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DUPLICATE INVERSIONS AND FOX-WRIGHT FUNCTION IDENTITIES

CHUANAN WEI

ABSTRACT. By means of duplicate inverse series relations and linear combinations, a class of closed formulas with respect to Fox-Wright function is established.

1. Introduction and notation. For a complex c and a natural number n, denote the shifted-factorial by

$$(c)_0 = 1 \quad \text{and} \quad (c)_n = c(c+1)\cdots(c+n-1) \quad \text{for } n = 1, 2, \dots .$$

Following Bailey [**1**], the hypergeometric series is defined by

$${}_{1+r}F_s\left[\begin{matrix} a_0, & a_1, & \cdots, & a_r \\ & b_1, & \cdots, & b_s \end{matrix}\,\middle|\, z\right] = \sum_{k=0}^{\infty} \frac{(a_0)_k (a_1)_k \cdots (a_r)_k}{k!(b_1)_k \cdots (b_s)_k} z^k$$

where $\{a_i\}$ and $\{b_j\}$ are complex parameters such that no zero factors appear in the denominators of the summand on the right hand side. If one of numerator parameters $\{a_i\}$ is a negative integer, then the series becomes terminating, which reduces to a polynomial in z. Thereunto, the famous Saalschütz's theorem (cf. [**1**, page 9]) can be stated as follows:

$$(1.1)\qquad {}_3F_2\left[\begin{matrix} a, & b, & -n \\ & c, & 1+a+b-c-n \end{matrix}\,\middle|\, 1\right] = \frac{(c-a)_n (c-b)_n}{(c)_n (c-a-b)_n}.$$

Define the Γ-function by the Euler integral

$$\Gamma(s) := \int_0^{\infty} t^{s-1} e^{-t}\, dt, \quad \Re(s) > 0.$$

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Then we have two transforms

$$\Gamma(s+n)=\Gamma(s)(s)_n \quad \text{and} \quad \Gamma(2s)=\frac{2^{2s-1}}{\sqrt{\pi}}\Gamma(s)\Gamma(s+1/2)$$

which will be used frequently without indication in this paper.

Recall the Fox-Wright function ${}_p\Psi_q$ (cf. [**6, 9, 10**]; see also [**8**, page 21]), which is defined by

$${}_p\Psi_q\left[\begin{matrix}(\alpha_1,A_1),\ldots, & (\alpha_p,A_p)\\ (\beta_1,B_1),\ldots, & (\beta_q,B_q)\end{matrix}\middle|\, z\right]:=\sum_{k=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma(\alpha_i+A_ik)}{\prod_{j=1}^{q}\Gamma(\beta_j+B_jk)}\frac{z^k}{k!}$$

and regarded as a generalization of hypergeometric series, where the coefficients $\{A_i\}$ and $\{B_j\}$ are positive real numbers such that

$$1+\sum_{j=1}^{q}B_j-\sum_{i=1}^{p}A_i>0.$$

Naturally, the domain where $\{A_i\}$ and $\{B_j\}$ take on values can be extended to the complex field. The corresponding function is also called the Fox-Wright function, which is denoted by ${}_p\Psi_q^*$.

Inversion techniques have been shown to be efficient in dealing with combinatorial identities (cf. [**2, 5**]). In order to prove the terminating balanced hypergeometric series, the same approach has been developed by Chu [**3, 4**] to the duplicate inverse series relations, which may be reproduced, for facilitating the subsequent use, as follows

Theorem. *For two complex variables x, y and four complex sequences $\{a_k,b_k,c_k,d_k\}_{k\geq 0}$, define two polynomial sequences by*

$$\phi(x;0)\equiv 1, \qquad \phi(x;m)=\prod_{i=0}^{m-1}(a_i+xb_i), \quad m=1,2,\ldots,$$
$$\psi(y;0)\equiv 1, \qquad \psi(y;n)=\prod_{j=0}^{n-1}(c_j+yd_j), \quad n=1,2,\ldots.$$

The system of equations

(1.2a)
$$\Omega_n = \sum_{k\geq 0} \binom{n}{2k} \frac{c_k + 2kd_k}{\phi(n;k)\psi(n;k+1)} f(k)$$

(1.2b)
$$- \sum_{k\geq 0} \binom{n}{1+2k} \frac{a_k + (1+2k)b_k}{\phi(n;1+k)\psi(n;k+1)} g(k)$$

is equivalent to the system of equations

(1.3a)
$$f(n) = \sum_{k=0}^{2n} (-1)^k \binom{2n}{k} \phi(k;n)\psi(k;n)\Omega_k$$

(1.3b)
$$g(n) = \sum_{k=0}^{1+2n} (-1)^k \binom{1+2n}{k} \phi(k;n)\psi(k;n+1)\Omega_k.$$

Otherwise, we have a third relation

(1.3c)
$$h(n) = \sum_{k=0}^{1+2n} (-1)^k \binom{1+2n}{k} \phi(k;1+n)\psi(k;n)\Omega_k$$

where

(1.4)
$$h(n) = f(n)\frac{(1+2n)\{a_n d_n - b_n c_n\}}{c_n + d_n(1+2n)} + g(n)\frac{a_n + b_n(1+2n)}{c_n + d_n(1+2n)}.$$

The purpose of this paper is to derive a class of closed formulas with respect to Fox-Wright function by employing this duplicate inversion theorem to the Saalschütz's theorem just displayed. The surprising fact is that each Fox-Wright identity includes countless hypergeometric series identities as the limiting cases.

2. Fox-Wright function identities. Let λ be a complex number. Then a special case of Saalschütz's theorem (1.1) may be stated as follows:

$${}_3F_2\left[\begin{matrix} -n/2, & (1-n)/2, & a-c \\ & 1/2-c-n-\lambda n, & 1+a+\lambda n \end{matrix}\,\middle|\, 1\right] = U(n)$$

where

$$U(n) := \frac{(1+2c+2\lambda n)_n(1/2+a+\lambda n)_n}{(1/2+c+\lambda n)_n(1+2a+2\lambda n)_n}.$$

This equation can be expressed as a binomial identity

(2.1)
$$\sum_{k\geq 0}\binom{n}{2k}\frac{1}{(1/2-c-n-\lambda n)_k(1+a+\lambda n)_k}\frac{(a-c)_k(2k)!}{4^k k!} = U(n).$$

Based on the identity just established, we are ready to derive several combinatorial evaluations.

2.1. It is clear that (2.1) can be rewritten in the form

$$\sum_{k\geq 0}\binom{n}{2k}\frac{a+k+2\lambda k}{(1/2-c-n-\lambda n)_k(a+\lambda n)_{k+1}}\frac{(a-c)_k(2k)!}{(a+k+2\lambda k)4^k k!} = \frac{U(n)}{a+\lambda n}$$

which is the case

$$\phi(x;n) := \left(\frac{1}{2}-c-x-\lambda x\right)_n \quad \text{and} \quad \psi(y;n) := (a+\lambda y)_n$$

of (1.2a)–(1.2b) with

$$\begin{aligned}
\Omega_n &:= \frac{U(n)}{a+\lambda n}\\
f(n) &:= \frac{(a-c)_n(2n)!}{(a+n+2\lambda n)4^n n!}\\
g(n) &:= 0\\
h(n) &:= \frac{\{a+n+\lambda(1/2+a-c+2n)\}(a-c)_n(1+2n)!}{(a+n+2\lambda n)(a+\lambda+n+2\lambda n)4^n n!}
\end{aligned}$$

where $h(n)$ is deduced from $f(n)$ and $g(n)$ via (1.4). Thus (1.3a), (1.3b) and (1.3c) give the following dual formulas:

$$\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}\frac{(1/2-c-k-\lambda k)_n(a+\lambda k)_n}{a+\lambda k}U(k)$$
$$=\frac{(a-c)_n(2n)!}{(a+n+2\lambda n)4^n n!},$$
$$\sum_{k=0}^{1+2n}(-1)^k\binom{1+2n}{k}\frac{(1/2-c-k-\lambda k)_n(a+\lambda k)_{n+1}}{a+\lambda k}U(k)$$
$$=0,$$
$$\sum_{k=0}^{1+2n}(-1)^k\binom{1+2n}{k}\frac{(1/2-c-k-\lambda k)_{n+1}(a+\lambda k)_n}{a+\lambda k}U(k)$$
$$=\frac{\{a+n+\lambda(1/2+a-c+2n)\}(a-c)_n(1+2n)!}{(a+n+2\lambda n)(a+\lambda+n+2\lambda n)4^n n!}.$$

They can be restated as the following Fox-Wright function identities, respectively.

Proposition 1 (Fox-Wright function identities).

(2.2a)
$$ {}_3\Psi_4^*\left[\begin{matrix}(a+n,\lambda), \quad (1/2+a,\lambda+1), \quad (1+2c,2\lambda+1)\\ (1+2n,-1),(1+c,\lambda),(1/2+c-n,\lambda+1),(1+2a,2\lambda+1)\end{matrix}\;\middle|\; -1\right]$$

(2.2b)
$$=\frac{1}{(a+n+2\lambda n)}\frac{(-1)^n(a-c)_n}{4^{a-c+n}n!},$$

(2.3a)
$$ {}_3\Psi_4^*\left[\begin{matrix}(1+a+n,\lambda), \quad (1/2+a,\lambda+1), \quad (1+2c,2\lambda+1)\\ (2+2n,-1),(1+c,\lambda),(1/2+c-n,\lambda+1),(1+2a,2\lambda+1)\end{matrix}\;\middle|\; -1\right]$$

(2.3b)
$$=0,$$

(2.4a)
$$ {}_3\Psi_4^*\left[\begin{matrix}(a+n,\lambda), \quad (1/2+a,\lambda+1), \quad (1+2c,2\lambda+1)\\ (2+2n,-1),(1+c,\lambda),(-1/2+c-n,\lambda+1),(1+2a,2\lambda+1)\end{matrix}\;\middle|\; -1\right]$$

(2.4b)

$$= -\frac{a+n+\lambda(1/2+a-c+2n)}{(a+n+2\lambda n)(a+\lambda+n+2\lambda n)}\frac{(-1)^n(a-c)_n}{4^{a-c+n}n!}.$$

2.2. Obviously, the relation (2.1) reads also as

$$\sum_{k\geq 0}\binom{n}{2k}\frac{-1/2-c-k-2\lambda k}{(1+a+\lambda n)_k(-1/2-c-n-\lambda n)_{k+1}}\frac{(a-c)_k(2k)!}{(1/2+c+k+2\lambda k)4^k k!}$$
$$= \frac{U(n)}{1/2+c+n+\lambda n}$$

which corresponds to the case

$$\phi(x;n) := (1+a+\lambda x)_n \quad \text{and} \quad \psi(y;n) := \left(-\frac{1}{2}-c-y-\lambda y\right)_n$$

of (1.2a)–(1.2b) with

$$\Omega_n := \frac{U(n)}{1/2+c+n+\lambda n}$$
$$f(n) := \frac{(a-c)_n(2n)!}{(1/2+c+n+2\lambda n)4^n n!}$$
$$g(n) := 0$$
$$h(n) := \frac{\{1+a+n+\lambda(1/2+a-c+2n)\}(a-c)_n(1+2n)!}{(1/2+c+n+2\lambda n)(3/2+c+\lambda+n+2\lambda n)4^n n!}$$

where $h(n)$ is derived from $f(n)$ and $g(n)$ via (1.4). Then (1.3a) and (1.3c) create the following dual formulas:

$$\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}\frac{(1+a+\lambda k)_n(-1/2-c-k-\lambda k)_n}{1/2+c+k+\lambda k}U(k)$$
$$= \frac{(a-c)_n(2n)!}{(1/2+c+n+2\lambda n)4^n n!},$$

$$\sum_{k=0}^{1+2n}(-1)^k\binom{1+2n}{k}\frac{(1+a+\lambda k)_{n+1}(-1/2-c-k-\lambda k)_n}{1/2+c+k+\lambda k}U(k)$$
$$= \frac{\{1+a+n+\lambda(1/2+a-c+2n)\}(a-c)_n(1+2n)!}{(1/2+c+n+2\lambda n)(3/2+c+\lambda+n+2\lambda n)4^n n!}.$$

They can be converted into the following Fox-Wright function identities, respectively.

Proposition 2 (Fox-Wright function identities).

(2.5a)
$$ {}_3\Psi_4^*\left[\begin{matrix} (1+a+n,\lambda), \quad (1/2+a,\lambda+1), \quad (1+2c,2\lambda+1) \\ (1+2n,-1),(1+c,\lambda),(3/2+c-n,\lambda+1),(1+2a,2\lambda+1) \end{matrix}\;\middle|\; -1\right] $$
(2.5b)
$$ = \frac{1}{(1/2+c+n+2\lambda n)}\frac{(-1)^n(a-c)_n}{4^{a-c+n}n!}, $$
(2.6a)
$$ {}_3\Psi_4^*\left[\begin{matrix} (2+a+n,\lambda), \quad (1/2+a,\lambda+1), \quad (1+2c,2\lambda+1) \\ (2+2n,-1),(1+c,\lambda),(3/2+c-n,\lambda+1),(1+2a,2\lambda+1) \end{matrix}\;\middle|\; -1\right] $$
(2.6b)
$$ = \frac{1+a+n+\lambda(1/2+a-c+2n)}{(1/2+c+n+2\lambda n)(3/2+c+\lambda+n+2\lambda n)}\frac{(-1)^n(a-c)_n}{4^{a-c+n}n!} $$
where the evaluation derived from (1.3b) is identical with (2.3a)–(2.3b) and so has been omitted.

2.3. Splitting the factor
$$ a+(1+2\lambda)k = (a+\lambda n+k) - \lambda(n-2k) $$
we may restate (2.1) as
$$ \begin{aligned} \sum_{k\geq 0}\binom{n}{2k} & \frac{-1/2-c-k-2\lambda k}{(a+\lambda n)_k(-1/2-c-n-\lambda n)_{k+1}} \\ & \times \frac{(a-c)_k(2k)!}{(a+k+2\lambda k)(1/2+c+k+2\lambda k)4^k k!} \\ & - \sum_{k\geq 0}\binom{n}{1+2k}\frac{a+\lambda+k+2\lambda k}{(a+\lambda n)_{1+k}(-1/2-c-n-\lambda n)_{k+1}} \\ & \times \frac{-\lambda(a-c)_k(1+2k)!}{(a+k+2\lambda k)(a+\lambda+k+2\lambda k)4^k k!} \\ & = \frac{U(n)}{(a+\lambda n)(1/2+c+n+\lambda n)}. \end{aligned} $$

The last equation fits into (1.2a)–(1.2b) rippingly under the following specifications:

$$\begin{aligned}
\phi(x;n) &:= (a+\lambda x)_n \\
\psi(y;n) &:= \Big(-\frac{1}{2}-c-y-\lambda y\Big)_n \\
\Omega_n &:= \frac{U(n)}{(a+\lambda n)(1/2+c+n+\lambda n)} \\
f(n) &:= \frac{(a-c)_n(2n)!}{(a+n+2\lambda n)(1/2+c+n+2\lambda n)4^n n!} \\
g(n) &:= \frac{-\lambda(a-c)_n(1+2n)!}{(a+n+2\lambda n)(a+\lambda+n+2\lambda n)4^n n!} \\
h(n) &:= \frac{(1+\lambda)(a-c)_n(1+2n)!}{(1/2+c+n+2\lambda n)(3/2+c+\lambda+n+2\lambda n)4^n n!}
\end{aligned}$$

where $h(n)$ is deduced from $f(n)$ and $g(n)$ via (1.4). So (1.3a), (1.3b) and (1.3c) yield the following dual formulas:

$$\begin{aligned}
&\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}\frac{(a+\lambda k)_n(-1/2-c-k-\lambda k)_n}{(a+\lambda k)(1/2+c+k+\lambda k)}U(k) \\
&\quad = \frac{(a-c)_n(2n)!}{(a+n+2\lambda n)(1/2+c+n+2\lambda n)4^n n!}, \\
&\sum_{k=0}^{1+2n}(-1)^k\binom{1+2n}{k}\frac{(a+\lambda k)_n(-1/2-c-k-\lambda k)_{n+1}}{(a+\lambda k)(1/2+c+k+\lambda k)}U(k) \\
&\quad = \frac{-\lambda(a-c)_n(1+2n)!}{(a+n+2\lambda n)(a+\lambda+n+2\lambda n)4^n n!}, \\
&\sum_{k=0}^{1+2n}(-1)^k\binom{1+2n}{k}\frac{(a+\lambda k)_{n+1}(-1/2-c-k-\lambda k)_n}{(a+\lambda k)(1/2+c+k+\lambda k)}U(k) \\
&\quad = \frac{(1+\lambda)(a-c)_n(1+2n)!}{(1/2+c+n+2\lambda n)(3/2+c+\lambda+n+2\lambda n)4^n n!}
\end{aligned}$$

which can be reformulated as the following Fox-Wright function identities, respectively.

Proposition 3 (Fox-Wright function identities).

(2.7a)
$$
{}_3\Psi_4^*\left[\begin{matrix}(a+n,\lambda), \quad (1/2+a,\lambda+1), \quad (1+2c,2\lambda+1)\\ \\ (1+2n,-1),(1+c,\lambda),(3/2+c-n,\lambda+1),(1+2a,2\lambda+1)\end{matrix}\ \middle|\ -1\right]
$$
(2.7b)
$$
=\frac{1}{(a+n+2\lambda n)(1/2+c+n+2\lambda n)}\frac{(-1)^n(a-c)_n}{4^{a-c+n}n!},
$$
(2.8a)
$$
{}_3\Psi_4^*\left[\begin{matrix}(a+n,\lambda), \quad (1/2+a,\lambda+1), \quad (1+2c,2\lambda+1)\\ \\ (2+2n,-1),(1+c,\lambda),(1/2+c-n,\lambda+1),(1+2a,2\lambda+1)\end{matrix}\ \middle|\ -1\right]
$$
(2.8b)
$$
=\frac{\lambda}{(a+n+2\lambda n)(a+\lambda+n+2\lambda n)}\frac{(-1)^n(a-c)_n}{4^{a-c+n}n!},
$$
(2.9a)
$$
{}_3\Psi_4^*\left[\begin{matrix}(1+a+n,\lambda), \quad (1/2+a,\lambda+1), \quad (1+2c,2\lambda+1)\\ \\ (2+2n,-1),(1+c,\lambda),(3/2+c-n,\lambda+1),(1+2a,2\lambda+1)\end{matrix}\ \middle|\ -1\right]
$$
(2.9b)
$$
=\frac{1+\lambda}{(1/2+c+n+2\lambda n)(3/2+c+\lambda+n+2\lambda n)}\frac{(-1)^n(a-c)_n}{4^{a-c+n}n!}.
$$

2.4. According to the factor-splitting

$$
a+(1+2\lambda)k=\frac{c+2\lambda k}{c+\lambda n}(a+\lambda n+k)+\frac{\lambda(a-c+k)}{c+\lambda n}(n-2k)
$$

the relation (2.1) may be expressed as

$$
\begin{aligned}
\sum_{k\geq 0}\binom{n}{2k}&\frac{-1/2-c-k-2\lambda k}{(a+\lambda n)_k(-1/2-c-n-\lambda n)_{k+1}}\\
&\qquad\times\frac{(c+2\lambda k)(a-c)_k(2k)!}{(a+k+2\lambda k)(1/2+c+k+2\lambda k)4^k k!}\\
&\qquad\quad-\sum_{k\geq 0}\binom{n}{1+2k}\frac{a+\lambda+k+2\lambda k}{(a+\lambda n)_{1+k}(-1/2-c-n-\lambda n)_{k+1}}
\end{aligned}
$$

$$\times \frac{\lambda(a-c)_{k+1}(1+2k)!}{(a+k+2\lambda k)(a+\lambda+k+2\lambda k)4^k k!}$$
$$= \frac{(c+\lambda n)U(n)}{(a+\lambda n)(1/2+c+n+\lambda n)}.$$

The last identity adapts to (1.2a)–(1.2b) ideally under the following specifications:

$$\phi(x;n) := (a+\lambda x)_n,$$
$$\psi(y;n) := \Big(-\frac{1}{2}-c-y-\lambda y\Big)_n$$
$$\Omega_n := \frac{(c+\lambda n)U(n)}{(a+\lambda n)(1/2+c+n+\lambda n)}$$
$$f(n) := \frac{(c+2\lambda n)(a-c)_n(2n)!}{(a+n+2\lambda n)(1/2+c+n+2\lambda n)4^n n!}$$
$$g(n) := \frac{\lambda(a-c)_{n+1}(1+2n)!}{(a+n+2\lambda n)(a+\lambda+n+2\lambda n)4^n n!}$$
$$h(n) := \frac{(c+\lambda n-\lambda/2)(a-c)_n(1+2n)!}{(1/2+c+n+2\lambda n)(3/2+c+\lambda+n+2\lambda n)4^n n!}$$

where $h(n)$ is derived from $f(n)$ and $g(n)$ via (1.4). In view of (1.3a), (1.3b) and (1.3c), we can have the following dual formulas:

$$\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}\frac{(c+\lambda k)(a+\lambda k)_n(-1/2-c-k-\lambda k)_n}{(a+\lambda k)(1/2+c+k+\lambda k)}U(k)$$
$$= \frac{(c+2\lambda n)(a-c)_n(2n)!}{(a+n+2\lambda n)(1/2+c+n+2\lambda n)4^n n!},$$
$$\sum_{k=0}^{1+2n}(-1)^k\binom{1+2n}{k}\frac{(c+\lambda k)(a+\lambda k)_n(-1/2-c-k-\lambda k)_{n+1}}{(a+\lambda k)(1/2+c+k+\lambda k)}U(k)$$
$$= \frac{\lambda(a-c)_{n+1}(1+2n)!}{(a+n+2\lambda n)(a+\lambda+n+2\lambda n)4^n n!},$$
$$\sum_{k=0}^{1+2n}(-1)^k\binom{1+2n}{k}\frac{(c+\lambda k)(a+\lambda k)_{n+1}(-1/2-c-k-\lambda k)_n}{(c+\lambda k)(1/2+c+k+\lambda k)}U(k)$$
$$= \frac{(c+\lambda n-\lambda/2)(a-c)_n(1+2n)!}{(1/2+c+n+2\lambda n)(3/2+c+\lambda+n+2\lambda n)4^n n!}$$

which can be recomposed as the following Fox-Wright function identities, respectively.

Proposition 4 (Fox-Wright function identities).

(2.10a)
$$ {}_3\Psi_4^*\left[\begin{matrix}(a+n,\lambda), & (1/2+a,\lambda+1), & (1+2c,2\lambda+1)\\ (1+2n,-1),(c,\lambda), & (3/2+c-n,\lambda+1), & (1+2a,2\lambda+1)\end{matrix}\,\middle|\,-1\right] $$
(2.10b)
$$ =\frac{c+2\lambda n}{(a+n+2\lambda n)(1/2+c+n+2\lambda n)}\frac{(-1)^n(a-c)_n}{4^{a-c+n}n!}, $$
(2.11a)
$$ {}_3\Psi_4^*\left[\begin{matrix}(a+n,\lambda), & (1/2+a,\lambda+1), & (1+2c,2\lambda+1)\\ (2+2n,-1),(c,\lambda), & (1/2+c-n,\lambda+1), & (1+2a,2\lambda+1)\end{matrix}\,\middle|\,-1\right] $$
(2.11b)
$$ =\frac{\lambda(c-a-n)}{(a+n+2\lambda n)(a+\lambda+n+2\lambda n)}\frac{(-1)^n(a-c)_n}{4^{a-c+n}n!}, $$
(2.12a)
$$ {}_3\Psi_4^*\left[\begin{matrix}(1+a+n,\lambda), & (1/2+a,\lambda+1), & (1+2c,2\lambda+1)\\ (2+2n,-1),(c,\lambda), & (3/2+c-n,\lambda+1), & (1+2a,2\lambda+1)\end{matrix}\,\middle|\,-1\right] $$
(2.12b)
$$ =\frac{c+\lambda n-\lambda/2}{(1/2+c+n+2\lambda n)(3/2+c+\lambda+n+2\lambda n)}\frac{(-1)^n(a-c)_n}{4^{a-c+n}n!}. $$

2.5. Now we consider a more general case. For a complex number β, the factor splitting

$$ a+(1+2\lambda)k=\frac{\beta+2k}{\beta+n}(a+\lambda n+k)-\frac{\beta\lambda-a-k}{\beta+n}(n-2k) $$

leads the relation (2.1) to

$$\begin{aligned}
\sum_{k\geq 0}\binom{n}{2k}&\frac{-1/2-c-k-2\lambda k}{(a+\lambda n)_k(-1/2-c-n-\lambda n)_{k+1}}\\
&\times\frac{(\beta+2k)(a-c)_k(2k)!}{(a+k+2\lambda k)(1/2+c+k+2\lambda k)4^k k!}\\
&\quad-\sum_{k\geq 0}\binom{n}{1+2k}\frac{a+\lambda+k+2\lambda k}{(a+\lambda n)_{1+k}(-1/2-c-n-\lambda n)_{k+1}}\\
&\times\frac{(a-\beta\lambda+k)(a-c)_k(1+2k)!}{(a+k+2\lambda k)(a+\lambda+k+2\lambda k)4^k k!}\\
&=\frac{(\beta+n)U(n)}{(a+\lambda n)(1/2+c+n+\lambda n)}.
\end{aligned}$$

The last relation matches with (1.2a)–(1.2b) perfectly under the following specifications:

$$\begin{aligned}
\phi(x;n)&:=(a+\lambda x)_n,\\
\psi(y;n)&:=\left(-\frac{1}{2}-c-y-\lambda y\right)_n\\
[illegible]&:=\frac{(\beta+n)U(n)}{(a+\lambda n)(1/2+c+n+\lambda n)}\\
f(n)&:=\frac{(\beta+2n)(a-c)_n(2n)!}{(a+n+2\lambda n)(1/2+c+n+2\lambda n)4^n n!}\\
g(n)&:=\frac{(a-\beta\lambda+n)(a-c)_n(1+2n)!}{(a+n+2\lambda n)(a+\lambda+n+2\lambda n)4^n n!}\\
h(n)&:=\frac{(n+\beta+\lambda\beta-c-1/2)(a-c)_n(1+2n)!}{(1/2+c+n+2\lambda n)(3/2+c+\lambda+n+2\lambda n)4^n n!}
\end{aligned}$$

where $h(n)$ is obtained from $f(n)$ and $g(n)$ via (1.4), we may write down, on account of (1.3a), (1.3b) and (1.3c), the following dual formulas:

$$\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}\frac{(\beta+k)(a+\lambda k)_n(-1/2-c-k-\lambda k)_n}{(a+\lambda k)(1/2+c+k+\lambda k)}U(k)$$
$$=\frac{(\beta+2n)(a-c)_n(2n)!}{(a+n+2\lambda n)(1/2+c+n+2\lambda n)4^n n!},$$
$$\sum_{k=0}^{1+2n}(-1)^k\binom{1+2n}{k}\frac{(\beta+k)(a+\lambda k)_n(-1/2-c-k-\lambda k)_{n+1}}{(a+\lambda k)(1/2+c+k+\lambda k)}U(k)$$
$$=\frac{(a-\beta\lambda+n)(a-c)_n(1+2n)!}{(a+n+2\lambda n)(a+\lambda+n+2\lambda n)4^n n!},$$
$$\sum_{k=0}^{1+2n}(-1)^k\binom{1+2n}{k}\frac{(\beta+k)(a+\lambda k)_n(-1/2-c-k-\lambda k)_n}{(a+\lambda k)(1/2+c+k+\lambda k)}U(k)$$
$$=\frac{(n+\beta+\lambda\beta-c-1/2)(a-c)_n(1+2n)!}{(1/2+c+n+2\lambda n)(3/2+c+\lambda+n+2\lambda n)4^n n!}$$

which may be expressed as the following Fox-Wright function identities with one additional parameter β, respectively.

Theorem 5 (Fox-Wright function identities).

(2.13a)
$${}_4\Psi_5^*\left[\begin{matrix}(1+\beta,1),(a+n,\lambda),(1/2+a,\lambda+1),(1+2c,2\lambda+1)\\ \\ (1+2n,-1),(\beta,1),(1+c,\lambda),(3/2+c-n,\lambda+1),(1+2a,2\lambda+1)\end{matrix}\middle|\,-1\right]$$

(2.13b)
$$=\frac{\beta+2n}{(a+n+2\lambda n)(1/2+c+n+2\lambda n)}\frac{(-1)^n(a-c)_n}{4^{a-c+n}n!},$$

(2.14a)
$${}_4\Psi_5^*\left[\begin{matrix}(1+\beta,1),(a+n,\lambda),(1/2+a,\lambda+1),(1+2c,2\lambda+1)\\ \\ (2+2n,-1),(\beta,1),(1+c,\lambda),(1/2+c-n,\lambda+1),(1+2a,2\lambda+1)\end{matrix}\middle|\,-1\right]$$

(2.14b)
$$=\frac{\lambda\beta-a-n}{(a+n+2\lambda n)(a+\lambda+n+2\lambda n)}\frac{(-1)^n(a-c)_n}{4^{a-c+n}n!},$$

(2.15a)

$$ {}_4\Psi_5^*\left[\begin{matrix}(1+\beta,1),(1+a+n,\lambda),(1/2+a,\lambda+1),(1+2c,2\lambda+1)\\ (2+2n,-1),(\beta,1),(1+c,\lambda),(3/2+c-n,\lambda+1),(1+2a,2\lambda+1)\end{matrix}\middle|\,-1\right] $$

(2.15b)

$$ =\frac{n+\beta+\lambda\beta-c-1/2}{(1/2+c+n+2\lambda n)(3/2+c+\lambda+n+2\lambda n)}\frac{(-1)^n(a-c)_n}{4^{a-c+n}n!}. $$

Propositions 1–4 are all limiting cases of Theorem 5. Their relations are displayed as follows:

$$ \begin{aligned} \beta=(1/2+c-n)/(1+\lambda):\ &(2.13a)\text{–}(2.13b)\Rightarrow(2.2a)-(2.2b)\\ &(2.15a)\text{–}(2.15b)\Rightarrow(2.3a)-(2.3b)\\ \beta=(-1/2+c-n)/(1+\lambda):\ &(2.14a)\text{–}(2.14b)\Rightarrow(2.4a)-(2.4b)\\ \beta=(a+n)/\lambda:\ &(2.13a)\text{–}(2.13b)\Rightarrow(2.5a)-(2.5b)\\ \beta=(1+a+n)/\lambda:\ &(2.15a)\text{–}(2.15b)\Rightarrow(2.6a)-(2.6b)\\ \beta\longrightarrow\infty:\ &(2.13a)\text{–}(2.15b)\Rightarrow(2.7a)-(2.9b)\\ \beta=c/\lambda:\ &(2.13a)\text{–}(2.15b)\Rightarrow(2.10a)\text{–}(2.12b). \end{aligned} $$

Taking $\beta=(1+2c)/(1+2\lambda)$, $\beta=(2a)/(1+2\lambda)$ and $\beta=(1+2a)/(2+2\lambda)$, respectively, in Theorem 5, we gain the remaining three cases.

Proposition 6 (Fox-Wright function identities).

$$ {}_3\Psi_4^*\left[\begin{matrix}(a+n,\lambda),\quad(1/2+a,\lambda+1),\quad(2+2c,2\lambda+1)\\ (1+2n,-1),(1+c,\lambda),(3/2+c-n,\lambda+1),(1+2a,2\lambda+1)\end{matrix}\middle|\,-1\right] $$

$$ =\frac{2}{(a+n+2\lambda n)}\frac{(-1)^n(a-c)_n}{4^{a-c+n}n!}, $$

$$ {}_3\Psi_4^*\left[\begin{matrix}(a+n,\lambda),\quad(1/2+a,\lambda+1),\quad(2+2c,2\lambda+1)\\ (2+2n,-1),(1+c,\lambda),(1/2+c-n,\lambda+1),(1+2a,2\lambda+1)\end{matrix}\middle|\,-1\right] $$

$$ =\frac{\lambda(1+2c-2a-2n)-a-n}{(a+n+2\lambda n)(a+\lambda+n+2\lambda n)}\frac{(-1)^n(a-c)_n}{4^{a-c+n}n!}, $$

$$ {}_3\Psi_4^*\left[\begin{matrix}(1+a+n,\lambda),\quad(1/2+a,\lambda+1),\quad(2+2c,2\lambda+1)\\ (2+2n,-1),(1+c,\lambda),(3/2+c-n,\lambda+1),(1+2a,2\lambda+1)\end{matrix}\middle|\,-1\right] $$

$$= \frac{1}{(3/2 + c + \lambda + n + 2\lambda n)} \frac{(-1)^n (a-c)_n}{4^{a-c+n} n!}.$$

Proposition 7 (Fox-Wright function identities).

$$ {}_3\Psi_4^* \left[\begin{matrix} (a+n, \lambda), \quad (1/2+a, \lambda+1), \quad (1+2c, 2\lambda+1) \\ \\ (1+2n, -1), (1+c, \lambda), (3/2+c-n, \lambda+1), (2a, 2\lambda+1) \end{matrix} \middle| \; -1 \right]$$

$$= \frac{2}{(1/2 + c + n + 2\lambda n)} \frac{(-1)^n (a-c)_n}{4^{a-c+n} n!},$$

$$ {}_3\Psi_4^* \left[\begin{matrix} (a+n, \lambda), \quad (1/2+a, \lambda+1), \quad (1+2c, 2\lambda+1) \\ \\ (2+2n, -1), (1+c, \lambda), (1/2+c-n, \lambda+1), (2a, 2\lambda+1) \end{matrix} \middle| \; -1 \right]$$

$$= \frac{-1}{(a + \lambda + n + 2\lambda n)} \frac{(-1)^n (a-c)_n}{4^{a-c+n} n!},$$

$$ {}_3\Psi_4^* \left[\begin{matrix} (1+a+n, \lambda), \quad (1/2+a, \lambda+1), \quad (1+2c, 2\lambda+1) \\ \\ (2+2n, -1), (1+c, \lambda), (3/2+c-n, \lambda+1), (2a, 2\lambda+1) \end{matrix} \middle| \; -1 \right]$$

$$= \frac{2a(1+\lambda) + (1+2\lambda)(a-c-1/2)}{(1/2 + c + n + 2\lambda n)(3/2 + c + \lambda + n + 2\lambda n)} \frac{(-1)^n (a-c)_n}{4^{a-c+n} n!}.$$

Proposition 8 (Fox-Wright function identities).

$$ {}_3\Psi_4^* \left[\begin{matrix} (a+n, \lambda), \quad (3/2+a, \lambda+1), \quad (1+2c, 2\lambda+1) \\ \\ (1+2n, -1), (1+c, \lambda), (3/2+c-n, \lambda+1), (1+2a, 2\lambda+1) \end{matrix} \middle| \; -1 \right]$$

$$= \frac{1/2 + a + 2n + 2\lambda n}{(a + n + 2\lambda n)(1/2 + c + n + 2\lambda n)} \frac{(-1)^n (a-c)_n}{4^{a-c+n} n!},$$

$$ {}_3\Psi_4^* \left[\begin{matrix} (a+n, \lambda), \quad (3/2+a, \lambda+1), \quad (1+2c, 2\lambda+1) \\ \\ (2+2n, -1), (1+c, \lambda), (1/2+c-n, \lambda+1), (1+2a, 2\lambda+1) \end{matrix} \middle| \; -1 \right]$$

$$= \frac{\lambda/2 - a - n - \lambda n}{(a + n + 2\lambda n)(a + \lambda + n + 2\lambda n)} \frac{(-1)^n (a-c)_n}{4^{a-c+n} n!},$$

$$ {}_3\Psi_4^* \left[\begin{matrix} (1+a+n, \lambda), \quad (3/2+a, \lambda+1), \quad (1+2c, 2\lambda+1) \\ \\ (2+2n, -1), (1+c, \lambda), (3/2+c-n, \lambda+1), (1+2a, 2\lambda+1) \end{matrix} \middle| \; -1 \right]$$

$$= \frac{(1+\lambda)(a-c+n)}{(1/2+c+n+2\lambda n)(3/2+c+\lambda+n+2\lambda n)} \frac{(-1)^n (a-c)_n}{4^{a-c+n} n!}.$$

3. Linear combinations and further Fox-Wright function identities. Performing replacement $n \to n-1$ for (2.14a)–(2.14b) and (2.15a)–(2.15b), respectively, the resulting expressions read as

(3.1a)
$$ {}_4\Psi_5^* \left[\begin{matrix} (1+\beta,1),(a-1+n,\lambda),(1/2+a,\lambda+1),(1+2c,2\lambda+1) \\ (2n,-1),(\beta,1),(1+c,\lambda),(3/2+c-n,\lambda+1),(1+2a,2\lambda+1) \end{matrix} \,\middle|\, -1 \right] $$
$$= \frac{4n(a-\lambda\beta-1+n)}{(a-c-1+n)(a-\lambda-1+n+2\lambda n)(a-2\lambda-1+n+2\lambda n)}$$

(3.1b)
$$\times \frac{(-1)^n (a-c)_n}{4^{a-c+n} n!},$$

(3.2a)
$$ {}_4\Psi_5^* \left[\begin{matrix} (1+\beta,1),(a+n,\lambda),(1/2+a,\lambda+1),(1+2c,2\lambda+1) \\ (2n,-1),(\beta,1),(1+c,\lambda),(3/2+c-n,\lambda+1),(1+2a,2\lambda+1) \end{matrix} \,\middle|\, -1 \right] $$
$$= \frac{4n(3/2+c-\beta-\lambda\beta-n)}{(a-c-1+n)(c-\lambda+1/2+n+2\lambda n)(c-2\lambda-1/2+n+2\lambda n)}$$

(3.2b)
$$\times \frac{(-1)^n (a-c)_n}{4^{a-c+n} n!}.$$

By means of the linear combinations of (2.13a)–(2.13b) with (3.1a)–(3.1b), (2.13a)–(2.13b) with (3.2a)–(3.2b) and (2.13a)–(2.13b) with (2.14a)–(2.14b), respectively, we can establish the following Fox-Wright function identities with two additional parameters β and γ.

Theorem 9 (Fox-Wright function identities).

$$ {}_5\Psi_6^* \left[\begin{matrix} (1+\beta,1),(1+\gamma,1),(a-1+n,\lambda),(1/2+a,\lambda+1),(1+2c,2\lambda+1) \\ (1+2n,-1),(\beta,1),(\gamma,1),(1+c,\lambda),(3/2+c-n,\lambda+1),(1+2a,2\lambda+1) \end{matrix} \,\middle|\, -1 \right] $$

$$= \frac{(\gamma+2n)R_n + (1+\lambda\gamma-a-n)T_n}{(a-1+n+2\lambda n)} \frac{(-1)^n (a-c)_n}{4^{a-c+n} n!},$$

$$_5\Psi_6^*\left[\begin{array}{c|c} (1+\beta,1),(1+\gamma,1),(a+n,\lambda),(1/2+a,\lambda+1),(1+2c,2\lambda+1) & \\ & -1 \\ (1+2n,-1),(\beta,1),(\gamma,1),(1+c,\lambda),(5/2+c-n,\lambda+1),(1+2a,2\lambda+1) & \end{array}\right]$$

$$= \frac{(\gamma+2n)R_n + (\gamma+\lambda\gamma-3/2-c+n)W_n}{(3/2+c+n+2\lambda n)} \frac{(-1)^n (a-c)_n}{4^{a-c+n} n!},$$

$$_5\Psi_6^*\left[\begin{array}{c|c} (1+\beta,1),(1+\gamma,1),(a+n,\lambda),(1/2+a,\lambda+1),(1+2c,2\lambda+1) & \\ & -1 \\ (2+2n,-1),(\beta,1),(\gamma,1),(1+c,\lambda),(3/2+c-n,\lambda+1),(1+2a,2\lambda+1) & \end{array}\right]$$

$$= \frac{(1+\gamma+2n)S_n + (\gamma+\lambda\gamma-1/2-c+n)R_n}{(3/2+c+\lambda+n+2\lambda n)} \frac{(-1)^n (a-c)_n}{4^{a-c+n} n!}$$

where

$$R_n = \frac{\beta+2n}{(a+n+2\lambda n)(1/2+c+n+2\lambda n)},$$

$$S_n = \frac{\lambda\beta - \text{[illegible]}}{(a+n+2\lambda n)(a+\lambda+n+2\lambda n)},$$

$$T_n = \frac{4n(a-\lambda\beta-1+n)}{(a-c-1+n)(a-\lambda-1+n+2\lambda n)(a-2\lambda-1+n+2\lambda n)},$$

$$W_n = \frac{4n(3/2+c-\beta-\lambda\beta-n)}{(a-c-1+n)(c-\lambda+1/2+n+2\lambda n)(c-2\lambda-1/2+n+2\lambda n)}.$$

The kind of methods used for establishing Theorem 9 can be further employed to deduce ${}_6\Psi_7^*$-function identities. From Theorem 9, we can derive $24 = 3\binom{8}{1}$ ${}_4\Psi_5^*$-function identities with one additional parameter and $108 = 3\binom{9}{2}$ ${}_3\Psi_4^*$-function identities. For the limit of space, the details will not be reproduced.

It is worth mentioning that all the Fox-Wright function identities appearing in this paper will reduce to hypergeometric series identities when λ takes arbitrary integer m. With the change of m, numerous hypergeometric series identities can be established. For instance, the case $\lambda = 1$ of Proposition 1 can be displayed as follows:

$$
{}_7F_6\left[\begin{array}{ccccccc} -2n, & a+n, & \frac{1+2c}{3}, & \frac{2+2c}{3}, & \frac{3+2c}{3}, & \frac{1+2a}{4}, & \frac{3+2a}{4} \\ & 1+c, & \frac{1+2a}{3}, & \frac{2+2a}{3}, & \frac{3+2a}{3}, & \frac{1+2c-2n}{4}, & \frac{3+2c-2n}{4} \end{array}\middle|\, 1\right]
= \frac{a}{a+3n}\frac{(1/2)_n(a-c)_n}{(1/2-c)_n(a)_n},
$$

$$
{}_7F_6\left[\begin{array}{ccccccc} -1-2n, & 1+a+n, & \frac{1+2c}{3}, & \frac{2+2c}{3}, & \frac{3+2c}{3}, & \frac{1+2a}{4}, & \frac{3+2a}{4} \\ & 1+c, & \frac{1+2a}{3}, & \frac{2+2a}{3}, & \frac{3+2a}{3}, & \frac{1+2c-2n}{4}, & \frac{3+2c-2n}{4} \end{array}\middle|\, 1\right]
= 0,
$$

$$
{}_7F_6\left[\begin{array}{ccccccc} -1-2n, & a+n, & \frac{1+2c}{3}, & \frac{2+2c}{3}, & \frac{3+2c}{3}, & \frac{1+2a}{4}, & \frac{3+2a}{4} \\ & 1+c, & \frac{1+2a}{3}, & \frac{2+2a}{3}, & \frac{3+2a}{3}, & \frac{1+2c-2n}{4}, & \frac{3+2c-2n}{4} \end{array}\middle|\, 1\right]
= \frac{a(1-2c+4a+6n)}{(1-2c)(a+3n)_2}\frac{(3/2)_n(a-c)_n}{(3/2-c)_n(a)_n}.
$$

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Department of Information Technology, Hainan Medical College, Haikou 571101, P.R. China
Email address: weichuanan@yahoo.com.cn