue may exceed 1. We shall extend the method of Pivovarchik to get a more refined asymptotic expression of eigenvalues and deal with different boundary conditions.

On each $i$th edge of the graph, problem (1)–(4) is reduced to a scalar Sturm-Liouville system. Define by $0 = x_0^{n,i} < x_1^{n,i} < x_2^{n,i} < \cdots < x_{n-1}^{n,i} < x_n^{n,i} = 1$ the $k$th zero of the eigenfunction on the $i$th edge of the graph, associated with the eigenvalue $\lambda_n$. The main theorem is the following.

**Theorem 1.2.** *Let* $\{x_k^{n,i}\}$ *be a nodal set of system* (1)–(4). *Denote by* $l_k^{n,i} \equiv x_{k+1}^{n,i} - x_k^{n,i}$ *the nodal length and* $j = j(x) \equiv \max\{k : x_k^{n,i} \le x < x_{k+1}^{n,i}\}$. *Assume* $|C| \ne 0, (\pi/2)$ *and* $\pi$. *In each of the following cases,* $F_i^n(x)$ *and* $\Theta_i^n$ *are defined in terms of the estimate of* $l_k^{n,i}$.

(a) *If* $l_k^{n,i} = (1/n) - (C/n^2\pi) + o(1/n^2)$, *then define*

$$\Theta_i^n = n\pi^2\left[k - \frac{1}{2} - \frac{k-1/2}{n\pi}C - nx_k^{n,i}\right],$$

$$F_i^n(x) = 2n^2\pi^2(nl_j^{n,i} - 1) - 2n^2\pi^2 \sum_{k=0}^{n-1} \left(l_k^{n,i} - \frac{1}{n}\right).$$

(b) *If* $l_k^{n,i} = (1/n) + o(1/n^2)$, *then define*

$$\Theta_i^n = n\pi^2 \left[k - \frac{1}{2} - nx_k^{n,i}\right],$$

$$F_i^n(x) = 2n^2\pi^2(nl_j^{n,i} - 1) - 2n^2\pi^2 \sum_{k=0}^{n-1} \left(l_k^{n,i} - \frac{1}{n}\right).$$

(c) *If* $l_k^{n,i} = [1/(n-(1/2))] + o(1/n^2)$, *then define*

$$\Theta_i^n = \left(n - \frac{1}{2}\right)\pi^2 \left[k - \frac{1}{2} - \left(n - \frac{1}{2}\right)x_k^{n,i}\right],$$

$$F_i^n(x) = 2\left(n - \frac{1}{2}\right)^2 \pi^2 \left(\left(n - \frac{1}{2}\right)l_k^{n,i} - 1\right) - 2\left(n - \frac{1}{2}\right)^2 \pi^2 \sum_{k=0}^{n-1} \left(l_k^{n,i} - \frac{1}{n-(1/2)}\right).$$

(d) *If* $l_k^{n,i} = [1/(n-1)] + o(1/n^2)$, *then define*

$$\Theta_i^n = (n-1)\pi^2 \left[k - \frac{1}{2} - (n-1)x_k^{n,i}\right],$$

$$F_i^n(x) = 2(n-1)^2\pi^2((n-1)l_j^{n,i} - 1) - 2(n-1)^2\pi^2 \sum_{k=0}^{n-1} \left(l_k^{n,i} - \frac{1}{n-1}\right).$$

*Then either* $\cot\alpha_i = \lim_{n\to\infty} \Theta_i^n$ *if the limit exists, or* $\alpha_i = 0$, *and* $F_i^n$ *converges to* $q_i - \int_0^1 q_i(t)\,dt$ *pointwisely almost everywhere in* $(0,1)$ *and in* $L^1$ *for* $i = 1, 2, \ldots, p$.

We remark that this reconstruction result for the Sturm-Liouville operator is complete in the sense that they cover both boundary conditions and potential functions. All reconstruction formulas are direct and explicit.

In Section 2, we consider the direct problem. A sharper estimate of eigenvalues is given. In Section 3, we reconstruct the boundary conditions and potential functions using the nodal points and obtain our main theorem. In Section 4, we derive the asymptotic expansions of eigenvalues.

**2. Direct problem.** In this section, we deal with the direct problem. We consider Sturm-Liouville operators on a $p$-star graph with general boundary conditions and obtain sharp estimates of eigenvalues.

First, by the argument in [**2**], eigenvalues of the system (1)–(4) are all discrete and real for $q_i \in L^1(0,1)$, $i = 1, 2, \dots, p$. Now, for every fixed $i = 1, 2, \dots, p$, let $u_i(x,\lambda)$ be the solution of the following initial value problem:

$$\begin{cases} -y_i'' + q_i(x) y_i = \lambda y_i, \\ y_i(0) = \sin\alpha_i, \quad y_i'(0) = -\cos\alpha_i. \end{cases}$$

Then $u_i(x,\lambda)$ satisfies the following integral equation

$$\begin{aligned} u_i(x,\lambda) = \sin\alpha_i \cos(\sqrt{\lambda}x) - \frac{\cos\alpha_i}{\sqrt{\lambda}} \sin(\sqrt{\lambda}x) \\ + \int_0^x \frac{\sin(\sqrt{\lambda}(x-t))}{\sqrt{\lambda}} q_i(t) u_i(t,\lambda)\, dt. \end{aligned} \tag{5}$$

If $y_i(x,\lambda)$ is an eigenfunction of system (1)–(4), then there is a nonzero real number $c_i$ such that $y_i(x,\lambda) = c_i u_i(x,\lambda)$. Moreover, $\lambda$ is an eigenvalue of (1)–(4) if and only if $\lambda$ is a zero of

$$\begin{aligned} \Phi(\lambda) \equiv &\begin{vmatrix} u_1(1,\lambda) & -u_2(1,\lambda) & 0 & 0 & \cdots & 0 \\ u_1(1,\lambda) & 0 & -u_3(1,\lambda) & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ u_1(1,\lambda) & 0 & \cdots & \cdots & \cdots & -u_p(1,\lambda) \\ u_1'(1,\lambda) & u_2'(1,\lambda) & \cdots & \cdots & \cdots & u_p'(1,\lambda) \end{vmatrix}, \\ = &\sum_{i=1}^{p} u_i'(1,\lambda) \prod_{\nu \neq i} u_\nu(1,\lambda). \end{aligned} \tag{6}$$

Hence, we can analyze function $\Phi(\lambda)$ to obtain the eigenvalue asymptotics.

In the following, we denote $A_1 = \prod_{i=1}^p \sin\alpha_i$, $A_2 = \sum_{i=1}^p \cot\alpha_i$ when $\alpha_i \neq 0$ for all $i = 1, 2, \dots, p$, and $A_3 = \sum_{j=T+1}^p \cot\alpha_j$ when $\alpha_j \neq 0$ for all $j = T+1, T+2, \dots, p$ where $1 \leq T \leq p-1$.

**Theorem 2.1.** I. *When each $\alpha_i = 0$, there are $p$ sequences of eigenvalues $\{\lambda_{n,\nu}\}$ $(\nu = 1, \dots, p)$ with asymptotics*

Ia
$$\sqrt{\lambda_{n,1}} = \left(n - \frac{1}{2}\right)\pi + \frac{1}{2p(n-(1/2))\pi} \times \int_0^1 (1-\cos((2n-1)\pi t))\left(\sum_{\nu=1}^p q_\nu(t)\right) dt + O\left(\frac{1}{n^2}\right),$$

Ib
$$\sqrt{\lambda_{n,\nu}} = n\pi + \frac{\Lambda_{n,\nu}}{n\pi} + O\left(\frac{1}{n^2}\right), \quad \nu = 2, 3, \dots, p.$$

*where $\Lambda_{n,\nu}$ is the $\nu$th root of the polynomial equation of degree $(p-1)$*

$$\sum_{i=1}^p \left[\prod_{\nu\neq i}\left(\Lambda - \frac{1}{2}\int_0^1 (1-\cos(2(n-1)\pi t))q_\nu(t)\,dt\right)\right] = 0. \tag{7}$$

II. *When each $\alpha_i \neq 0$, there are $p$ sequences of eigenvalues $\{\lambda_{n,\nu}\}$ $(\nu = 1, \dots, p)$ with asymptotics*

IIa
$$\sqrt{\lambda_{n,1}} = (n-1)\pi + \frac{1}{(n-1)p\pi}\left(-A_2 + \frac{1}{2}\int_0^1 (1+\cos(2(n-1)t)) \times \left(\sum_{\nu=1}^p q_\nu(t)\right) dt\right) + O\left(\frac{1}{n^2}\right),$$

IIb
$$\sqrt{\lambda_{n,\nu}} = \left(n - \frac{1}{2}\right)\pi - \frac{\Lambda_{n,\nu}}{(n-(1/2))\pi} + O\left(\frac{1}{n^2}\right), \quad \nu = 2, 3, \dots, p.$$

*where $\Lambda_{n,\nu}$ is the $\nu$th root of the polynomial equation of degree $(p-1)$*

$$\sum_{i=1}^p \left[\prod_{\nu\neq i}(\Lambda - \cot\alpha_\nu + \frac{1}{2}\int_0^1 (1+\cos((2n-1)\pi t))q_\nu(t)\,dt)\right] = 0. \tag{8}$$

III. *When* $\alpha_i = 0$, $i = 1, 2, \ldots, T$ *and* $\alpha_i \neq 0$, $i = T+1, T+2, \ldots, p$ *where* $1 \leq T \leq p-1$, *there are* $p$ *sequences of eigenvalues* $\{\lambda_{n,\nu}\}$ $(\nu = 1, \ldots, p)$, *with asymptotics*

IIIa

$$\sqrt{\lambda_{n,\nu}} = n\pi + (-1)^{\nu} \sin^{-1} \sqrt{\frac{T}{p}} + \delta_n + o\left(\frac{1}{n}\right), \quad \nu = 1, 2,$$

IIIb

$$\sqrt{\lambda_{n,\nu}} = n\pi + \frac{\Lambda^0_{n,\nu}}{n\pi} + O\left(\frac{1}{n^2}\right), \quad \nu = 3, 4, \ldots, T+1,$$

IIIc

$$\sqrt{\lambda_{n,\nu}} = \left(n - \frac{1}{2}\right)\pi - \frac{\Lambda^1_{n,\nu}}{(n-(1/2))\pi} + O\left(\frac{1}{n^2}\right), \\ \nu = T+2, T+3, \ldots, p,$$

*where*

$$\delta_n = -\frac{1}{4pn\pi}\left(2TA_3 - (p-T)\int_0^1 \left(\sum_{\nu=1}^{T} q_\nu(t)\right) dt \\ - T\int_0^1 \left(\sum_{\nu=T+1}^{p} q_\nu(t)\right) dt\right),$$

$\Lambda^0_{n,\nu}$ *is the* $\nu$*th root of the polynomial equation of degree* $(T-1)$

$$\sum_{i=1}^{T}\left[\prod_{\substack{\nu \neq i \\ \nu=1}}^{T}\left(\Lambda - \frac{1}{2}\int_0^1 (1-\cos(2n\pi t))q_\nu(t)\right)\right] = 0, \tag{9}$$

*and* $\Lambda^1_{n,\nu}$ *is the* $\nu$*th root of the polynomial equation of degree* $(p-T-1)$

$$\sum_{i=T+1}^{p}\left[\prod_{\substack{\nu \neq i \\ \nu=T+1}}^{p}\left(\Lambda - \cot\alpha_\nu + \frac{1}{2} \\ \times \int_0^1 (1+\cos((2n-1)\pi t)q_\nu(t))\, dt\right)\right] = 0. \tag{10}$$

So there are three cases dependent upon the boundary conditions. In each case, all the eigenvalues can be grouped into $p$ sequences ($\lambda_{n,\nu}$, $\nu = 1, 2, \dots, p$) which might have distinct asymptotic behavior. The proof of Theorem 2.1 is tedious and will be delayed to Section 4.

Now, we denote by $x_{k,\nu}^{n,i}$, $k = 1, 2, \dots, n-1$, the $k$th zero of the $n$th eigenfunction corresponding to the eigenvalue $\lambda_{n,\nu}$ at the $i$th branch of the graph. Referring to [**10**], one may obtain the asymptotic expansion of the nodal points, with $s_{n,\nu} = \sqrt{\lambda_{n,\nu}}$:

I. If $\alpha_i = 0$, then as $n \to \infty$,

$$x_{k,\nu}^{n,i} = \frac{k\pi}{s_{n,\nu}} + \frac{1}{2s_{n,\nu}^2}\int_0^{x_{k,\nu}^{n,i}} (1 - \cos(2s_{n,\nu}t))q_i(t)\,dt + o\left(\frac{1}{s_{n,\nu}^3}\right). \tag{11}$$

II. If $\alpha_i \neq 0$, then as $n \to \infty$,

$$\begin{aligned} x_{k,\nu}^{n,i} &= \frac{(k-(1/2))\pi}{s_{n,\nu}} - \frac{1}{s_{n,\nu}^2}\cot\alpha_i \\ &\quad + \frac{1}{2s_{n,\nu}^2}\int_0^{x_{k,\nu}^{n,i}} (1 + \cos(2s_{n,\nu}t))q_i(t)\,dt + o\left(\frac{1}{s_{n,\nu}^3}\right). \end{aligned} \tag{12}$$

The above estimates are uniform for $k = 1, 2, \dots, n-1$. Therefore, in both cases, the nodal length $l_{k,\nu}^{n,i}$ is given by

$$l_{k,\nu}^{n,i} = \frac{\pi}{s_{n,\nu}} + \frac{1}{2s_{n,\nu}^2}\int_{x_{k+1,\nu}^{n,i}}^{x_{k,\nu}^{n,i}} (1 + \cos(2s_{n,\nu}t))q_i(t)\,dt + o\left(\frac{1}{s_{n,\nu}^3}\right). \tag{13}$$

## 3. Inverse problem.

*Proof of Theorem* 1.2. For the inverse nodal problem, if we have a set of nodal points $\{x_k^{n,i}\}$ for the $i$th branch of the $p$-star graph, we shall classify the nodal points and then use them to build up the reconstruction formulas.

First, by the first terms in the estimate of eigenvalues, the eigenvalues can be classified into four cases, and in each case we have a different estimate of nodal length:

$$l_k^{n,i} = \begin{cases} (A)\ \frac{1}{n-(1/2)} + o(\frac{1}{n^2}) & \text{if } \sqrt{\lambda_n} = (n-(1/2))\pi + O(1/n), \\ (B)\ \frac{1}{n} + o(1/n^2) & \text{if } \sqrt{\lambda_n} = n\pi + O(1/n), \\ (C)\ \frac{1}{n-1} + o(1/n^2) & \text{if } \sqrt{\lambda_n} = (n-1)\pi + O(1/n), \\ (D)\ \frac{1}{n} - \frac{C}{n^2\pi} + o(1/n^2) & \text{if } \sqrt{\lambda_n} = n\pi + C + O(1/n), \end{cases}$$

where $C$ is a constant such that $|C| \neq 0$, $\pi/2$, $\pi$.

For case (A), $l_k^{n,i} = [1/(n-(1/2))] + o(1/n^2)$, there are three possible asymptotic expansions for the corresponding eigenvalues: (Ia), (IIb) and (IIIc) as in Theorem 2.1. If (Ia) is valid, then $\alpha_i = 0$ for $i = 1, 2, \dots, p$. Also,

$$\sqrt{\lambda_n} = \left(n - \frac{1}{2}\right)\pi + o\left(\frac{1}{n}\right).$$

Furthermore, from the reconstruction formula of $q$ in Theorem 1.1 and (13), we obtain

$$\begin{aligned} F_n(x) &= 2\left(n - \frac{1}{2}\right)^2 \pi^2 \left(\left(n - \frac{1}{2}\right) l_k^{n,i} - 1\right) \\ &\quad - 2\left(n - \frac{1}{2}\right)^2 \pi^2 \sum_{k=0}^{n-1} \left(l_k^{n,i} - \frac{1}{n-(1/2)}\right) \end{aligned}$$

converges to $q_i - \int_0^1 q_i(t)\,dt$ pointwisely almost everywhere in $(0,1)$ and in $L^1$. If (IIb) or (IIIc) is valid, then for some constant $\Lambda_n$,

$$\sqrt{\lambda_n} = \left(n - \frac{1}{2}\right)\pi - \frac{\Lambda_n}{(n-(1/2))\pi} + o\left(\frac{1}{n}\right). \tag{14}$$

By the reconstruction formula of the boundary conditions in Theorem 1.1,

$$\cot\alpha_i = \lim_{n\to\infty} \left(n - \frac{1}{2}\right)\pi\left(\left(k - \frac{1}{2}\right)\pi - \left(n - \frac{1}{2}\right)\pi x_k^{n,i}\right)$$

if the limit exists. Otherwise $\alpha_i = 0$. Also, from (13) and (14),

$$\begin{aligned} l_k^{n,i} &= \frac{1}{n-(1/2)} + \frac{1}{2(n-(1/2))^2\pi^2} \\ &\quad \times \int_{x_k^{n,i}}^{x_{k+1}^{n,i}} (1+\gamma_i \cos((2n-1)\pi t)) q_i(t)\, dt \\ &\quad + \frac{\Lambda_n}{(n-(1/2))^3\pi^2} + o\left(\frac{1}{n^3}\right). \end{aligned}$$

Since $q \in L^1(0,1)$, we have

$$\begin{aligned} \Lambda_n &= \frac{(n-(1/2))^3\pi^2}{n} \sum_{k=0}^{n-1} \left( l_k^{n,i} - \frac{1}{n-(1/2)} \right) \\ &\quad - \frac{1}{2} \int_0^1 q_i(t)\, dt + o(1). \end{aligned}$$

Hence, by Theorem 1.1 and the estimates of nodal length,

$$\begin{aligned} F_i^n &= 2\left(n-\frac{1}{2}\right)^2 \pi^2 \left( \left(n - \frac{1}{2}\right) l_k^{n,i} - 1 \right) \\ &\quad - 2\left(n-\frac{1}{2}\right)^2 \pi^2 \sum_{k=0}^{n-1} \left( l_k^{n,i} - \frac{1}{n-(1/2)} \right) \end{aligned}$$

converges to $q_i - \int_0^1 q_i(t)\, dt$ pointwise almost everywhere in $(0,1)$ and in $L^1$.

For case (D), we have $l_k^{n,i} = 1/n - [C/(n^2\pi)] + o(1/n^2)$. This is associated with case (IIIa), where

$$\sqrt{\lambda_n} = n\pi + (-1)^\nu \sin^{-1} \sqrt{\frac{T}{p}} + \delta_n + o\left(\frac{1}{n}\right).$$

For this case,

$$\cot \alpha_i = \lim_{n\to\infty} n\pi \left( \left(k - \frac{1}{2}\right)\pi - n\pi x_k^{n,i} \right)$$

if the limit exists. Otherwise $\alpha_i = 0$. On the other hand, by (13),

$$l_k^{n,i} = \frac{1}{n} + \frac{1}{2n^2\pi^2}\int_{x_k^{n,i}}^{x_{k+1}^{n,i}} (1-\cos(2n\pi t))q_i(t)\,dt \\ - \frac{(-1)^\nu \sin^{-1}\sqrt{T/p} + \delta_n}{n^2\pi} + \frac{(\sin^{-1}\sqrt{T/p})^2}{n^3\pi^2} + o\bigg(\frac{1}{n^3}\bigg).$$

Thus,

$$C = (-1)^\nu \sin^{-1}\sqrt{\frac{T}{p}}, \quad \nu = 1, 2,$$

$$\delta_n = -n\pi \sum_{k=0}^{n-1}\bigg(l_k^{n,i} - \frac{1}{n}\bigg) + \frac{1}{2n\pi}\int_0^1 q_i(t)\,dt \\ - C + \frac{C^2}{n\pi} + o\bigg(\frac{1}{n}\bigg).$$

Note that, for (IIIa), $0 < T < p$. So $|C| \neq 0, \pi/2, \pi$. Finally,

$$F_i^n(x) = 2n^2\pi^2(nl_j^{n,i} - 1) - 2n^2\pi^2 \sum_{k=0}^{n-1}\bigg(l_k^{n,i} - \frac{1}{n}\bigg)$$

converges to $q_i - \int_0^1 q_i(t)\,dt$ pointwisely almost everywhere in $(0,1)$ and in $L^1$.

Cases (B) and (C) are similar, and their proofs are omitted. $\square$

We make a final remark about Theorem 1.2. We do not need to have all the nodal data to achieve the reconstruction. According to the theorem, we only need to find out all (or most of) the nodal data associated with the $\nu$th family of eigenvalues and eigenfunctions as discussed in Theorem 2.1. That is, for the $i$th branch, once we have an infinite number of nodal sequence $X_{n,i} = \{x_k^{n,i} : k = 1, 2, \dots, n\}$ which are associated with one of the seven cases (Ia, Ib, IIa, IIb, IIIa–IIIc) in Theorem 2.1, then the reconstruction can be achieved. We emphasize that we need not know which case it is; we can use the asymptotics of $l_k^{n,i}$ and the value of $\alpha_i$ to determine the case, as shown in the above proof.

**4. Proof of Theorem 2.1.** In the following, we analyze the function $\Phi(\lambda)$ defined as (6) to obtain estimates of the eigenvalues. Lemma 4.1 is a counting lemma proved by Rouché's theorem and is an extension of the Dirichlet problem in [**14, 15**], so it is omitted here.

**Lemma 4.1.** I. *When each* $\alpha_i = 0$*, there are* $p$ *sequences of eigenvalues* $\{\lambda_{n,\nu}\}$ $(\nu = 1, \dots, p)$*, with asymptotics*

$$\sqrt{\lambda_{n,1}} = \left(n - \frac{1}{2}\right)\pi + o(1),$$
$$\sqrt{\lambda_{n,\nu}} = n\pi + o(1), \quad \nu = 2, 3, \dots, p.$$

II. *When each* $\alpha_i \neq 0$*, there are* $p$ *sequences of eigenvalues* $\{\lambda_{n,\nu}\}$ $(\nu = 1, \dots, p)$*, with asymptotics*

$$\sqrt{\lambda_{n,1}} = (n-1)\pi + o(1),$$
$$\sqrt{\lambda_{n,\nu}} = \left(n - \frac{1}{2}\right)\pi + o(1), \quad \nu = 2, 3, \dots, p.$$

III. *When* $\alpha_i = 0$*,* $i = 1, 2, \dots, T$ *and* $\alpha_i \neq 0$*,* $i = T+1, T+2, \dots, p$ *where* $1 \leq T \leq p-1$*, there are* $p$ *sequences of eigenvalues* $\{\lambda_{n,\nu}\}$ $(\nu = 1, \dots, p)$*, with asymptotics*

$$\sqrt{\lambda_{n,\nu}} = n\pi + (-1)^\nu \sin^{-1}\sqrt{\frac{T}{p}} + o(1), \quad \nu = 1, 2,$$
$$\sqrt{\lambda_{n,\nu}} = n\pi + o(1), \quad \nu = 3, 4, \dots, T+1,$$
$$\sqrt{\lambda_{n,\nu}} = \left(n - \frac{1}{2}\right)\pi + o(1), \quad \nu = T+2, T+3, \dots, p.$$

Now, using the rough estimates, we can obtain shaper eigenvalue asymptotics.

*Proof of Theorem* 2.1. The proofs of I, II and III are similar. We shall prove III only. For $\alpha_i = 0$, $i = 1, 2, \dots, T$ and $\alpha_i \neq 0$,

$i = T+1, T+2, \ldots, p$, let $\lambda$ be an eigenvalue and sufficiently large. According to Lemma 4.1, we can sort the eigenvalues by the their estimates. For $\sqrt{\lambda_n} = n\pi + O(1)$, $\sqrt{\lambda_n} = (n-1)\pi + o(1)$ and $\sqrt{\lambda_n} = n - 1/2 + o(1)$, the argument is similar and then we skip the last two here.

(i) $\sqrt{\lambda_n} = n\pi + O(1)$. In this case, it follows from (6) that

$$\sum_{i=1}^{T}\left(-\cos\sqrt{\lambda} + O\left(\frac{1}{\sqrt{\lambda}}\right)\right)\prod_{\nu\neq i,\nu=1}^{T} G_\nu^2(\lambda)\prod_{\nu=T+1}^{p} G_\nu^1(\lambda)$$
$$+\sum_{i=T+1}^{p}\left(-\sqrt{\lambda}\sin\alpha_i\sin\sqrt{\lambda} + O(1)\right)\prod_{\nu=1}^{T} G_\nu^2(\lambda)\prod_{\substack{\nu\neq i\\ \nu=T+1}}^{p} G_\nu^1(\lambda) = 0,$$

where

$$\begin{aligned} G_\nu^1(\lambda) &= \sin\alpha_\nu\cos\sqrt{\lambda} + O\left(\frac{1}{\sqrt{\lambda}}\right),\\ G_\nu^2(\lambda) &= -\frac{\sin\sqrt{\lambda}}{\sqrt{\lambda}} + \frac{\cos\sqrt{\lambda}}{2\lambda}\int_0^1 (1-\cos(2\sqrt{\lambda}t))q_\nu(t)\,dt\\ &\qquad - \frac{\sin\sqrt{\lambda}}{2\lambda}\int_0^1 \sin(2\sqrt{\lambda}t)q_\nu(t)\,dt + O\left(\frac{1}{\lambda^{3/2}}\right). \end{aligned}$$

This implies

$$\sum_{i=1}^{T}\prod_{\substack{\nu\neq i\\ \nu=1}}^{T}\Bigg[\sqrt{\lambda}\tan\sqrt{\lambda} - \frac{1}{2}\int_0^1 (1-\cos(2\sqrt{\lambda}t))q_\nu(t)\,dt$$
$$- \frac{\tan\sqrt{\lambda}}{2}\int_0^1 \sin(2\sqrt{\lambda}t)q_\nu(t)\,dt\Bigg] + O\left(\frac{1}{\sqrt{\lambda}}\right) = 0.$$

From the above equation, we obtain $\sqrt{\lambda}\tan\sqrt{\lambda} = O(1)$, and hence

$$\sum_{i=1}^{T}\prod_{\substack{\nu\neq i\\ \nu=1}}^{T}\left[\sqrt{\lambda}\tan\sqrt{\lambda} - \frac{1}{2}\int_0^1 (1-\cos(2\sqrt{\lambda}t))q_\nu(t)\,dt\right]$$
$$+ O\left(\frac{1}{\sqrt{\lambda}}\right) = 0.$$

Denote by $\Lambda_{n,\nu}$, $\nu = 3, 4, \ldots, T+1$, the roots of the following equation

$$\sum_{i=1}^{T} \prod_{\substack{\nu \neq i \\ \nu=1}}^{T} \left[ \Lambda - \frac{1}{2} \int_0^1 (1 - \cos(2n\pi t)) q_\nu(t)\, dt \right] = 0.$$

Then $\sqrt{\lambda_{n,\nu}} \tan \sqrt{\lambda_{n,\nu}} = \Lambda_{n,\nu} + O(1/\sqrt{\lambda_{n,\nu}})$. Hence, we have

$$\sqrt{\lambda_{n,\nu}} = n\pi + \frac{\Lambda_{n,\nu}}{\sqrt{\lambda_{n,\nu}}} + O\left(\frac{1}{n^2}\right), \quad \nu = 3, 4, \ldots, T-1.$$

(ii) $\sqrt{\lambda_n} = n\pi + (-1)^\nu \sin^{-1} \sqrt{T/p} + o(1)$. In this case, (6) implies

$$\sum_{i=1}^{T} \sum_{j=T+1}^{p} \left\{ H_{i,j}(\lambda) \prod_{\substack{\nu=1 \\ \nu \neq i}}^{T} G_\nu^3(\lambda) \prod_{\substack{\nu=T+1 \\ \nu \neq j}}^{p} G_\nu^4(\lambda) \right\} = 0\ ,$$

where

$$\begin{aligned}
G_\nu^3(\lambda) &= 1 - \frac{\cot \sqrt{\lambda}}{2\sqrt{\lambda}} \int_0^1 (1 - \cos(2\sqrt{\lambda} t)) q_\nu(t)\, dt \\
&\quad + \frac{1}{2\sqrt{\lambda}} \int_0^1 \sin(2\sqrt{\lambda} t) q_\nu(t) dt + O\left(\frac{1}{\lambda}\right), \\
G_\nu^4(\lambda) &= 1 - \frac{\cot \alpha_\nu}{\sqrt{\lambda}} \tan \sqrt{\lambda} + \frac{\tan \sqrt{\lambda}}{2\sqrt{\lambda}} \int_0^1 (1 + \cos(2\sqrt{\lambda} t)) q_\nu(t)\, dt \\
&\quad - \frac{1}{2\sqrt{\lambda}} \int_0^1 \sin(2\sqrt{\lambda} t) q_\nu(t)\, dt + O\left(\frac{1}{\lambda}\right), \\
H_{i,j}(\lambda) &= \left(p \sin^2 \sqrt{\lambda} - T\right) \left(1 + \frac{1}{2\sqrt{\lambda}} \int_0^1 \sin(2\sqrt{\lambda} t)(q_i(t) - q_j(t))\, dt\right) \\
&\quad + \frac{p \sin \sqrt{\lambda} \cos \sqrt{\lambda}}{2\sqrt{\lambda}} \\
&\quad \times \left(2 \cot \alpha_j - \int_0^1 (1 - \cos(2\sqrt{\lambda} t))(q_i(t) + q_j(t))\, dt\right),
\end{aligned}$$

or equivalently,

$$\sum_{i=1}^{T} \sum_{j=T+1}^{p} \frac{H_{i,j}(\lambda)}{G_i^3(\lambda) G_j^4(\lambda)} = 0.$$

Hence,

$$
\begin{aligned}
p\sin^2\sqrt{\lambda}-T = -\frac{\sin\sqrt{\lambda}\cos\sqrt{\lambda}}{2\sqrt{\lambda}}\Bigg(2TA_3-(p-T)\\
&\times\int_0^1(1-\cos(2\sqrt{\lambda}t))\bigg(\sum_{\nu=1}^{T}q_\nu(t)\bigg)\,dt\\
&-T\int_0^1(1+\cos(2\sqrt{\lambda}t))\bigg(\sum_{\nu=T+1}^{p}q_\nu(t)\bigg)\,dt\Bigg)+O\bigg(\frac{1}{\lambda}\bigg).
\end{aligned}
$$

Now, let $\sqrt{\lambda_n}=n\pi+\sin^{-1}\sqrt{T/p}+\delta_n$, where $\delta_n=o(1)$. Then

$$
\begin{aligned}
p\sin^2\sqrt{\lambda_n}-T &= 2\sqrt{T(p-T)}\;\delta_n+o(\delta_n),\\
&= -\frac{\sin\sqrt{\lambda_n}\cos\sqrt{\lambda_n}}{2\sqrt{\lambda_n}}\\
&\quad\times\Bigg(2TA_3-(p-T)\int_0^1\bigg(\sum_{\nu=1}^{T}q_\nu(t)\bigg)\,dt\\
&\quad-T\int_0^1\bigg(\sum_{\nu=T+1}^{p}q_\nu(t)\bigg)\,dt\Bigg)+o\bigg(\frac{1}{n}\bigg),\\
&= -\frac{\sqrt{T(p-T)}}{2pn\pi}\Bigg(2TA_3-(p-T)\int_0^1\bigg(\sum_{\nu=1}^{T}q_\nu(t)\bigg)\,dt\\
&\quad-T\int_0^1\bigg(\sum_{\nu=T+1}^{p}q_\nu(t)\bigg)\,dt\Bigg)+o\bigg(\frac{1}{n}\bigg),
\end{aligned}
$$

This implies

$$
\begin{aligned}
\delta_n = -\frac{1}{4pn\pi}\Bigg(2TA_3-(p-T)\int_0^1\bigg(\sum_{\nu=1}^{T}q_\nu(t)\bigg)\,dt\\
-T\int_0^1\bigg(\sum_{\nu=T+1}^{p}q_\nu(t)\bigg)\,dt\Bigg)+o\bigg(\frac{1}{n}\bigg).\qquad\square
\end{aligned}
$$

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