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ONE-SIDED L^p NORM AND BEST APPROXIMATION IN ONE-SIDED L^p NORM

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ABSTRACT. Let f be a p integrable function on K, a compact subset of $\mathbf{R}$, and μ a σ-finite positive measure. For $p > 1$, the one-sided L^p norm is defined as follows:

$$||f||_p = \max\left\{\left(\int_{\{f>0\}} |f|^p d\mu\right)^{1/p}, \left(\int_{\{f<0\}} |f|^p d\mu\right)^{1/p}\right\}.$$

We first show that the above definition is indeed a norm and then study the best approximation in the one-sided L^p norms. Among others, characterization and uniqueness of best approximation are discussed.

1. The one-sided L^p norm. In the classical L^1 norm, The norm of a function $f(x)$ equals the sum of the area above the x-axis and area below the x-axis bounded by $f(x)$. In approximation, this area bounded by the two functions measures distance between the two functions. Suppose we prefer a more balanced approximation from above and below, but still like to use the area as a measure. Then the following norm, which I call 'one-sided L^1 norm,' may be a good choice. The norm equals the larger one of the above two areas. For example, if the cost of an error is based more on the larger one of the two areas than the total area. Also, these one-sided norms are more consistent with the supremum norm which is the larger one of the deviations from above and below the x-axis instead of the sum of the deviations. In [**7**] this one-sided L^1 norm and the best approximation in this norm are studied. In this paper, I introduce 'one-sided L^p norm' defined similarly, and discuss the best approximation in the one-sided L^p norm.

Let K be a compact subset of R and μ a σ-finite positive measure. (It can be easily generalized to a metric space.)

Definition 1. Let $f \in L^P$, the set of all real functions whose pth power is integrable over K. For $p \geq 1$, the one-sided L^p norm is defined

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as follows:

$$\|f\|_p = \max\left\{\left(\int_{\{f>0\}} |f|^p d\mu\right)^{1.p}, \ \left(\int_{\{f<0\}} |f|^p d\mu\right)^{1/p}\right\}.$$

In the following discussion, $d\mu$ will be omitted in all formulas.

We need first to show that the above definition does define a norm.

Lemma 2. *For any nonnegative real numbers a, b, c, d, p and q with $(1/p)+(1/q)=1$, the following inequality holds*:

$$a^{1/p}b^{1/q} + c^{1/p}d^{1/q} \le (a+c)^{1/p}(b+d)^{1/q}.$$

Proof. For nonnegative real numbers z and x with $0 \le x \le 1$, define the function $f(t)$, $t \ge 0$ as follows:

$$f(t) = (1+z)\left(\frac{1+zt}{1+z}\right)^x - (1+zt^x).$$

Then

$$\begin{aligned} f'(t) &= (1+z)x\left(\frac{1+zt}{1+z}\right)^{x-1}\left(\frac{z}{1+z}\right) - zxt^{x-1} \\ &= \frac{xz}{(1+z)^{x-1}}[(1+zt)^{x-1} - (1+z)^{x-1}t^{x-1}]. \end{aligned}$$

Since $1+zt \ge (1+z)t$ when $0 \le t \le 1$, $1+zt \le (1+z)t$ when $t \ge 1$, and $x-1 \le 0$, $f'(t) \le 0$ for $0 \le t \le 1$ and $f'(t) \ge 0$ for $t \ge 1$. This shows that $f(t)$ has a minimal value 0 at $t=1$, and thus $f(t) \ge 0$ for all $t \ge 0$.

Letting $z = c/a$, $t = (ad)/(bc)$ and $x = 1/q$, we have

$$\begin{aligned} f\left(\frac{ad}{bc}\right) &= \left(1+\frac{c}{a}\right)\left(\frac{1+(c/a)(ad/bc)}{1+(c/a)}\right)^{1/q} - \left(1+\frac{c}{a}\left(\frac{ad}{bc}\right)^{1/q}\right) \\ &= \left(1+\frac{c}{a}\right)\left(\frac{a+(ad/b)}{a+c}\right)^{1/q} - \left(1+\frac{c}{a}\left(\frac{ad}{bc}\right)^{1/q}\right) \ge 0, \end{aligned}$$

or

$$1+\frac{c}{a}\left(\frac{ad}{bc}\right)^{1/q} \leq \left(1+\frac{c}{a}\right)\left(\frac{a+(ad/b)}{a+c}\right)^{1/q}.$$

Multiplying both sides by $a(b/a)^{1/q} = a^{1/p}b^{1/q}$, we get

$$a^{1/p}b^{1/q} + c^{1/p}d^{1/q} \leq (a+c)^{1/p}(b+d)^{1/q}. \qquad \square$$

Theorem 3. *For $p \geq 1$, the one-sided L^p norm defined above is indeed a norm.*

Proof. $||f||_p \geq 0$ and $||cf||_p = |c|||f||_p$ are obvious. Clearly the norm of f equal 0 implies f is 0. We only need to verify $||f+g||_p \leq ||f||_p + ||g||_p$ for any f and g. Let $D_1 = \{f+g>0\} \cap \{f>0\}$, $D_2 = \{f+g>0\} \cap \{f \leq 0\}$, $D_3 = \{f+g>0\} \cap \{g>0\}$, and $D_4 = \{f+g>0\} \cap \{g \leq 0\}$. Then

$$\begin{aligned}
\int_{\{f+g>0\}} |f+g|^p &= \int_{D_1\cap D_3} |f+g|^p + \int_{D_1\cap D_4} |f+g|^p \\
&\quad + \int_{D_2\cap D_3} |f+g|^p + \int_{D_2\cap D_4} |f+g|^p \\
&= \left(\int_{D_1\cap D_3} |f+g|^p\right)^{1/p}\left(\int_{D_1\cap D_3} |f+g|^p\right)^{1/q} \\
&\quad + \left(\int_{D_1\cap D_4} |f+g|^p\right)^{1/p}\left(\int_{D_1\cap D_4} |f+g|^p\right)^{1/q} \\
&\quad + \left(\int_{D_2\cap D_3} |f+g|^p\right)^{1/p}\left(\int_{D_2\cap D_3} |f+g|^p\right)^{1/q} \\
&\leq \left[\left(\int_{D_1\cap D_3} |f|^p\right)^{1/p} + \left(\int_{D_1\cap D_3} |g|^p\right)^{1/p}\right] \\
&\quad \times \left(\int_{D_1\cap D_3} |f+g|^p\right)^{1/q} \\
&\quad + \left(\int_{D_1\cap D_4} |f|^p\right)^{1/p}\left(\int_{D_1\cap D_4} |f+g|^p\right)^{1/q} \\
&\quad + \left(\int_{D_2\cap D_3} |g|^p\right)^{1/p}\left(\int_{D_2\cap D_3} |f+g|^p\right)^{1/q}
\end{aligned}$$

$$
\begin{aligned}
&= \left(\int_{D_1 \cap D_3} |f|^p \right)^{1/p} \left(\int_{D_1 \cap D_3} |f+g|^p \right)^{1/q} \\
&\quad + \left(\int_{D_1 \cap D_4} |f|^p \right)^{1/p} \left(\int_{D_1 \cap D_4} |f+g|^p \right)^{1/q} \\
&\quad + \left(\int_{D_1 \cap D_3} |g|^p \right)^{1/p} \left(\int_{D_1 \cap D_3} |f+g|^p \right)^{1/q} \\
&\quad + \left(\int_{D_2 \cap D_3} |g|^p \right)^{1/p} \left(\int_{D_2 \cap D_3} |f+g|^p \right)^{1/q} \\
&\le \left(\int_{D_1} |f|^p \right)^{1/p} \left(\int_{D_1} |f+g|^p \right)^{1/q} \\
&\quad + \left(\int_{D_3} |g|^p \right)^{1/p} \left(\int_{D_3} |f+g|^p \right)^{1/q} \\
&\le \left(\int_{D_1} |f|^p \right)^{1/p} \left(\int_{\{f+g>0\}} |f+g|^p \right)^{1/q} \\
&\quad + \left(\int_{D_3} |g|^p \right)^{1/p} \left(\int_{\{f+g>0\}} |f+g|^p \right)^{1/q}.
\end{aligned}
$$

The second to the last inequality uses Lemma 2. Divide both sides by

$$\left(\int_{\{f+g>0\}} |f+g|^p \right)^{1/q}$$

and get

$$
\begin{aligned}
\left(\int_{\{f+g>0\}} |f+g|^p \right)^{1/p} &\le \left(\int_{D_1} |f|^p \right)^{1/p} + \left(\int_{D_3} |g|^p \right)^{1/p} \\
&\le ||f||_p + ||g||_p.
\end{aligned}
$$

Similarly we can show

$$\left(\int_{\{f+g<0\}} |f+g|^p \right)^{1/p} \le ||f||_p + ||g||_p.$$

This proves the theorem. $\square$

Remark. In the above proof, if $(\int_{\{f+g>0\}} |f+g|^p)^{1/p} = ||f+g||_p = ||f||_p + ||g||_p$, then all inequalities become equalities. From

$(\int_{D_1} |f|^p)^{1/p} + (\int_{D_3} |g|^p)^{1/p} = (\int_{\{f>0\}} |f|^p)^{1/p} + (\int_{\{g>0\}} |g|^p)^{1/p}$, we can get $\{f > 0\} \subset \{f + g > 0\}$ and $\{g > 0\} \subset \{f + g > 0\}$. From $(\int_{D_1} |f+g|^p)^{1/q} = (\int_{\{f+g>0\}} |f+g|^p)^{1/q}$ and $(\int_{D_3} |f+g|^p)^{1/q} = (\int_{\{f+g>0\}} |f+g|^p)^{1/q}$, we can get $\{f + g > 0\} \subset \{f > 0\}$ and $\{f + g > 0\} \subset \{g > 0\}$. This shows $\{f + g > 0\} = \{f > 0\} = \{g > 0\}$. Finally, from $(\int_{\{f+g>0\}} |f+g|^p)^{1/p} = (\int_{\{f>0\}} |f|^p)^{1/p} + (\int_{\{g>0\}} |g|^p)^{1/p}$, we can get $g(x) = c_1 f(x)$ almost everywhere on $\{f > 0\} = \{g > 0\} = \{f + g > 0\}$. Similarly, we can get $g(x) = c_2 f(x)$ almost everywhere on $\{f < 0\} = \{g < 0\} = \{f + g < 0\}$, if $(\int_{\{f+g<0\}} |f + g|^p)^{1/p} = (\int_{\{f<0\}} |f|^p)^{1/p} + (\int_{\{g<0\}} |g|^p)^{1/p}$.

From the definition, the following properties are easy to see:

Theorem 4. *Let $||f||_{L^p}$ denote the regular L^p norm of f and $||f||_\infty$ the supremum norm. Then*

$$||f||_p \le ||f||_{L^p} \le \sqrt[p]{2}||f||_p, \qquad \lim_{p\to 1} ||f||_p = ||f||_1,$$

and

$$\lim_{p\to\infty} ||f||_p = ||f||_\infty.$$

Next we characterize the dual space of L^p in the one-sided L^p norm.

Theorem 5. *Let X be the space of all p integrable functions on K with the one-sided L^p norm. $(p > 1)$. Then $X^* = L^q$ $(1/p + 1/q = 1)$ and for each $h \in L^q$, $H \in X^*$ is defined by*

$$\begin{aligned} H(g) = \int hg \text{ and } |||H||| \\ = \left(\int_{\{h>0\}} |h|^q d\mu\right)^{1/q} + \left(\int_{\{h<0\}} |h|^q d\mu\right)^{1/q}. \end{aligned}$$

Proof. Since $(L^p)^* = L^q$ is well known in the usual norm and by Theorem 4 these two norms are equivalent, we need only to verify the norm. For any $g \in L^q$,

$$\begin{aligned}
|h(g)| = \left|\int hg\right| &= \left|\int_{\{h>0\}\cap\{g>0\}} hg + \int_{\{h>0\}\cap\{g<0\}} hg \right.\\
&\quad \left. + \int_{\{h<0\}\cap\{g>0\}} hg + \int_{\{h<0\}\cap\{g<0\}} hg\right|\\
&\leq \max\left\{\left|\int_{\{h>0\}\cap\{g>0\}} hg + \int_{\{h<0\}\cap\{g<0\}} hg\right|,\right.\\
&\qquad \left.\left|\int_{\{h<0\}\cap\{g>0\}} hg + \int_{\{h>0\}\cap\{g<0\}} hg\right|\right\}\\
&\leq \max\left\{\left(\int_{\{h>0\}\cap\{g>0\}} |g|^p\right)^{1/p}\left(\int_{\{h>0\}\cap\{g>0\}} |h|^q\right)^{1/q}\right.\\
&\quad + \left(\int_{\{h<0\}\cap\{g<0\}} |g|^p\right)^{1/p}\left(\int_{\{h<0\}\cap\{g<0\}} |h|^q\right)^{1/q},\\
&\quad \left(\int_{\{h>0\}\cap\{g<0\}} |g|^p\right)^{1/p}\left(\int_{\{h>0\}\cap\{g<0\}} |h|^q\right)^{1/q}\\
&\quad \left. + \left(\int_{\{h<0\}\cap\{g>0\}} |g|^p\right)^{1/p}\left(\int_{\{h<0\}\cap\{g>0\}} |h|^q\right)^{1/q}\right\}\\
&\leq ||g||_p \max\left\{\left(\int_{\{h>0\}\cap\{g>0\}} |h|^q\right)^{1/q} + \left(\int_{\{h<0\}\cap\{g<0\}} |h|^q\right)^{1/q},\right.\\
&\quad \left.\left(\int_{\{h>0\}\cap\{g<0\}} |h|^q\right)^{1/q} + \left(\int_{\{h<0\}\cap\{g>0\}} |h|^q\right)^{1/q}\right\}\\
&\leq ||g||_p\left[\left(\int_{\{h>0\}} |h|^q\right)^{1/q} + \left(\int_{\{h<0\}} |h|^q\right)^{1/q}\right].
\end{aligned}$$

This shows that

$$|||H||| \leq \left[\left(\int_{\{h>0\}} |h|^q\right)^{1/q} + \left(\int_{\{h<0\}} |h|^q\right)^{1/q}\right].$$

On the other hand, let $h \in L^q$ and assume $\int_{\{h>0\}} |h|^q \geq \int_{\{h<0\}} |h|^q$. Define

$$g(x) = \begin{cases} |h|^{q-1} & x \in \{h > 0\} \\ -c|h|^{q-1} & x \in \{h < 0\}, \end{cases}$$

where $c = 0$ if $h \geq 0$ almost everywhere, otherwise choose $c > 0$ such that

$$\int_{\{g>0\}} |g|^p = \int_{\{g<0\}} |g|^p.$$

Then

$$\begin{aligned}
|h(g)| = \left|\int hg\right| &= \int_{\{h>0\}} |h|^q + c\int_{\{h<0\}} |h|^q \\
&= \left(\int_{\{h>0\}} |h|^q\right)^{1/q} \left(\int_{\{h>0\}} |h|^q\right)^{1/p} \\
&\quad + c\left(\int_{\{h<0\}} |h|^q\right)^{1/q} \left(\int_{\{h<0\}} |h|^q\right)^{1/p} \\
&= \left(\int_{\{h>0\}} |h|^q\right)^{1/q} \left(\int_{\{g>0\}} |g|^p\right)^{1/p} \\
&\quad + \left(\int_{\{h<0\}} |h|^q\right)^{1/q} \left(\int_{\{g<0\}} |g|^p\right)^{1/p} \\
&= ||g||_p \left[\left(\int_{\{h>0\}} |h|^q\right)^{1/q} + \left(\int_{\{h<0\}} |h|^q\right)^{1/q}\right].
\end{aligned}$$

This shows that

$$|||H||| \geq \left(\int_{\{h>0\}} |h|^q\right)^{1/q} + \left(\int_{\{h<0\}} |h|^q\right)^{1/q}. \qquad \square$$

2. Best approximation in the one-sided L^p norm.

Definition 6. Let L^p be the space of all real-valued p integrable functions with respect to μ on K, and let V be a linear subspace of L^p. Let $f(x) \in L^p$. A function $v \in V$ is called a best approximant to f from V in the one-sided L^p norm, if

$$||f - v||_p \leq ||f - u||_p \quad \text{for all } u \in V.$$

This approximation in one-sided L^p norm is different from the one sided approximation in L^p norm which studies approximation of a function from above or below in L^p norm.

In order to establish a characterization theorem of the best approximation similar to the one in the regular L^p norms, we need a characterization theorem independent of norm.

Theorem 7 [**4**]. *Let M be a subspace of a normed linear space X and $f \in X \backslash M$. X^* is the duel space of X. Then $v \in M$ is a best approximant to f from M if and only if there exists $H \in X^*$, for which $H(g) = 0$ for all $g \in M$, $|||H||| = 1$, and $H(f - v) = ||f - v||$.*

Theorem 8. *$v \in V$ is a best approximant to f from V in the one-sided L^p $(p > 1)$ norm if and only if there exist two constants $c_+ \geq 0$ and $c_- \geq 0$ such that $c_+ + c_- > 0$, $c_+ = 0$ if $(\int_{\{f-v>0\}} |f - v|^p)^{1/p} < ||f - v||_p$, $c_- = 0$ if $(\int_{\{f-v<0\}} |f - v|^p)^{1/p} < ||f - v||_p$, and for any $u \in V$,*

$$c_+ \int_{\{f-v>0\}} |f - v|^{p-1} u \; - \; c_- \int_{\{f-v<0\}} |f - v|^{p-1} u = 0.$$

Proof. ($\Rightarrow$). Let v be a best approximant to f from V. By Theorem 5 and Theorem 7 there exists $H \in (L^p)^*$ with $H(g) = \int hg$ and $h \in L^q$ such that

$$|||H||| = \left(\int_{\{h>0\}} |h|^q \right)^{1/q} + \left(\int_{\{h<0\}} |h|^q \right)^{1/q} = 1.$$

Then

$$\begin{aligned} ||f - v||_p = \int (f - v)h &= \int_{\{f-v>0\}} (f - v)h + \int_{\{f-v<0\}} (f - v)h \\ &\leq \int_{\{f-v>0\} \cap \{h>o\}} (f - v)h + \int_{\{f-v<0\} \cap \{h<o\}} (f - v)h \\ &\leq \left(\int_{\{f-v>0\}} |f - v|^p \right)^{1/p} \left(\int_{\{h>0\}} |h|^q \right)^{1/q} \\ &\quad + \left(\int_{\{f-v<0\}} |f - v|^p \right)^{1/p} \left(\int_{\{h<0\}} |h|^q \right)^{1/q} \end{aligned}$$

$$
\begin{aligned}
&\le ||f-v||_p\left[\left(\int_{\{h>0\}}|h|^q\right)^{1/q}+\left(\int_{\{h<0\}}|h|^q\right)^{1/q}\right]\\
&=||f-v||_p.
\end{aligned}
$$

All above inequalities must be the equalities. This is only possible when $|h|^q = c_1|f-v|^p$ almost everywhere, $\operatorname{sgn}(f-v)=\operatorname{sgn}(h)\ge 0$ almost everywhere on $\{f-v>0\}$, $|h|^q = c_2|f-v|^p$ almost everywhere, and $\operatorname{sgn}(f-v)=\operatorname{sgn}(h)\le 0$ almost everywhere on $\{f-v>0\}$. Also, If either

$$\left(\int_{\{f-v<0\}}|f-v|^p\right)^{1/p} < \left(\int_{\{f-v>0\}}|f-v|^p\right)^{1/p} = ||f-v||_p,$$

or

$$\left(\int_{\{f-v>0\}}|f-v|^p\right)^{1/p} < \left(\int_{\{f-v<0\}}|f-v|^p\right)^{1/p} = ||f-v||_p,$$

then $h(x)=0$ almost everywhere on $\{f<0\}$ or on $\{f>0\}$ respectively, because otherwise the last $\le$ would be $<$. So one may let c_1 or c_2 be 0 in these cases. Thus,

$$h(x)=\begin{cases} c_1|f(x)-v(x)|^{p-1} & x\in\{f-v>0\}\\ -c_2|f(x)-v(x)|^{p-1} & x\in\{f-v\le 0\}\end{cases}$$

($\Leftarrow$). Define

$$h(x)=\begin{cases} \frac{c_1/(c_1+c_2)}{|f(x)-v(x)|^{p-1}/||f(x)-v(x)||_p^{p-1}} & x\in\{f-v>0\}\\ -\frac{c_2}{c_1+c_2}\frac{|f(x)-v(x)|^{p-1}||f(x)-v(x)||_p^{p-1}}{} & x\in\{f-v\le 0\}.\end{cases}$$

$H(u)=\int hu=0$ for all $u\in V$, $|||H|||=1$, and $\int(f-v)h=||f-v||_p$. By Theorem 7, v is a best approximant to f from V. □

It is well known that the approximating set is a convex set in a normed linear space, and is so for this norm.

Theorem 9. *The set of all best approximants to f in the one-sided L^p norm is a convex set.*

Theorem 10. *If V contains a function $u > 0$ almost everywhere, then for any f and any of its best approximants v in the one-sided L^p norm,*

$$\left(\int_{\{f-v>0\}} |f-v|^p\right)^{1/p} = ||f-v||_p = \left(\int_{\{f-v<0\}} |f-v|^p\right)^{1/p}.$$

Proof. Let $u \in V$ such that $u(x) > 0$ almost everywhere, and let v be a best approximant to f from V. Suppose

$$\left(\int_{\{f-v>0\}} |f-v|^p\right)^{1/p} = ||f-v||_p > \left(\int_{\{f-v<0\}} |f-v|^p\right)^{1/p}.$$

Then, for small $\varepsilon > 0$,

$$\int_{\{f-v-\varepsilon u>0\}} |f-v-\varepsilon u|^p < \int_{\{f-v>0\}} |f-v|^p.$$

On the other hand, let

$$\delta = \left(\int_{\{f-v>0\}} |f-v|^p\right)^{1/p} - \left(\int_{\{f-v<0\}} |f-v|^p > 0\right)^{1/p} > 0.$$

There exists an $M > 0$ such that

$$\int_{\{u\geq M\}} |f-v|^p < \left(\frac{\delta}{3}\right)^p,$$

and there exists a $0 < \varepsilon_1 \leq 1$ such that when $0 < \varepsilon \leq \varepsilon_1$

$$\int |\varepsilon u|^p < \left(\frac{\delta}{3}\right)^p,$$

and

$$\int_{|f-v|\leq\varepsilon\}} |f-v|^p < \left(\frac{\delta}{3}\right)^p.$$

Then, for $0 < \varepsilon \le \varepsilon_2 = \min\{\varepsilon_1, \varepsilon_1/M\}$

$$\begin{aligned}\int_{\{f-v-\varepsilon u<0\}} |f-v|^p &\le \int_{\{f-v<0\}} |f-v|^p + \int_{\{0\le f-v\le \varepsilon M\}} |f-v|^p \\ &\quad + \int_{\{u\ge M\}} |f-v|^p \\ &< \int_{\{f-v<0\}} |f-v|^p + 2\left(\frac{\delta}{3}\right)^p,\end{aligned}$$

and

$$\begin{aligned}&\left(\int_{\{f-v-\varepsilon u<0\}} |f-v-\varepsilon u|^p\right)^{1/p} \\ &\qquad \le \left(\int_{\{f-v-\varepsilon u<0\}} |f-v|^p\right)^{1/p} + \left(\int_{\{f-v-\varepsilon u<0\}} |\varepsilon u|^p\right)^{1/p} \\ &\qquad < \left(\int_{\{f-v-\varepsilon u<0\}} |f-v-\varepsilon u|^p\right)^{1/p} + \frac{\delta}{3} \\ &\qquad < \left(\int_{\{f-v<0\}} |f-v|^p\right)^{1/p} + \delta \\ &\qquad \le \left(\int_{\{f-v>0\}} |f-v|^p\right)^{1/p}.\end{aligned}$$

Hence,

$$||f-(v+\varepsilon u)||_p < ||f-v||_p$$

for small $\varepsilon > 0$. □

A similar argument leads to:

Theorem 11. *If V contains a function $u \ge 0$ almost everywhere, then for each function f which has at least one best approximant from V in the one-sided L^p norm, there exists a best approximant, v, such that*

$$\left(\int_{\{f-v>0\}} |f-v|^p\right)^{1/p} = ||f-v||_p = \left(\int_{\{f-v<0\}} |f-v|^p\right)^{1/p}.$$

Theorem 12. *Let $f \in L^p$ be the space of all p integrable functions on K, and $v \in V$ a subspace of L^p, $p > 1$.*

1) *If $||f-v||_p = (\int_{\{f-v>0\}} |f-v|^p)^{1/p} > (\int_{\{f-v<0\}} |f-v|^p)^{1/p}$, $v \in V$ is a best approximant to f from V in the one-sided L^p norm if and only if v is a best approximant to f from V in L^p norm on $\{f - v \geq 0\}$.*

2) *If $||f-v||_p = (\int_{\{f-v<0\}} |f-v|^p)^{1/p} > (\int_{\{f-v>0\}} |f-v|^p)^{1/p}$, $v \in V$ is a best approximant to f from V in the one-sided L^p norm if and only if v is a best approximant to f from V in L^p norm on $\{f - v \leq 0\}$.*

3) *If $||f-v||_p = (\int_{\{f-v>0\}} |f-v|^p)^{1/p} = (\int_{\{f-v<0\}} |f-v|^p)^{1/p}$, and $v \in V$ is a best approximant to f from V in L^p norm, then v is a best approximant to f from V in the one-sided L^p norm.*

Proof. By replacing f by $f - v$, we may assume $v = 0$. For Case 1, 0 is the best approximant to f in the regular L^p norm on $\{f \geq 0\}$ if and only if

$$\int_{\{f>0\}} |f|^{p-1} u = 0 \quad \text{for any } u \in V$$

which in turn is true if and only if 0 is the best approximant to f in the one-sided L^p norm by Theorem 7. Case 2 is similar to Case 1. For Case 3, it can be easily seen by the definition of one-sided L^p norm. □

Now, let us consider the uniqueness of the best approximation.

Definition 13. Let U be an n dimensional subspace of $C[a, b]$. U is called a unicity space of $C[a, b]$ in the one-sided L^p norm if for every function $f \in C[a, b]$, the best approximant to f in the one-sided L^p norm is unique.

Theorem 14. *Let U be a subspace of $C[a, b]$. If either U contains a function $u > 0$ almost everywhere, or no nontrivial $u \in U$ vanishes on an interval, then U is a unicity space in the one-sided L^p norm for $p > 1$.*

Proof. Suppose f has two best approximants from V. Without loss of generality, let v and $-v$ be the two best approximants. Then 0 is also a best approximant by Theorem 9, and

$$||f||_p = ||f+v||_p = ||f-v||_p.$$

By the remark following Theorem 3, $f+v = c_1(f-v)$ almost everywhere on $\{f>0\}$ if $||f||_p = (\int_{\{f>0\}} |f|^p)^{1/p}$. This leads to $v = c_2 f$ and $v=0$ on $\{f>0\}$. If no nontrivial $u \in U$ vanishes on an interval, it leads to $v=0$. If U contains a function $u>0$ almost everywhere, by Theorem 8

$$||f||_p = \left(\int_{\{f>0\}} |f|^P\right)^{1/p} = \left(\int_{\{f<0\}} |f|^P\right)^{1/p}.$$

This also leads to $v=0$. □

The following example shows that without the additional condition in Theorem 13 the subspace U may not be a unicity space.

Example 1. Let $[a,b] = [-2,2]$,

$$u_1(x) = \begin{cases} x & -2 \le x \le 0 \\ 0 & 0 \le x \le 2 \end{cases},$$

and

$$u_2(x) = \begin{cases} 0 & -2 \le x \le 0 \\ x(x-1) & 0 \le x \le 1 \\ \alpha(x-1)(x-2) & 1 \le x \le 2, \end{cases}$$

where $\alpha < 0$ is determined by

$$\int_{-2}^{0} |x(x-4)|^{p-1} u_2(x) = 0.$$

Let $U = \text{span}\,\{u_1, u_2\}$. Define

$$f(x) = \begin{cases} 0 & -2 \le x \le 0 \\ x(x-4) & 0 \le x \le 2. \end{cases}$$

Then $cu_1(x)$ is a best approximant to f from U in the one-sided L^1 norm for any $0 \leq c \leq 1$.

It is easy to see that the above result holds for any one-sided L^p norm with a positive weight function $w(x)$, i.e., if we define

$$||f|| = \max\left\{\left(\int_{\{f>0\}} w|f|^p\right)^{1/p}, \left(\int_{\{f<0\}} w|f|^p\right)^{1/p}\right\},$$

then the conclusions of the above theorems still hold.

Example 2. If $Z(U) = \{x|u(x) = 0 \text{ for all } u \in U\}$ contains an open interval (c, d), then U is not a unicity space for any one-sided L^p norm with any positive weight function $w(x)$. Choose a nontrivial $u \in U$. Define $f(x) = -|u(x)|$ for $x \in [a, b]\backslash[c, d]$ and $f(x) \geq 0$ for $x \in [c, d]$ such that

$$\int_{[c,d]} w|f|^p \geq 2\int w|u|^p + 1.$$

Then all $cu(x)$, $-1 \leq c \leq 1$, are best approximants to f from U.

Finally, we consider the interpolation property of best approximations. In [**5, 6**], I studied the interpolation property of linear projections. The following result was proved.

Theorem 15. *Let $V = \operatorname{span}\{v_1, v_2, \dots, v_n\}$ be a subspace of $C[a, b]$ and a projection from $C[a, b]$ onto V is defined by*

$$P(f) = \sum_{i=1}^{n}\left(\int_a^b h_i f\right)v_i,$$

where $h_1, \dots, h_n$ are n linearly independent Borel measurable functions satisfying $\int_a^b h_i v_j = \delta_{ij}$. Then $P(f)$ interpolates f at n or more points (whether counting multiplicity or not) for any $f \in C[a, b]$ if and only if V is a weak Chebyshev space defined below.

Definition 16. A function is said to have n sign changes on $[a, b]$, if there exists $n+1$ points $a \leq x_0 < x_1 < \cdots < x_n \leq b$ such that

$$f(x_{i-1})f(x_i) < 0, \quad i = 1, 2, \dots, n.$$

Let U be an n dimensional subspace of $C[a,b]$. If every $u \in U$ has at most $n-1$ sign changes, then U is called a weak Chebyshev space.

Using these results we have

Theorem 17. *If V is an n dimensional weak Chebyshev space and a unicity space in best approximation in the one-sided L^p norm, then the best approximant to f interpolates f at n or more points for any $f \in C[a,b]$.*

Proof. Suppose there exists f which has only k $(< n)$ zeros and 0 is its best approximant. Since V is a weak Chebyshev space, there exists a nontrivial u V such that $\mathrm{sgn}(v)\mathrm{sgn}(f) \geq 0$. Then for small $c > 0$, $||f - cu||_p \leq ||f||$ which contradicts the uniqueness of the best approximation. □

Theorem 18. *Let V be an n dimensional subspace of $C[a,b]$. If the best approximant to f in one-sided L^p norm interpolates f at n or more points for any $f \in C[a,b]$, then V is a weak Chebyshev space.*

Proof. Suppose V is not a weak Chebyshev space. Let $v_1, v_2, \dots, v_n$ be a basis of V and $\int_a^b v_i v_j = \delta_{ij}$. Define the linear projection

$$P(f) = \sum_{i=1}^{n} \left(\int_a^b v_i f \right) v_i.$$

Then, by Theorem 15, there exists an $f \in C[a,b]$ with $P(f) = 0$ and f has fewer than n zeros. Define

$$g(x) = \begin{cases} |f|^{1/(p-1)} & x \in \{f \geq 0\} \\ -c|f|^{1/(q-1)} & x \in \{f < 0\}, \end{cases}$$

where $c > 0$ is so chosen such that

$$||g||_p = \left(\int_{\{g>0\}} |g|^p d\mu \right)^{1/p} = \left(\int_{\{g<0\}} |g|^p d\mu \right)^{1/p}.$$

g has the same number of zeros as f has. Let $c_1 = 1$ and $c_2 = 1/|c|^{p-1}$. Then, for $i = 1, \dots n$,

$$c_1 \int_{\{g>0\}} |g|^{p-1} v_i - c_2 \int_{\{g<0\}} |g|^{p-1} v_i = \int_{\{f>0\}} f v_i + \int_{\{f<0\}} f v_i$$
$$= \int f v_i = 0.$$

By Theorem 8, 0 is a best approximant to g which has fewer than n zeros. $\square$

Under the uniqueness condition Theorems 17 and 18 show the equivalence of U being a weak Chebyshev space and every best approximant interpolating the function at n or more points. This uniqueness condition can not be dropped, but I suspect it could be replaced by '$Z(U)$ does not contain an open interval'. I cannot prove it now.

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