

A DIRICHLET ANALOGUE OF THE FREE MONOGENIC INVERSE SEMIGROUP VIA MÖBIUS INVERSION

EMIL DANIEL SCHWAB AND GEORGE STOIANOV

ABSTRACT. The Möbius inversion formula of the free monogenic inverse semigroup is represented by the Möbius function for Cauchy product. In this short note we describe a Dirichlet analogue of this inverse semigroup.

1. Introduction. The Cauchy product and the Dirichlet product are familiar convolutions of arithmetical functions. The corresponding Möbius functions μ_C and μ_D (as convolution inverses of the zeta function) are the following ones:

$$(1.1) \quad \mu_C(n) = \begin{cases} 1 & \text{if } n = 0 \\ -1 & \text{if } n = 1 \\ 0 & \text{if } n > 1 \end{cases}$$

$$(1.2) \quad \mu_D(n) = \begin{cases} 1 & \text{if } n = 1 \\ (-1)^k & \text{if } n = p_1 \cdots p_k \text{ where } p_i \text{ are distinct primes} \\ 0 & \text{if } p^2 \mid n \text{ for some prime } p \end{cases}$$

The reduced standard division category $C_F(S)$ of an inverse monoid S relative to an idempotent transversal F of the D -classes of S with $1 \in F$, $(\text{Ob } C_F(S) = F; \text{Hom}(e, f) = \{(s, e) \in S \times F \mid s^{-1}s \leq e, ss^{-1} = f\}; e \xrightarrow{(s,e)} f \xrightarrow{(t,f)} g = e \xrightarrow{(t,f)(s,e)=(ts,e)} g)$ is a Möbius category if and only if S is combinatorial and $(E(S), \leq)$ is locally finite (see [10]). If an inverse semigroup S is without identity it may be converted into an inverse monoid by adjoining an identity. We call the Möbius inversion formula of such Möbius category $C_F(S)$, the Möbius inversion formula of the inverse monoid (semigroup) S . In [11] there are given Möbius inversion formulas of some inverse semigroups S :

$$(1.3) \quad F, G : \mathbf{N} \longrightarrow \mathbf{C}, \quad F(n) = \sum_{i=0}^n G(i) \Leftrightarrow G(n) = \sum_{i=0}^n \mu_C(n-i)F(i)$$

2010 AMS *Mathematics subject classification*. Primary 20M18, 20M50, 05A15.
Received by the editors on May 3, 2005, and in revised form on February 4, 2009.

is the Möbius inversion formula of the bicyclic semigroup B . The bicyclic semigroup is a combinatorial, E -unitary and bisimple inverse monoid. From [10, 11] the classical Möbius inversion formula

$$(1.4) \quad F, G : \mathbf{N}^* \longrightarrow \mathbf{C}, \quad F(n) = \sum_{d|n} G(d) \Leftrightarrow G(n) = \sum_{d|n} \mu_D\left(\frac{n}{d}\right) F(d)$$

is the Möbius inversion formula of the inverse monoid $B_M = \mathbf{N}^* \times \mathbf{N}^*$ with the multiplication

$$(1.5) \quad (a, b) \cdot (c, d) = \left(\frac{ma}{b}, \frac{md}{c} \right),$$

where m is the least common multiple of b and c . The multiplicative (or Dirichlet) analogue B_M of the bicyclic semigroup B is combinatorial, E -unitary and bisimple like B . In [11] it was shown that the Möbius inversion formula of the free monogenic inverse semigroup I (using Scheiblich's [9] representation with identity adjoined) is the following one:

$$(1.6) \quad \begin{aligned} F, G : \mathbf{N} \times \mathbf{N} &\longrightarrow \mathbf{C} \\ F(a, b) &= \sum_{u=0}^a \sum_{v=0}^b G(u, v) \iff \\ G(a, b) &= \sum_{u=0}^a \sum_{v=0}^b \mu_C(a-u) \mu_C(b-v) F(u, v) \end{aligned}$$

The free monogenic inverse semigroup is E -unitary, combinatorial and completely semisimple.

Starting with the Möbius category of the integer affine maps, the purpose of this paper is to give a Dirichlet analogue I_D (via the Möbius inversion formula (1.6)) different from the standard Dirichlet analogue, namely the multiplicative analogue of the free monogenic inverse semigroup. This “new” Dirichlet analogue I_D is combinatorial, E -unitary and completely semisimple inverse semigroup like the free monogenic inverse semigroup. It arises as the Leech inverse monoid of a triangular bimorphic division category.

2. The Möbius category of integer affine maps. A Möbius category is a decomposition-finite category C (i.e., a small category in

which any morphism α has only finitely many nontrivial factorizations $\alpha = \beta\gamma$ such that an incidence function $\xi : \text{Mor } C \rightarrow \mathbf{C}$ has a convolution inverse if and only if $\xi(\alpha) \neq 0$ for each identity morphism α . The convolution $\xi * \eta$ of two incidence functions ξ and η is defined by

$$(2.1) \quad (\xi * \eta)(\alpha) = \sum_{\beta\gamma=\alpha} \xi(\beta)\eta(\gamma).$$

A triangular category is a special Möbius category. A category C is called triangular if the set $\text{Ob } C$ of objects of C is the set of non-negative integers $\mathbf{N}$ (or the set of positive integers $\mathbf{N}^*$) and the cardinalities $A(k, n) = |\text{Hom}(k, n)|$ of sets of morphisms constitute a triangular family of numbers:

$$(2.2) \quad A(n, n) = 1 \text{ for all } n; \quad A(k, n) = 0 \text{ if } k > n.$$

For further details on Möbius categories and triangular categories, we refer the reader to [1, 6].

Now, let C_a be the category of integer affine maps defined by:

- $\text{Ob } C_a = \mathbf{N}^*$;
- $\{\text{Hom}(k, n) = (f, k) \mid f : \mathbf{R} \rightarrow \mathbf{R}, f(x) = ax + b \text{ with } a \in \mathbf{N}^*, b \in \mathbf{N} \text{ and } f(k) = n\}$;
- The composition $(g, n)(f, k)$ of two morphisms $(f, k) : k \rightarrow n$ and $(g, n) : n \rightarrow m$ is given by: $(g, n)(f, k) = (g \circ f, k)$, where $g \circ f$ is the usual composition of maps.

Straightforward verification shows that:

Proposition 1. C_a is a triangular category in which every morphism is a bimorphism (with the corresponding family of triangular numbers: $A(k, n) = [n/k]$, where $[n/k]$ is the integer part of n/k).

Proposition 2. The convolution $\xi * \eta$ of two incidence functions, $\xi, \eta : \text{Mor } C_a \rightarrow \mathbf{C}$ is given by

$$(2.3) \quad (\xi * \eta)(ax + b, k) = \sum_{u|a} \sum_{v=0}^{[ub/a]} \xi\left(\frac{a}{u}x + b - \frac{a}{u}v, uk + v\right) \eta(ux + v, k).$$

Proof. Let $(ax + b, k) : k \rightarrow n$ be a morphism of C_a and $m \in \mathbf{N}^*$ such that $k \leq m \leq n$. Given a morphism $(ux + v, k) : k \rightarrow m$, there exists (a necessarily unique) morphism $(sx + t, m) : m \rightarrow n$ such that $(ax + b, k) = (sx + t, m)(ux + v, k)$ if and only if $u \mid a$ and $v \leq (ub)/a$. Then $s = a/u$ and $t = b - (a/u)v$. Thus the convolution $\xi * \eta$ of two incidence functions ξ and η of C_a is given by (2.3). $\square$

An (abstract) division category D is a small category with pushouts and a quasi initial object I (i.e., an object I with at least one morphism into each object of D) in which every morphism is an epimorphism (see [2, page 268]). We have:

Proposition 3. *The category C_a is a division category.*

Proof. 1 is a quasi initial object and the following square is a pushout

$$(2.4) \quad \begin{array}{ccc} k & \xrightarrow{(f_2=a_2x+b_2,k)} & n \\ (f_1=a_1x+b_1,k) \downarrow & & \downarrow (g_2=c_2x+d_2,n_2) \\ n_1 & \xrightarrow{(g_1=c_1x+d_1,n_1)} & n \end{array}$$

where $c_1 = m/a_1$, $c_2 = m/a_2$, $m = l.c.m.\{a_1, a_2\}$, $n = \max\{c_1n_1, c_2n_2\}$ and $d_1 = n - c_1n_1$, $d_2 = n - c_2n_2$. $\square$

In what follows we shall evaluate the Möbius function of the triangular-division category C_a .

If C is a Möbius category, the convolution inverse ξ^{-1} of an incidence function ξ of C with $\xi(\alpha) \neq 0$ for each identity morphism α , is given recursively by:

$$(2.5) \quad \xi^{-1}(\alpha) = \begin{cases} 1/\xi(\alpha) & \text{if } \alpha \text{ is an identity} \\ -\xi^{-1}(1_{\text{Codom } \alpha}) \sum_{\beta\gamma=\alpha; \gamma \neq \alpha} \xi(\beta)\xi^{-1}(\gamma) & \text{otherwise} \end{cases}$$

(the convolution identity is δ , where $\delta(\alpha) = 1$ if α is an identity morphism and $\delta(\alpha) = 0$ otherwise).

The Möbius function μ is the convolution inverse of the incidence function ζ defined by: $\zeta(\alpha) = 1$ for each morphism α of the Möbius category C .

Proposition 4. *The Möbius function of C_a is given by:*

$$(2.6) \quad \mu(ax + b, k) = \begin{cases} \mu_D(a) & \text{if } b = 0 \\ -\mu_D(a) & \text{if } b = a \\ 0 & \text{otherwise} \end{cases}$$

Proof. In C_a , for $\xi = \zeta$, formula (2.5) becomes

$$(2.7) \quad \mu(ax + b, k) = \begin{cases} 1 & \text{if } a = 1, b = 0 \\ -\sum_{u|a; u \neq a} \sum_{v=0}^{[ub/a]} \mu(ux + v, k) - \sum_{v=0}^{b-1} \mu(ax + v, k) & \text{otherwise.} \end{cases}$$

Now, using (2.7) we prove (2.6) by induction on the positive integer a . If $a = 1$, it is easy to see that (2.6) holds for $b = 0, 1, 2$. Assume $b > 2$ and $\mu(x + b_1, k) = 0$ for $b_1 = 2, \dots, b-1$. Then:

$$(2.8) \quad \mu(x + b, k) = -\sum_{b_1=0}^{b-1} \mu(x + b_1, k) = -\mu(x, k) - \mu(x + 1, k) = 0.$$

If $a \geq 2$, it is useful to consider the following cases: $b = 0$, $0 < b < a$, $b = a$ and $b > a$. In each case it is easy to check that (2.6) holds. $\square$

3. The Leech inverse monoid of C_a . The category C_a is a Möbius category and a division category. We call such categories *MD* categories (Möbius-division categories, see [12]). A *MD* category is a small category M with the following properties:

- (i) M has pushouts;
- (ii) M has a quasi initial object I ;
- (iii) Every morphism of M is an epimorphism;
- (iv) The identity morphisms are indecomposable (i.e., $1 = \beta\gamma \implies \beta = \gamma = 1$)

(v) M is decomposition-finite.

If M is an MD -category with quasi initial object I , then

$$(3.1) \quad L(M, I) = \{(\alpha, \beta) \in \text{Mor } C \times \text{Mor } C \mid \text{Dom } \alpha = \text{Dom } \beta = I, \text{Codom } \alpha = \text{Codom } \beta\}$$

with the multiplication

$$(3.2) \quad (\alpha_1, \beta_1) \cdot (\alpha_2, \beta_2) = (\gamma_1 \alpha_1, \gamma_2 \beta_2), \text{ where } [\beta_1, \alpha_2, \gamma_1, \gamma_2] \text{ is a pushout,}$$

is the Leech inverse monoid of M .

Theorem 1 [12]. *Every MD -category M with a quasi initial object I is isomorphic to the reduced standard division category $C_F(L(M, I))$ for any idempotent transversal F of the D -classes of $L(M, I)$ with $(1_I, 1_I) \in F$.*

For us, the significance of this theorem follows from the fact that the forthcoming Leech inverse monoid $I_D \cong L(C_a, 1)$ arises from an MD category, namely, C_a . Thus the reduced standard division category $C_F(I_D)$ is isomorphic to C_a . So, the Möbius inversion formula of the inverse monoid I_D is the Möbius inversion formula of C_a .

If in C_a the inner square

$$(3.3) \quad \begin{array}{ccc} 1 & \xrightarrow[f_2]{g_2} & f_2(1) = g_2(1) \\ f_1 \downarrow & & \downarrow q \\ f_1(1) = g_1(1) & \xrightarrow[p]{} & \bullet \end{array}$$

is a pushout, then the set of integer affine maps pairs $\{(f, g) \mid f(1) = g(1)\}$ with the multiplication

$$(3.4) \quad (f_1, g_1)(f_2, g_2) = (p \circ f_1, q \circ g_2)$$

is the Leech inverse monoid $L(C_a, 1)$. We can now easily check (taking into account (2.4), that the Leech inverse monoid $L(C_a, 1)$ is isomorphic to I' given by:

$$(3.5) \quad I' = \{(a, b, a', b') \in \mathbf{N}^4 \mid a \neq 0, a' \neq 0, a + b = a' + b'\},$$

$$(3.6) \quad (a, b, a', b')(c, d, c', d') \\ = \begin{cases} \left(\frac{ma}{a'}, \frac{mb}{b'}, \frac{mc'}{c}, m \left(\frac{d'}{c} + \frac{b'}{a'} - \frac{d}{c} \right) \right) & \text{if } \frac{b'}{a'} \geq \frac{d}{c} \\ \left(\frac{ma}{a'}, m \left(\frac{b}{a'} + \frac{d}{c} - \frac{b'}{a'} \right), \frac{mc'}{c}, \frac{md'}{c} \right) & \text{if } \frac{b'}{a'} \leq \frac{d}{c} \end{cases}$$

where $m = \text{l.c.m.}(a', c)$. By $(u, r, v, s) \rightarrow (u, v, r, s)$, as in [9], and by an adjustment of (3.6), we obtain the following isomorphic copy of $L(C_a, 1)$:

Proposition 5. *The Leech inverse monoid $L(C_a, 1)$ of the MD-category of integer affine maps is isomorphic to the inverse monoid I_D given by*

$$(3.7) \quad I_D = \{(a, b, a', b') \in \mathbf{N}^{*2} \times \mathbf{N}^2 \mid a + a' = b + b'\}$$

$$(3.8) \quad (a, b, a', b')(c, d, c', d') \\ = \left(\frac{ma}{b}, \frac{md}{c}, m' + \frac{ma'}{b} - \frac{mb'}{b}, m' + \frac{md'}{c} - \frac{mc'}{c} \right)$$

where $m = \text{l.c.m.}(b, c)$ and $m' = \max(\frac{mb'}{b}, \frac{mc'}{c})$.

Now, the Möbius inversion formula of I_D is the Möbius inversion formula of C_a . For two incidence functions $\xi, \eta : \text{Mor } C_a \rightarrow \mathbf{C}$ we have $\xi = \zeta * \eta$ if and only if $\eta = \mu * \xi$. That is,

$$(3.9) \quad \xi(ax + b, k) = \sum_{u|a} \sum_{v=0}^{[(ub)/a]} \eta(ux + v, k)$$

if and only if

$$(3.10) \quad \eta(ax + b, k) = \sum_{u|a} \sum_{v=0}^{[(ub)/a]} \mu\left(\frac{a}{u}x + b - \frac{a}{u}v, uk + v\right) \xi(ux + v, k).$$

By (2.6) it follows that (3.9) holds if and only if

$$(3.11) \quad \eta(ax + b, k) = \sum_{u|a; a|ub} \mu_D\left(\frac{a}{u}\right) \xi\left(ux + \frac{ub}{a}, k\right) \\ - \sum_{u|a; a|ub; b \neq 0} \mu_D\left(\frac{a}{u}\right) \xi\left(ux + \frac{ub}{a} - 1, k\right).$$

So, we obtain the following Möbius inversion formula. Given functions $F, G : \mathbf{N}^* \times \mathbf{N} \rightarrow \mathbf{C}$,

$$(3.12) \quad F(a, b) = \sum_{u|a} \sum_{v=0}^{[(ub)/a]} G(u, v)$$

for all $(a, b) \in \mathbf{N}^* \times \mathbf{N}$, if and only if

$$(3.13) \quad G(a, b) = \begin{cases} \sum_{u|a} \mu_D\left(\frac{a}{u}\right) F(u, 0) & \text{if } b = 0 \\ \sum_{u|a; a|ub} \mu_D\left(\frac{a}{u}\right) \left[F\left(u, \frac{ub}{a}\right) - F\left(u, \frac{ub}{a} - 1\right)\right] & \text{if } b \neq 0. \end{cases}$$

By an adjustment of (3.13) we have:

Theorem 2. (The Möbius inversion formula for I_D .) *Given functions $F, G : \mathbf{N}^* \times \mathbf{N} \rightarrow \mathbf{C}$,*

$$(3.14) \quad F(a, b) = \sum_{u|a} \sum_{v=0}^{[(ub)/a]} G(u, v)$$

for all $(a, b) \in \mathbf{N}^* \times \mathbf{N}$, if and only if

$$(3.15) \quad G(a, b) = \sum_{u|a; a|ub} \sum_{v=0}^{ub/a} \mu_D\left(\frac{a}{u}\right) \mu_C\left(\frac{ub}{a} - v\right) F(u, v).$$

We say that the inverse semigroup I_D is a Dirichlet analogue of the free monogenic inverse semigroup I via Möbius inversion. The Möbius functions and the Möbius inversion formulas of these semigroups have a pronounced similarity as we can see in the table at the end.

By [10, Theorem 3.3, Corollary 3.4 and Theorem 4.1], as for I , it follows

Corollary 1. *The Dirichlet analogue I_D of I is a combinatorial completely semisimple E -unitary inverse monoid.*

The free monogenic inverse semi-group I (Scheiblich's [9] representation with identity adjoined)	The Dirichlet analogue I_D	The multiplicative analogue I_M
$I = \{(a, b, a', b') \in \mathbf{N}^4 \mid a + a' = b + b'\},$ $(a, b, a', b') (c, d, c', d') = (ma/b, md/c, m'a'/b', m'd'/c'),$ where $m = \max(b, c),$ $m' = \max(b', c')$	$I_D = \{(a, b, a', b') \in \mathbf{N}^{*2} \times \mathbf{N}^2 \mid a + a' = b + b'\},$ $(a, b, a', b') (c, d, c', d') = (ma/b, md/c, (m'b + ma' - mb')/b, (m'c + md' - mc')/c),$ where $m = l.c.m.(b, c)$ $m' = \max(mb'/b, mc'/c)$	$I_M = \{(a, b, a', b') \in \mathbf{N}^{*4} \mid aa' = bb'\},$ $(a, b, a', b') (c, d, c', d') = (ma/b, md/c, m'a'/b', m'd'/c'),$ where $m = l.c.m.(b, c)$ $m' = l.c.m.(b', c').$
The Möbius function of I : $\mu : \mathbf{N} \times \mathbf{N} \rightarrow \mathbf{C}$ $\mu(a, b) = \mu_C(a)\mu_C(b)$ $= \begin{cases} \mu_C(a), & \text{if } b = 0 \\ -\mu_C(a) & \text{if } b = 1 \\ 0 & \text{otherwise} \end{cases}$	The Möbius function of I_D : $\mu : \mathbf{N}^* \times \mathbf{N} \rightarrow \mathbf{C}$ $\mu(a, b) = \begin{cases} \mu_D(a), & \text{if } b = 0 \\ -\mu_D(a) & \text{if } b = 1 \\ 0 & \text{otherwise} \end{cases}$	The Möbius function of I_M : $\mu : \mathbf{N}^* \times \mathbf{N}^* \rightarrow \mathbf{C}$ $\mu(a, b) = \mu_D(a)\mu_D(b)$
The Möbius inversion formula for I : $F, G : \mathbf{N} \times \mathbf{N} \rightarrow \mathbf{C},$ $F(a, b) = \sum_{u=0}^a \sum_{v=0}^b G(u, v) \Leftrightarrow$ $G(a, b) = \sum_{u=0}^a \sum_{v=0}^b \mu_C(a-u)\mu_C(b-v)F(u, v).$	The Möbius inversion formula for I_D : $F, G : \mathbf{N}^* \times \mathbf{N} \rightarrow \mathbf{C}$ $F(a, b) = \sum_{u a} \sum_{v=0}^{[ub/a]} G(u, v) \Leftrightarrow$ $G(a, b) = \sum_{u a} \sum_{v=0}^{ub/a} \mu_D(a/u)\mu_C((ub - va)/a)F(u, v)$	The Möbius inversion formula for I_M : $F, G : \mathbf{N}^* \times \mathbf{N}^* \rightarrow \mathbf{C}$ $F(a, b) = \sum_{u a} \sum_{v b} G(u, v) \Leftrightarrow$ $G(a, b) = \sum_{u a} \sum_{v b} \mu_D(a/u)\mu_D(b/v)F(u, v).$

Remark 1. The multiplicative analogue of I , denoted I_M :

$$I_M = \{(a, b, a', b') \in \mathbf{N}^{*4} \mid aa' = bb'\}$$

with multiplication:

$$(a, b, a', b')(c, d, c', d') = \left(\frac{ma}{b}, \frac{md}{c}, \frac{m'a'}{b'}, \frac{m'd'}{c'} \right),$$

where $m = l.c.m.(b, c)$ and $m' = l.c.m.(b', c')$, is also a Leech inverse monoid arises from an MD category, namely, the truly standard division category $D(\mathbf{N}^*)$ of the multiplicative monoid of all positive integers, $\mathbf{N}^*$. Since this is triangular and bimorphic (cancellative), the multiplicative analogue is also a combinatorial, completely semisimple

and E -unitary inverse semigroup. We can obtain, in a way similar to the above, the Möbius function and the Möbius inversion formula of I_M . We included them in the table on the previous page where the central place is allocated for the Dirichlet analogue I_D .

Acknowledgments. The authors would like to thank the referee for very valuable suggestions.

REFERENCES

1. M. Content, F. Lemay and P. Leroux, *Catégories de Möbius et fonctorialités: un cadre général pour l'inversion de Möbius*, J. Combin. Theory **28** (1980), 169–190.
2. P.A. Grillet, *Semigroups. An introduction to the structure theory*, Marcel Dekker, Inc., New York, 1995.
3. H. James and M.V. Lawson, *An application of groupoids of fractions to inverse semigroups*, Per. Math. Hungar. **38** (1999), 43–54.
4. M.V. Lawson, *Inverse semigroups. The theory of partial symmetries*, World Scientific, Singapore, 1998.
5. J. Leech, *Constructing inverse monoids from small categories*, Semigroup Forum **36** (1987), 89–116.
6. P. Leroux, *Catégories triangulaires: Exemples, applications et problèmes*, Research report, Univ. of Quebec at Montreal, 1980.
7. M. Petrich, *Inverse semigroups*, John Wiley & Sons, New York, 1984.
8. G.C. Rota, *On the foundations of combinatorial theory. Theory of Möbius functions*, Z. Wahrscheinlich. **2** (1964), 340–368.
9. H.E. Scheiblich, *A characterization of a free elementary inverse semigroup*, Semigroup Forum **2** (1974), 76–79.
10. E.D. Schwab, *Möbius categories as reduced standard division categories of combinatorial inverse monoids*, Semigroup Forum **69** (2004), 30–40.
11. ———, *The Möbius category of some combinatorial inverse semigroups*, Semigroup Forum **69** (2004), 41–50.
12. ———, *On incidence algebras of combinatorial inverse monoids*, Comm. Algebra **38** (2010), 1779–1789.

DEPARTMENT OF MATHEMATICAL SCIENCES, UNIVERSITY OF TEXAS AT EL PASO,
EL PASO, TX 79968

Email address: eschwab@utep.edu

CAG, ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE, CH-1015 LAUSANNE

Email address: george.stoianov@epfl.ch