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ON SOME FUNCTIONAL EQUATIONS ARISING FROM (m,n)-JORDAN DERIVATIONS AND COMMUTATIVITY OF PRIME RINGS

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ABSTRACT. The purpose of this paper is to prove the following result. Let $m, n \geq 1$ be some fixed integers with $m \neq n$, and let R be a prime ring with $(m+n)^2 < \text{char}(R)$. Suppose a nonzero additive mapping $D : R \to R$ exists satisfying the relation $(m+n)^2 D(x^3) = m(3m+n)D(x)x^2 + 4mnxD(x)x + n(3n+m)x^2D(x)$ for all $x \in R$. In this case D is a derivation and R is commutative.

1. Introduction. Throughout, R will represent an associative ring with center $Z(R)$. Given an integer $n \geq 2$, a ring R is said to be n-torsion free, if, for $x \in R$, $nx = 0$ implies $x = 0$. As usual, the commutator $xy - yx$ will be denoted by $[x, y]$. Recall that a ring R is prime if, for $a, b \in R$, $aRb = (0)$ implies that either $a = 0$ or $b = 0$ and is semiprime in the case where $aRa = (0)$ implies $a = 0$. An additive mapping $D : R \to R$, where R is an arbitrary ring, is called a derivation if $D(xy) = D(x)y + xD(y)$ holds for all pairs $x, y \in R$, and is called a Jordan derivation in the case where $D(x^2) = D(x)x + xD(x)$ is fulfilled for all $x \in R$. Obviously, any derivation is a Jordan derivation. The converse in general is not true. Herstein [**14**] has proved that any Jordan derivation on a prime ring with $\text{char}(R) \neq 2$ is a derivation. A brief proof of Herstein's result can be found in [**9**]. Cusack [**11**] has proved Herstein's theorem for 2-torsion free semiprime rings (see [**5**] for an alternative proof). It should be mentioned that Herstein's theorem has been fairly generalized by Beidar, Brešar, Chebotar and Martindale in [**1**]. An additive mapping $D : R \to R$ is called a left derivation if $D(xy) = yD(x) + xD(y)$ holds for all pairs $x, y \in R$ and is called a left Jordan derivation (or Jordan left derivation) in the case where $D(x^2) = 2xD(x)$ is fulfilled for all $x \in R$. The concepts of left derivation and left Jordan derivation were introduced by Brešar and

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Vukman in [**10**]. One can easily prove (see [**10**]) that the existence of a nonzero left derivation on a prime ring forces the ring to be commutative. Moreover, we have the following result.

Let R be a prime ring, and let $D : R \to R$ be a nonzero left Jordan derivation. If $\mathrm{char}\,(R) \neq 2$, then D is a derivation and R is commutative.

The result we have just mentioned was first proved by Brešar and Vukman [**10**] under the additional assumption that $\mathrm{char}\,(R) \neq 3$. Later on, Deng [**12**] proved that the assumption $\mathrm{char}\,(R) \neq 3$ is superfluous.

Vukman [**17**] has proved that in the case where a left Jordan derivation $D : R \to R$ exists, where R is a 2-torsion free semiprime ring, then D is a derivation which maps R into $Z(R)$.

The concept of left Jordan derivation and related results are connected with the theory of commuting and centralizing mappings. A mapping F, which maps a ring R into itself, is called centralizing on R in the case where $[F(x), x] \in Z(R)$ holds for all $x \in R$. In a special case where $[F(x), x] = 0$ is fulfilled for all $x \in R$, F is called *commuting* on R. A classical result of Posner (Posner's second theorem) [**15**] states that the existence of a nonzero centralizing derivation $D : R \to R$, where R is a prime ring, forces the ring to be commutative.

Let $m \geq 0$, $n \geq 0$ with $m + n \neq 0$ some fixed integers. An additive mapping $D : R \to R$, where R is an arbitrary ring, is called an (m, n)-Jordan derivation in the case where

$$(m + n)D(x^2) = 2mD(x)x + 2nxD(x)$$

holds for all $x \in R$.

2. Results. The concept of the (m, n)-Jordan derivation was introduced by Vukman in [**18**]. This concept covers the concept of left Jordan derivation as well as the concept of Jordan derivation. More precisely, $(0, 1)$-Jordan derivation is a left Jordan derivation and $(1, 1)$-Jordan derivation on a 2-torsion free ring is a Jordan derivation.

Vukman [**18**] has recently proved the following result.

Theorem 1. *Let $m \geq 1$, $n \geq 1$ be some fixed integers with $m \neq n$, and let R be a prime ring with* $\mathrm{char}\,(R) \neq 2mn(m+n)(m-n)$. *Suppose*

$D : R \to R$ is a nonzero (m, n)-Jordan derivation. If $\operatorname{char}(R) = 0$ *or* $\operatorname{char}(R) > 3$, *then D is a derivation and R is commutative.*

One can prove (see [**18**] for the details) that any (m, n)-Jordan derivation on arbitrary 2-torsion free ring R satisfies the following relation

$$(1)\qquad \begin{aligned}(m+n)^2 D(x^3) &= m(3m+n)D(x)x^2 \\ &\quad + 4mnxD(x)x + n(3n+m)x^2 D(x), \quad x \in R.\end{aligned}$$

In the case $m = n \neq 0$, R is 2 and m-torsion the torsion free ring, the above relation reduces to

$$(2)\qquad D(x^3) = D(x)x^2 + xD(x)x + x^2 D(x), \quad x \in R.$$

Beidar, Brešar, Chebotar and Martindale [**1**, Theorem 4.4] have proved that in the case where there exists an additive mapping $D : R \to R$, where R is a prime ring with $\operatorname{char}(R) \neq 2$ satisfying relation (2) for all $x \in R$, then D is a derivation (actually they proved a much more general result). In this paper we consider the functional equation (1) in case $m \neq n$. More precisely, it is our aim in this paper to prove the following result.

Theorem 2. *Let $m \geq 1$, $n \geq 1$ be some fixed integers with $m \neq n$, and let R be a prime ring with $(m+n)^2 <$* $\operatorname{char}(R)$. *Suppose that $D : R \to R$ is a nonzero additive mapping satisfying the relation*

$$(3)\qquad \begin{aligned}(m+n)^2 D(x^3) &= m(3m+n)D(x)x^2 \\ &\quad + 4mnxD(x)x + n(3n+m)x^2 D(x)\end{aligned}$$

for all $x \in R$. In this case D is a derivation and R is commutative.

For the proof of Theorem 2 we need Theorem 3 below, which is of independent interest. Our result is obtained as an application of the theory of functional identities (Brešar-Beidar-Chebotar theory). We refer the reader to [**7**] for an introductory account on functional identities and to [**8**] for full treatment of this theory.

Let R be a ring, and let X be a subset of R. By $C(X)$ we denote the set $\{r \in R \mid [r, X] = 0\}$. Let $m \in \mathbf{N}$, and let $E : X^{m-1} \to R$, $p : X^{m-2} \to R$ be arbitrary mappings. In the case where $m = 1$, this should be understood as that E is an element in R and $p = 0$. Let $1 \leq i < j \leq m$, and define $E^i, p^{ij}, p^{ji} : X^m \to R$ by

$$E^i(\overline{x}_m) = E(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_m),$$
$$p^{ij}(\overline{x}_m) = p^{ji}(\overline{x}_m) = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_{j-1}, x_{j+1}, \dots, x_m),$$

where $\overline{x}_m = (x_1, \dots, x_m) \in X^m$.

Let $I, J \subseteq \{1, \dots, m\}$, and, for each $i \in I$, $j \in J$ let $E_i, F_j : X^{m-1} \to R$ be arbitrary mappings. Consider the functional identities

$$\sum_{i\in I} E_i^i(\overline{x}_m)x_i + \sum_{j\in J} x_j F_j^j(\overline{x}_m) = 0, \qquad (\overline{x}_m \in X^m), \tag{4}$$

$$\sum_{i\in I} E_i^i(\overline{x}_m)x_i + \sum_{j\in J} x_j F_j^j(\overline{x}_m) \in C(X), \quad (\overline{x}_m \in X^m). \tag{5}$$

A natural possibility when (4) and (5) are fulfilled is when there exist mappings $p_{ij} : X^{m-2} \to R$, $i \in I$, $j \in J$, $i \neq j$, and $\lambda_k : X^{m-1} \to C(X)$, $k \in I \cup J$, such that

$$\begin{aligned} E_i^i(\overline{x}_m) &= \sum_{j\in J, j\neq i} x_j p_{ij}^{ij}(\overline{x}_m) + \lambda_i^i(\overline{x}_m), \\ F_j^j(\overline{x}_m) &= -\sum_{i\in I, j\neq i} p_{ij}^{ij}(\overline{x}_m)x_i - \lambda_j^j(\overline{x}_m), \\ \lambda_k &= 0 \quad \text{if} \quad k \notin I \cap J \end{aligned} \tag{6}$$

for all $\overline{x}_m \in X^m$, $i \in I$, $j \in J$. We shall say that every solution of form (6) is a standard solution of (4) and (5).

The case where one of the sets I or J is empty is not excluded. The sum over the empty set of indices should be simply read as zero. So, when $J = 0$ (respectively $I = 0$) (4) and (5) reduce to

$$\sum_{i\in I} E_i^i(\overline{x}_m)x_i = 0 \tag{7}$$

$$\left(\text{respectively } \sum_{j\in J} x_j F_j^j(\overline{x}_m) = 0\right), \quad (\overline{x}_m \in X^m),$$

$$\sum_{i\in I} E_i^i(\overline{x}_m)x_i \in C(X) \tag{8}$$

$$\left(\text{respectively } \sum_{j\in J} x_j F_j^j(\overline{x}_m) \in C(X)\right), \quad (\overline{x}_m \in X^m).$$

In that case the (only) standard solution is

$$E_i = 0, \quad i \in I \quad (\text{respectively } F_j = 0, \quad j \in J). \tag{9}$$

A d-freeness of X will play an important role in this paper. For a definition of d-freeness, we refer the reader to [**4**]. Let us mention that a prime ring R is a d-free subset of its maximal right ring of quotients, unless R satisfies the standard polynomial identity of degree less than $2d$ (see [**2**, Theorem 2.4]).

Let R be a ring, and let

$$p(x_1, x_2, x_3) = \sum_{\pi\in S_3} x_{\pi(1)}x_{\pi(2)}x_{\pi(3)}$$

be a fixed multilinear polynomial in noncommutative indeterminates x_1, x_2, x_3. Further, let L be a subset of R closed under p, i.e., $p(\overline{x}_3) \in L$ for all $x_1, x_2, x_3 \in L$, where $\overline{x}_3 = (x_1, x_2, x_3)$. We shall consider a mapping $D : L \to R$ satisfying

$$\begin{aligned}(m+n)^2 D(p(\overline{x}_3)) &= m(3m+n)\sum_{\pi\in S_3} D(x_{\pi(1)})x_{\pi(2)}x_{\pi(3)} \\ &\quad + 4mn\sum_{\pi\in S_3} x_{\pi(1)}D(x_{\pi(2)})x_{\pi(3)} \\ &\quad + n(3n+m)\sum_{\pi\in S_3} x_{\pi(1)}x_{\pi(2)}D(x_{\pi(3)})\end{aligned} \tag{10}$$

for all $x_1, x_2, x_3 \in L$. Let us mention that the idea of considering the expression $[p(\overline{x}_3), p(\overline{y}_3)]$ in its proof is taken from [**3**] and was used in [**13**] as well.

Theorem 3. *Let L be a 6-free Lie subring of R closed under p. If $D : L \to R$ is an additive mapping satisfying* (10)*, then $p \in C(L)$ and $\lambda : L \to C(L)$ exist such that $(m+n)^2 n(3n+m)D(x) = px + \lambda(x)$ for all $x \in L$.*

Proof. Note that, for any $a \in L$ and $\overline{x}_3 \in L^3$, we have

$$[p(\overline{x}_3), a] = p([x_1, a], x_2, x_3) + p(x_1, [x_2, a], x_3) + p(x_1, x_2, [x_3, a]).$$

Thus,

$$\begin{aligned}(m+n)^2 D[p(\overline{x}_3), a] = (m+n)^2 &D(p([x_1, a], x_2, x_3)) \\ &+ (m+n)^2 D(p(x_1, [x_2, a], x_3)) \\ &+ (m+n)^2 D(p(x_1, x_2, [x_3, a])).\end{aligned}$$

Using (10), it follows that

$$\begin{aligned}(m+n)^2 D[p(\overline{x}_3), a] = m(3m+n) &\sum_{\pi \in S_3} D[x_{\pi(1)}, a] x_{\pi(2)} x_{\pi(3)} \\ &+ 4mn \sum_{\pi \in S_3} [x_{\pi(1)}, a] D(x_{\pi(2)}) x_{\pi(3)} \\ &+ n(3n+m) \sum_{\pi \in S_3} [x_{\pi(1)}, a] x_{\pi(2)} D(x_{\pi(3)}) \\ &+ m(3m+n) \sum_{\pi \in S_3} D(x_{\pi(1)}) [x_{\pi(2)}, a] x_{\pi(3)} \\ &+ 4mn \sum_{\pi \in S_3} x_{\pi(1)} D[x_{\pi(2)}, a] x_{\pi(3)} \\ &+ n(3n+m) \sum_{\pi \in S_3} x_{\pi(1)} [x_{\pi(2)}, a] D(x_{\pi(3)}) \\ &+ m(3m+n) \sum_{\pi \in S_3} D(x_{\pi(1)}) x_{\pi(2)} [x_{\pi(3)}, a] \\ &+ 4mn \sum_{\pi \in S_3} x_{\pi(1)} D(x_{\pi(2)}) [x_{\pi(3)}, a] \\ &+ n(3n+m) \sum_{\pi \in S_3} x_{\pi(1)} x_{\pi(2)} D[x_{\pi(3)}, a].\end{aligned}$$

Thus,

$$
\begin{aligned}
(m+n)^2 D[p(\overline{x}_3), a] &= m(3m+n) \sum_{\pi\in S_3} D[x_{\pi(1)}, a] x_{\pi(2)} x_{\pi(3)} \\
&+ 4mn \sum_{\pi\in S_3} [x_{\pi(1)}, a] D(x_{\pi(2)}) x_{\pi(3)} \\
&+ n(3n+m) \sum_{\pi\in S_3} [x_{\pi(1)} x_{\pi(2)}, a] D(x_{\pi(3)}) \\
&+ m(3m+n) \sum_{\pi\in S_3} D(x_{\pi(1)}) [x_{\pi(2)} x_{\pi(3)}, a] \\
&+ 4mn \sum_{\pi\in S_3} x_{\pi(1)} D[x_{\pi(2)}, a] x_{\pi(3)} \\
&+ 4mn \sum_{\pi\in S_3} x_{\pi(1)} D(x_{\pi(2)}) [x_{\pi(3)}, a] \\
&+ n(3n+m) \sum_{\pi\in S_3} x_{\pi(1)} x_{\pi(2)} D[x_{\pi(3)}, a].
\end{aligned} \tag{11}
$$

In particular,

$$
\begin{aligned}
&(m+n)^2 D[p(\overline{x}_3), p(\overline{y}_3)] \\
&= m(3m+n) \sum_{\pi\in S_3} D[x_{\pi(1)}, p(\overline{y}_3)] x_{\pi(2)} x_{\pi(3)} \\
&+ 4mn \sum_{\pi\in S_3} [x_{\pi(1)}, p(\overline{y}_3)] D(x_{\pi(2)}) x_{\pi(3)} \\
&+ n(3n+m) \sum_{\pi\in S_3} [x_{\pi(1)} x_{\pi(2)}, p(\overline{y}_3)] D(x_{\pi(3)}) \\
&+ m(3m+n) \sum_{\pi\in S_3} D(x_{\pi(1)}) [x_{\pi(2)} x_{\pi(3)}, p(\overline{y}_3)] \\
&+ 4mn \sum_{\pi\in S_3} x_{\pi(1)} D[x_{\pi(2)}, p(\overline{y}_3)] x_{\pi(3)} \\
&+ 4mn \sum_{\pi\in S_3} x_{\pi(1)} D(x_{\pi(2)}) [x_{\pi(3)}, p(\overline{y}_3)] \\
&+ n(3n+m) \sum_{\pi\in S_3} x_{\pi(1)} x_{\pi(2)} D[x_{\pi(3)}, p(\overline{y}_3)]
\end{aligned} \tag{12}
$$

for all $\overline{x}_3, \overline{y}_3 \in L^3$. For $i = 1, 2, 3$ we also have (by (11))

$$
\begin{aligned}
\varphi(x_{\pi(i)}) &= (m+n)^2 D[x_{\pi(i)}, p(\overline{y}_3)] \\
&= -(m+n)^2 D[p(\overline{y}_3), x_{\pi(i)}] \\
&= m(3m+n) \sum_{\sigma\in S_3} D[x_{\pi(i)}, y_{\sigma(1)}] y_{\sigma(2)} y_{\sigma(3)} \\
&\quad + 4mn \sum_{\sigma\in S_3} [x_{\pi(i)}, y_{\sigma(1)}] D(y_{\sigma(2)}) y_{\sigma(3)} \\
&\quad + n(3n+m) \sum_{\sigma\in S_3} [x_{\pi(i)}, y_{\sigma(1)} y_{\sigma(2)}] D(y_{\sigma(3)}) \\
&\quad + m(3m+n) \sum_{\sigma\in S_3} D(y_{\sigma(1)}) [x_{\pi(i)}, y_{\sigma(2)} y_{\sigma(3)}] \\
&\quad + 4mn \sum_{\sigma\in S_3} y_{\sigma(1)} D[x_{\pi(i)}, y_{\sigma(2)}] y_{\sigma(3)}
\end{aligned}
$$

$$
\begin{aligned}
&+ 4mn \sum_{\sigma\in S_3} y_{\sigma(1)} D(y_{\sigma(2)}) [x_{\pi(i)}, y_{\sigma(3)}] \\
&+ n(3n+m) \sum_{\sigma\in S_3} y_{\sigma(1)} y_{\sigma(2)} D[x_{\pi(i)}, y_{\sigma(3)}]
\end{aligned}
$$

for all $\overline{y}_3 \in L^3$. Therefore, (12) can be written as

(13)

$$
\begin{aligned}
&(m+n)^4 D[p(\overline{x}_3), p(\overline{y}_3)] \\
&\qquad = m(3m+n) \sum_{\pi\in S_3} \varphi(x_{\pi(1)}) x_{\pi(2)} x_{\pi(3)} \\
&\qquad\quad + (m+n)^2 4mn \sum_{\pi\in S_3} [x_{\pi(1)}, p(\overline{y}_3)] D(x_{\pi(2)}) x_{\pi(3)} \\
&\qquad\quad + (m+n)^2 n(3n+m) \sum_{\pi\in S_3} [x_{\pi(1)} x_{\pi(2)}, p(\overline{y}_3)] D(x_{\pi(3)}) \\
&\qquad\quad + (m+n)^2 m(3m+n) \sum_{\pi\in S_3} D(x_{\pi(1)}) [x_{\pi(2)} x_{\pi(3)}, p(\overline{y}_3)] \\
&\qquad\quad + 4mn \sum_{\pi\in S_3} x_{\pi(1)} \varphi(x_{\pi(2)}) x_{\pi(3)}
\end{aligned}
$$

$$+ (m+n)^2 4mn \sum_{\pi\in S_3} x_{\pi(1)} D(x_{\pi(2)})[x_{\pi(3)}, p(\overline{y}_3)]$$
$$+ n(3n+m) \sum_{\pi\in S_3} x_{\pi(1)} x_{\pi(2)} \varphi(x_{\pi(3)})$$

for all $\overline{x}_3, \overline{y}_3 \in L^3$. On the other hand, using $[p(\overline{x}_3), p(\overline{y}_3)] = -[p(\overline{y}_3), p(\overline{x}_3)]$, we get from the above identity

$$\begin{aligned}
(14)\quad & (m+n)^4 D[p(\overline{x}_3), p(\overline{y}_3)] \\
&= m(3m+n) \sum_{\sigma\in S_3} \varphi'(y_{\sigma(1)}) y_{\sigma(2)} y_{\sigma(3)} \\
&\quad + (m+n)^2 4mn \sum_{\sigma\in S_3} [p(\overline{x}_3), y_{\sigma(1)}] D(y_{\sigma(2)}) y_{\sigma(3)} \\
&\quad + (m+n)^2 n(3n+m) \sum_{\sigma\in S_3} [p(\overline{x}_3), y_{\sigma(1)} y_{\sigma(2)}] D(y_{\sigma(3)}) \\
&\quad + (m+n)^2 m(3m+n) \sum_{\sigma\in S_3} D(y_{\sigma(1)}) [p(\overline{x}_3), y_{\sigma(2)} y_{\sigma(3)}] \\
&\quad + 4mn \sum_{\sigma\in S_3} y_{\sigma(1)} \varphi'(y_{\sigma(2)}) y_{\sigma(3)} \\
&\quad + (m+n)^2 4mn \sum_{\sigma\in S_3} y_{\sigma(1)} D(y_{\sigma(2)}) [p(\overline{x}_3), y_{\sigma(3)}] \\
&\quad + n(3n+m) \sum_{\sigma\in S_3} y_{\sigma(1)} y_{\sigma(2)} \varphi'(y_{\sigma(3)})
\end{aligned}$$

for all $\overline{x}_3, \overline{y}_3 \in L^3$, where

$$\begin{aligned}
\varphi(y_{\sigma(i)})' &= m(3m+n) \sum_{\pi\in S_3} D[x_{\pi(1)}, y_{\sigma(i)}] x_{\pi(2)} x_{\pi(3)} \\
&\quad + 4mn \sum_{\pi\in S_3} [x_{\pi(1)}, y_{\sigma(i)}] D(x_{\pi(2)}) x_{\pi(3)} \\
&\quad + n(3n+m) \sum_{\pi\in S_3} [x_{\pi(1)} x_{\pi(2)}, y_{\sigma(i)}] D(x_{\pi(3)}) \\
&\quad + m(3m+n) \sum_{\pi\in S_3} D(x_{\pi(1)}) [x_{\pi(2)} x_{\pi(3)}, y_{\sigma(i)}] \\
&\quad + 4mn \sum_{\pi\in S_3} x_{\pi(1)} D[x_{\pi(2)}, y_{\sigma(i)}] x_{\pi(3)}
\end{aligned}$$

$$+ 4mn \sum_{\pi \in S_3} x_{\pi(1)} D(x_{\pi(2)})[x_{\pi(3)}, y_{\sigma(i)}]$$
$$+ n(3n+m) \sum_{\pi \in S_3} x_{\pi(1)} x_{\pi(2)} D[x_{\pi(3)}, y_{\sigma(i)}]$$

for all $\overline{x}_3 \in L^3$. Let $s : \mathbf{Z} \to \mathbf{Z}$ be a mapping defined by $s(i) = i - 3$. For each $\sigma \in S_3$ the mapping $s^{-1}\sigma s : \{4,5,6\} \to \{4,5,6\}$ is denoted by $\overline{\sigma}$. Comparing identities (13) and (14) and writing x_{3+i} instead of y_i, $i = 1,2,3$, we can express the so-obtained relation as

$$\sum_{i=1}^{6} E_i^i(\overline{x}_6) x_i + \sum_{j=1}^{6} x_j F_j^j(\overline{x}_6) = 0,$$

for all $\overline{x}_6 = (x_1, x_2, x_3, x_4, x_5, x_6) \in L^6$. We can prove that $p \in L$ and a mapping $\lambda : L \to C(L)$ exist such that

$$(m+n)^2 m(3m+n) D(x) = xp + \lambda(x) \tag{15}$$

for all $x \in L$. Similarly, we can show that $q \in L$ and a mapping $\mu : L \to C(L)$ exist such that

$$(m+n)^2 n(3n+m) D(x) = qx + \mu(x) \tag{16}$$

for all $x \in L$. Thus,

$$n(3n+m)(m+n)^2 m(3m+n) D(x) = n(3n+m)xp + n(3n+m)\lambda(x),$$

$$m(3m+n)(m+n)^2 n(3n+m) D(x) = m(3m+n)qx + m(3m+n)\mu(x),$$

for all $x \in L$. Comparing these two identities, we arrive at

$$n(3n+m)xp - m(3m+n)qx \in C(L)$$

for all $x \in L$. It follows that $n(3n+m)p = m(3m+n)q \in C(L)$, which yields $p, q \in C(L)$. Thereby, the proof is completed. $\square$

We are now in a position to prove Theorem 2.

Proof. The complete linearization of (3) gives us (10). First suppose that R is not a PI ring (satisfying the standard polynomial identity of degree less than 6). According to Theorem 3, $p \in C$ and $\lambda : R \to C$ exist such that

$$(m+n)^2 m(3m+n)D(x) = px + \lambda(x)$$

for all $x \in R$. Thus,

$$x^2(2(m+n)^2 px + 3(m+n)^2\lambda(x)) = (m+n)^2\lambda(x^3),$$

which yields

$$x^2(2px + 3\lambda(x)) = \lambda(x^3)$$

for all $x \in R$. A complete linearization of this identity leads to

$$\sum_{\pi \in S_3} x_{\pi(1)}x_{\pi(2)}\Big(2px_{\pi(3)} + 3\lambda(x_{\pi(3)})\Big) = \lambda(p(\overline{x}_3))$$

for all $x_1, x_2, x_3 \in R$. Since R is not a PI ring, it follows that

$$2px + 3\lambda(x) = 0 \tag{17}$$

for all $x \in R$. Thus, $[2px, y] = 0$ for all $x, y \in R$, which in turn implies $[x, y]zp = 0$ for all $x, y, z \in R$. By the primeness of R, it follows that R is commutative or $p = 0$. The second relation gives us $\lambda(x) = 0$ for all $x \in R$ by (17). Thus, $D = 0$. Suppose now that $[x, y] = 0$ for all $x, y \in R$. Using (17) it follows that $\lambda(x)y - \lambda(y)x = 0$ for all $x, y \in R$, which implies $\lambda = 0$. Consequently, $D = 0$.

Assume now that R is a PI ring. It is well known, that in this case, R has a nonzero center (see [**16**]). Let c be a nonzero central element. Pick any $x \in R$, and set $x_1 = x_2 = cx$ and $x_3 = x$ in (10). We arrive at

$$\begin{aligned}(m+n)^2 D(6c^2x^3) &= m(3m+n)c(4D(cx)x^2 + 2cD(x)x^2)\\ &\quad + 4mnc(4xD(cx)x + 2cxD(x)x)\\ &\quad + n(3n+m)c(4x^2D(cx) + 2cx^2D(x)).\end{aligned}$$

On the other hand, setting $x_1 = x_2 = c$ and $x_3 = x^3$ in (10), we obtain

$$\begin{aligned}(m+n)^2 D(6c^2x^3) = {} & m(3m+n)c(4D(c)x^3 + 2cD(x^3)) \\ & + 4mnc(2D(c)x^3 + 2cD(x^3) + 2x^3D(c)) \\ & + n(3n+m)c(2cD(x^3) + 4x^3D(c)).\end{aligned}$$

Comparing the so-obtained relations, we get

$$\begin{aligned}0 = {} & m(3m+n)(D(cx)x^2 - cD(x)x^2 - D(c)x^3) \\ & + 2mn(2xD(cx)x - 2cxD(x)x - D(c)x^3 - x^3D(c)) \\ & + n(3n+m)(x^2D(cx) - cx^2D(x) - x^3D(c))\end{aligned} \tag{18}$$

for all $x \in R$. In the case where $x = c$, we have

$$D(c^2) = 2cD(c). \tag{19}$$

The complete linearization of (18) and setting $x_1 = x$ and $x_2 = x_3 = c$ in the so-obtained identity yields

$$\begin{aligned}6(m+n)^2 D(cx) = {} & (2m(3m+n) + 4mn)D(c)x \\ & + (2n(3n+m) + 4mn)xD(c) \\ & + 6(m+n)^2 cD(x)\end{aligned}$$

for all $x \in R$. Hence,

$$(m+n)(D(cx) - cD(x)) = mD(c)x + nxD(c) \tag{20}$$

for all $x \in R$.

Putting cx instead of x in (3), we get

$$\begin{aligned}(m+n)^2 D(c^3x^3) = {} & m(3m+n)c^2D(cx)x^2 \\ & + 4mnc^2xD(cx)x \\ & + n(3n+m)c^2x^2D(cx)\end{aligned} \tag{21}$$

for all $x \in R$. On the other hand, setting $x_1 = x_2 = c$ and $x_3 = cx^3$ in (10), we obtain

$$\begin{aligned}(m+n)^2 D(c^3x^3) = {} & (m+n)^2 c^2 D(cx^3) \\ & + 2m(m+n)c^2D(c)x^3 \\ & + 2n(m+n)c^2x^3D(c)\end{aligned} \tag{22}$$

for all $x \in R$. Note that, by (20),

$$\begin{aligned}(m+n)^2 D(cx^3) &= (m+n)((m+n)cD(x^3)\\ &\quad + mD(c)x^3 + nx^3D(c))\\ &= m(3m+n)cD(x)x^2\\ &\quad + 4mncxD(x)x + n(3n+m)cx^2D(x)\\ &\quad + m(m+n)D(c)x^3 + n(m+n)x^3D(c).\end{aligned}$$

Comparing identities (21) and (22), we arrive at

$$\begin{aligned}m(3m+n)(D(cx) - cD(x))x^2&\\ + 4mnx(D(cx) - cD(x))x&\\ + n(3n+m)x^2(D(cx) - cD(x))&\\ = 3m(m+n)D(c)x^3 + 3n(m+n)x^3D(c)&\end{aligned} \tag{23}$$

for all $x \in R$. Multiplying this relation by $(m+n)$ and using (20), it follows that

$$\begin{aligned}m(3m+n)(mD(c)x + nxD(c))x^2&\\ + 4mnx(mD(c)x + nxD(c))x&\\ + n(3n+m)x^2(mD(c)x + nxD(c))&\\ = 3m(m+n)^2D(c)x^3 + 3n(m+n)^2x^3D(c),&\end{aligned}$$

which in turn implies

$$\begin{aligned}(5m+3n)D(c)x^3 + (3m+5n)x^3D(c)&\\ = (m+7n)x^2D(c)x + (7m+n)xD(c)x^2&\end{aligned}$$

for all $x \in R$. After a complete linearization and putting $x_1 = x_2 = x$ and $x_3 = c$ in this new identity, we obtain $[[x, D(c)], x] = 0$ for all $x \in R$. Using Posner's second theorem, it follows that $[x, D(c)] = 0$ for all $x \in R$. From (20), we get

$$D(cx) = D(c)x + cD(x) \tag{24}$$

for all $x \in R$. Pick any $x \in R$, and set $x_1 = c$ and $x_2 = x_3 = x$ in (10). We arrive at

$$\begin{aligned}6(m+n)^2D(cx^2) &= m(3m+n)(4D(x)xc + 2D(c)x^2)\\ &\quad + 4mn(2cD(x)x + 2xD(x)c + 2xD(c)x)\\ &\quad + n(3n+m)(4cxD(x) + 2x^2D(c))\end{aligned}$$

for all $x \in R$. By (24), we have $6(m+n)^2 D(cx^2) = 6(m+n)^2(D(c)x^2 + cD(x^2))$ for all $x \in R$. Comparing the so-obtained identities, we arrive at

$$(m+n)D(x^2) = 2mD(x)x + 2nxD(x) \tag{25}$$

for all $x \in R$.

The linearization of relation (25) gives us

$$\begin{aligned}(m+n)D(xy+yx) = 2mD(x)y \\ + 2mD(y)x + 2nxD(y) + 2nyD(x)\end{aligned} \tag{26}$$

for all $x, y \in R$. Now, putting $(m+n)^2 x^3$ for y in relation (26) and applying (3), we obtain after some calculations

$$\begin{aligned}(m+n)^3 D(x^4) = {}& (4m^3 + 3m^2n + mn^2)D(x)x^3 \\ & + (7m^2n + mn^2)xD(x)x^2 + (7mn^2 + m^2n)x^2D(x)x \\ & + (4n^3 + 3mn^2 + m^2n)x^3D(x)\end{aligned} \tag{27}$$

for all $x \in R$. On the other hand, putting $(m+n)x^2$ for x in (21), we obtain

$$\begin{aligned}(m+n)^3 D(x^4) = {}& 4m^2(m+n)D(x)x^3 + 4mn(m+n)xD(x)x^2 \\ & + 4mn(m+n)x^2D(x)x + 4n^2(m+n)x^3D(x)\end{aligned} \tag{28}$$

for all $x \in R$. By comparing (23) and (24), we obtain

$$\begin{aligned} & mn(n-m)D(x)x^3 + 3mn(m-n)xD(x)x^2 + \\ & 3mn(n-m)x^2D(x)x + mn(m-n)x^3D(x) = 0\end{aligned} \tag{29}$$

for all $x \in R$. Whence, it follows that

$$D(x)x^3 - 3xD(x)x^2 + 3x^2D(x)x - x^3D(x) = 0$$

for all $x \in R$, which can be written in the form

$$[[[D(x), x], x], x] = 0$$

for all $x \in R$. By the result of Brešar [**6**], it follows that $[D(x), x] = 0$ holds for all $x \in R$, which makes it possible to replace $D(x)x$ in (25) with $xD(x)$. We therefore have $(m+n)D(x^2) = 2(m+n)xD(x)$ for all $x \in R$, which reduces to $D(x^2) = 2xD(x)$, $x \in R$. Again applying the fact that D is commuting on R, we arrive at $D(x^2) = D(x)x + xD(x)$ for all $x \in R$. In other words, D is a Jordan derivation, whence it follows that D is a derivation by Herstein's theorem. Thus, D is a nonzero commuting derivation. By Posner's second theorem, R is commutative. Thereby the proof of the theorem is complete. $\square$

REFERENCES

1. K.I. Beidar, M. Brešar, M.A. Chebotar and W.S. Martindale 3rd, *On Herstein's Lie map conjectures* II, J. Algebra **238** (2001), 239–264.

2. K.I. Beidar and M.A. Chebotar, *On functional identities and d-free subsets of rings* I, Comm. Algebra **28** (2000), 3925–3952.

3. K.I. Beidar and Y. Fong, *On additive isomorphisms of prime rings preserving polynomials*, J. Algebra **217** (1999), 650–667.

4. K.I. Beidar, A.V. Mikhalev and M.A. Chebotar, *Functional identities in rings and their applications*, Russian Math. Surv. **59** (2004), 403–428.

5. M. Brešar, *Jordan derivations on semiprime rings*, Proc. Amer. Math. Soc. **104** (1988), 1003–1006.

6. ———, *On a generalization of the notion of centralizing mappings*, Proc. Amer. Math. Soc. **114** (1992), 641–649.

7. ———, *Functional identities*: *A survey*, Contemp. Math. **259** (2000), 93–109.

8. M. Brešar, M.A. Chebotar and W.S. Martindale 3rd, *Functional identities*, Birkhauser Verlag, Basel, 2007.

9. M. Brešar and J. Vukman, *Jordan derivations on prime rings*, Bull. Austral. Math. Soc. **37** (1988), 321–322.

10. ———, *On left derivations and related mappings*, Proc. Amer. Math. Soc. **110** (1990), 7–16.

11. J. Cusack, *Jordan derivations on rings*, Proc. Amer. Math. Soc. **53** (1975), 321–324.

12. Q. Deng, *On Jordan left derivations*, Math. J. Okayama Univ. **34** (1992), 145–147.

13. M. Fošner and J. Vukman, *Equations related to derivations on prime rings*, to appear.

14. I.N. Herstein, *Jordan derivations of prime rings*, Proc. Amer. Math. Soc. **8** (1957), 1104–1110.

15. E.C. Posner, *Derivations in prime rings*, Proc. Amer. Math. Soc. **8** (1957), 1093–1100.

16. L.H. Rowen, *Some results on the center of a ring with polynomial identity*, Bull. Amer. Math. Soc. **79** (1973), 219–223.

17. J. Vukman, *On left Jordan derivations of rings and Banach algebras*, Aequat. Math. **75** (2008), 260–266.

18. J. Vukman, *On (m, n)-Jordan derivations and commutativity of prime rings*, Demonstr. Math. **41** (2008), 773–778.

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