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DIVISIBLY NORM-PRESERVING MAPS BETWEEN COMMUTATIVE BANACH ALGEBRAS

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ABSTRACT. Let $\mathcal{A}$ and $\mathcal{B}$ be unital commutative Banach algebras. Suppose that $\mathcal{A}$ is semi-simple. Let $\rho : \mathcal{A} \to \mathcal{A}$ and $\tau : \mathcal{B} \to \mathcal{B}$ be bijections. If $T : \mathcal{A} \to \mathcal{B}$ is a surjection with, for some $\alpha \in \mathbf{C} \setminus \{0\}$, $\mathrm{r}(T(f)\tau(T(g)) - \alpha) = \mathrm{r}(f\rho(g) - \alpha)$ for all $f, g \in \mathcal{A}$, then $\mathcal{B}$ is semi-simple and $\mathrm{r}(T(f)T(g)^{-1} - 1) = \mathrm{r}(fg^{-1} - 1)$ for every $f \in \mathcal{A}$ and $g \in \mathcal{A}^{-1}$. As a consequence, $T(1)$ is invertible and $T(1)^{-1}T$ is a real-algebra isomorphism. If, in addition, $T(1)^{-1}T(i) = i$, then $T(1)^{-1}T$ is a complex-algebra isomorphism. This result unifies and generalizes [**3**, Theorem 7.4] and [**4**, Theorem 3.2 and 6.2].

1. Introduction. Let $C(X)$ be the commutative Banach algebra of all complex-valued continuous functions on a compact Hausdorff space X with respect to pointwise operations and the supremum norm. In 2001, Molnár [**10**] introduced "spectral multiplicativity conditions" for a surjection T from $C(X)$ onto itself, which is not necessarily linear nor continuous: $\sigma(T(f)T(g)) = \sigma(fg)$ for all $f, g \in C(X)$; or $\sigma(T(f)\overline{T(g)}) = \sigma(f\overline{g})$ for all $f, g \in C(X)$. Here, $\sigma(f)$ and $\overline{f}$ are the spectrum and the complex conjugate function of $f \in C(X)$, respectively. In particular, he proved that if X is first countable, then $T(1)^{-1}T$ is an algebra automorphism. These results can be extended for uniform algebras on arbitrary compact Hausdorff spaces and commutative Banach algebras.

For uniform algebras, the corresponding result was proven by Rao and Roy [**11**]. Hatori, Miura and Takagi [**2**] replaced the spectra of elements by their ranges. Further work has been done analyzing maps with $\sigma_\pi(T(f)T(g)) = \sigma_\pi(fg)$, where $\sigma_\pi(\cdot)$ is the peripheral spectrum, by Luttman and Tonev [**9**]. Generalizing further to norm conditions, Hatori, Miura and Takagi [**3**, Corollary 7.5] showed that if

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a surjection $T : A \to B$ satisfies, for some non-zero complex number α, $\|T(f)T(g) - \alpha\|_\infty = \|fg - \alpha\|_\infty$ for all $f, g \in A$, then $T(1)^{-1}T$ is a real-algebra isomorphism. Lambert, Luttman and Tonev [**7**, Theorem 4] also showed that a surjection $T : A \to B$ is an algebra isomorphism if T satisfies $T(1) = 1$ and $\|T(f)T(g) + \alpha\|_\infty = \|fg + \alpha\|$ for all $f, g \in A$ and for all $\alpha \in \mathbf{C}$ with $|\alpha| = 1$. In the case when $A = C(X)$ and $B = C(Y)$, Honma [**5**, Theorem 1.1] has shown that a surjection $T : C(X) \to C(Y)$ is an algebraic isomorphism whenever $T(\lambda) = \lambda$ for $\lambda = \pm 1, \pm i$ and $\|T(f)\overline{T(g)} - 1\|_\infty = \|f\overline{g} - 1\|$ for all $f, g \in C(X)$. Most recently, Luttman and Lambert [**8**] gave a representation of such a surjection $T : A \to B$ that $\|\alpha T(f)T(g) + \beta\|_\infty = \|\alpha fg + \beta\|_\infty$ for all $f, g \in A$, where α, β are fixed non-zero complex numbers.

For unital commutative Banach algebras $\mathcal{A}$ and $\mathcal{B}$, Hatori, Miura and Takagi [**3**, Theorem 7.4] have shown that if $\mathcal{A}$ is semi-simple and $T : \mathcal{A} \to \mathcal{B}$ is a surjection such that, for some non-zero complex number α, $\mathrm{r}(T(f)T(g) - \alpha) = \mathrm{r}(fg - \alpha)$ for all $f, g \in \mathcal{A}$, then $T(1)^{-1}T$ is a real-algebra isomorphism. Here, $\mathrm{r}(f)$ is the spectral radius of $f \in \mathcal{A}$. They [**3**] have also extended the results of Molnár for a surjection $T : \mathcal{A} \to \mathcal{B}$ between two unital commutative Banach algebras with symmetric involutions $*$ and $\star$, respectively. An involution $*$ on $\mathcal{A}$ is symmetric if $\widehat{f^*} = \overline{\widehat{f}}$ for every $f \in \mathcal{A}$, where $\widehat{\cdot}$ is the Gelfand transform. In particular, it was shown that if $\mathcal{A}$ is semi-simple and $\sigma(T(f)T(g)^\star) = \sigma(fg^*)$ for all $f, g \in \mathcal{A}$, then $T(1)^{-1}T$ is an algebraic isomorphism [**4**, Theorem 6.2].

Let $\rho : \mathcal{A} \to \mathcal{A}$ and $\tau : \mathcal{B} \to \mathcal{B}$ be bijections. Suppose that $\mathcal{A}$ is semi-simple and $T : \mathcal{A} \to \mathcal{B}$ is a surjection with, for some non-zero complex number α, $\mathrm{r}(T(f)\tau(T(g)) - \alpha) = \mathrm{r}(f\rho(g) - \alpha)$ for all $f, g \in \mathcal{A}$. In this paper, we will show that $\mathcal{B}$ is semi-simple and $\mathrm{r}(T(f)T(g)^{-1} - 1) = \mathrm{r}(fg^{-1} - 1)$ for every $f \in \mathcal{A}$ and $g \in \mathcal{A}^{-1}$. Furthermore, we will prove that $T(\mathcal{A}^{-1}) = \mathcal{B}^{-1}$ and $T(1)^{-1}T$ is a real-algebra isomorphism. If, in addition, $T(1)^{-1}T(i) = i$, then we show that $T(1)^{-1}T$ is a complex-algebra isomorphism.

2. Preliminaries. Let A and B be uniform algebras on compact Hausdorff spaces X and Y, respectively. Suppose that A^{-1} is the group of invertible elements in A and $\exp A = \{\exp f : f \in A\}$ is the exponentials of A. Let $(\mathcal{G}_A, \mathcal{G}_B) = (\exp A, \exp B)$ or (A^{-1}, B^{-1}).

We denote by $\sigma(f)$ and $\mathrm{r}(f)$ the spectrum and the spectral radius of $f \in A$, respectively. It is well known that $\mathrm{r}(f) = \|f\|_\infty$ for each $f \in A$, where $\|\cdot\|_\infty$ is the supremum norm on X. Define the subset of $\sigma(f)$ for $f \in A$ by $\sigma_\pi(f) = \{\lambda \in \sigma(f) : |\lambda| = \mathrm{r}(f)\}$. A subset K of X is called a peak set for A if there exists an $f_K \in A$ with $\sigma_\pi(f_K) = \{1\}$ and ${f_K}^{-1}(\{1\}) = K$. We say that f_K is a peak function of A and peaks on K. If $\{x\}$ for $x \in X$ is the intersection of a family of peak sets for A, x is called a peak point in the weak sense for A. Let $\mathrm{Ch}\,(A)$ be the Choquet boundary of A. Recall that $\mathrm{Ch}\,(A)$ is the set of all peak points in the weak sense for A. Set $P_{\mathcal{G}_A}(x) = \{u \in \mathcal{G}_A : \sigma_\pi(u) = \{1\}, u(x) = 1\}$ for each $x \in \mathrm{Ch}\,(A)$. Note that $P_{\mathcal{G}_A}(x)$ is not empty for any $x \in \mathrm{Ch}\,(A)$. We denote by $\mathbf{C}$ and $\mathbf{N}$ the complex number field and the set of all positive integers, respectively.

The following lemma is a well-known extension of a theorem of Bishop [**1**, Theorem 2.4.1]. See also [**4**, Lemma 2.3].

Lemma 2.1. *Suppose that $f \in A$ and $x_0 \in Ch\,(A)$ with $|f(x_0)| \neq 0$. For a closed set $F \subset X$ with $x_0 \notin F$, there exists a $u \in P_{\exp A}(x_0)$ such that $\sigma_\pi(fu) = \{f(x_0)\}$ and $|fu(x)| < |f(x_0)|$ for all $x \in F$.*

Note that $f(x_0) \neq 0$ for every $f \in \mathcal{G}_A$. If $f \in \mathcal{G}_A$ satisfies $|f(x_0)| < 1$ for some $x_0 \in \mathrm{Ch}\,(A)$, then by Lemma 2.1 there exists a $u_0 \in P_{\mathcal{G}_A}(x_0)$ such that $\sigma_\pi(fu_0) = \{f(x_0)\}$. Since $\|fu_0\|_\infty = |f(x_0)| < 1$, we get the following lemma.

Lemma 2.2. *Let $f \in \mathcal{G}_A$ with $\|f\|_\infty = 1$ and $x_0 \in Ch\,(A)$. If $\|fu\|_\infty = 1$ for all $u \in P_{\mathcal{G}_A}(x_0)$, then $|f(x_0)| = 1$.*

For $x \in \mathrm{Ch}\,(A)$ and $y \in \mathrm{Ch}\,(B)$, define the subsets of A, B by

$$V_x = \{f \in \mathcal{G}_A : |f(x)| = 1 = \|f\|_\infty\},$$
$$W_y = \{\mathfrak{f} \in \mathcal{G}_B : |\mathfrak{f}(y)| = 1 = \|\mathfrak{f}\|_\infty\}.$$

Note that $P_{\mathcal{G}_A}(x) \subset V_x$ for any $x \in \mathrm{Ch}\,(A)$. Let $x_1, x_2 \in \mathrm{Ch}\,(A)$ with $x_1 \neq x_2$. By Lemma 2.1 with $f = 1$, there exists a $u \in P_{\mathcal{G}_A}(x_1)$ such that $u(x_1) = 1$ and $|u(x_2)| < 1$. Since $u \in V_{x_1} \setminus V_{x_2}$, we have $V_{x_1} \not\subset V_{x_2}$. We have the following:

Lemma 2.3. *If $V_{x_1} \subset V_{x_2}$ for some x_1, $x_2 \in Ch(A)$, then $x_1 = x_2$.*

Let $\rho : \mathcal{G}_A \to \mathcal{G}_A$ and $\tau : \mathcal{G}_B \to \mathcal{G}_B$ be bijections. For some fixed $\alpha \in \mathbf{C} \setminus \{0\}$, let T be a surjection from $\mathcal{G}_A$ onto $\mathcal{G}_B$ such that

$$\|T(f)\tau(T(g)) - \alpha\|_\infty = \|f\rho(g) - \alpha\|_\infty$$

for all $f, g \in \mathcal{G}_A$.

Remark 2.1. Note that $\tau(T(\rho^{-1}(\alpha g^{-1}))) = \alpha T(g)^{-1}$ for every $g \in \mathcal{G}_A$ since

$$\|T(g)\tau(T(\rho^{-1}(\alpha g^{-1}))) - \alpha\|_\infty = \|g\rho(\rho^{-1}(\alpha g^{-1})) - \alpha\|_\infty = 0.$$

Then we have

$$\left\|\frac{T(f)}{T(g)} - 1\right\|_\infty = \left\|\frac{f}{g} - 1\right\|_\infty \tag{2.1}$$

for all $f, g \in \mathcal{G}_A$ since

$$\begin{aligned}\left\|\frac{\alpha T(f)}{T(g)} - \alpha\right\|_\infty &= \|T(f)\tau(T(\rho^{-1}(\alpha g^{-1}))) - \alpha\|_\infty \\ &= \|f\rho(\rho^{-1}(\alpha g^{-1})) - \alpha\|_\infty = \left\|\frac{\alpha f}{g} - \alpha\right\|_\infty.\end{aligned}$$

Remark 2.2. By (2.1), the map T is injective. Thus $T^{-1} : \mathcal{G}_B \to \mathcal{G}_A$ is a well-defined surjection satisfying

$$\left\|\frac{T^{-1}(\mathfrak{f})}{T^{-1}(\mathfrak{g})} - 1\right\|_\infty = \left\|\frac{\mathfrak{f}}{\mathfrak{g}} - 1\right\|_\infty$$

for all $\mathfrak{f}, \mathfrak{g} \in \mathcal{G}_B$. The equation (2.1) also implies that

$$\begin{aligned}\left\|\frac{f}{g} - \frac{1}{n}\right\|_\infty &= \frac{1}{n}\left\|\frac{nf}{g} - 1\right\|_\infty = \frac{1}{n}\left\|\frac{T(nf)}{T(g)} - 1\right\|_\infty \\ &\le \frac{1}{n}\left\{\left\|\frac{T(nf)}{T(f)}\right\|_\infty \left\|\frac{T(f)}{T(g)}\right\|_\infty + 1\right\}\end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{n}\left\{\left(\left\|\frac{T(nf)}{T(f)}-1\right\|_\infty+1\right)\left\|\frac{T(f)}{T(g)}\right\|_\infty+1\right\}\\
&=\frac{1}{n}\left\{\left(\left\|\frac{nf}{f}-1\right\|_\infty+1\right)\left\|\frac{T(f)}{T(g)}\right\|_\infty+1\right\}\\
&=\left\|\frac{T(f)}{T(g)}\right\|_\infty+\frac{1}{n}
\end{aligned}$$

for all $n\in\mathbf{N}$ and $f,g\in\mathcal{G}_A$. Applying similar arguments to T^{-1}, we get

$$\left\|\frac{T(f)}{T(g)}-\frac{1}{n}\right\|_\infty\leq\left\|\frac{T^{-1}(T(f))}{T^{-1}(T(g))}\right\|_\infty+\frac{1}{n}=\left\|\frac{f}{g}\right\|_\infty+\frac{1}{n}$$

for all $n\in\mathbf{N}$ and $f,g\in\mathcal{G}_A$. We conclude that

$$\left\|\frac{T(f)}{T(g)}\right\|_\infty=\left\|\frac{f}{g}\right\|_\infty \tag{2.2}$$

for all $f,g\in\mathcal{G}_A$.

Until the end of this section, we will assume that $T(1)=1$. By (2.2), we have

$$\|T(f)\|_\infty=\|f\|_\infty \tag{2.3}$$

for every $f\in\mathcal{G}_A$.

Lemma 2.4. *For any $y\in Ch(B)$, there exists a unique $x\in Ch(A)$ such that $T^{-1}(W_y)=V_x$.*

Proof. Let $y\in \mathrm{Ch}(B)$ and set $G_y=\cap_{f\in T^{-1}(W_y)}|f|^{-1}(\{1\})$. First, we show that G_y is not empty. By the finite intersection property, it is enough to show that $\cap_{j=1}^n|f_j|^{-1}(\{1\})\neq\varnothing$ for any finite number of functions $f_1,f_2,\dots,f_n\in T^{-1}(W_y)$. Since T is surjective, there exists an $f_0\in\mathcal{G}_A$ with $T(f_0)=\Pi_{j=1}^nT(f_j)$. Let $j=1,2,\dots,n$. By the definition of W_y, we have $|T(f_j)(y)|=1=\|T(f_j)\|_\infty$. Hence $\|f_0\|_\infty=\|T(f_0)\|_\infty=1$ by (2.3), and $|T(f_0)|^{-1}(\{1\})\subset|T(f_j)|^{-1}(\{1\})$. Since $\mathrm{Ch}(A)$ is a boundary for A, there exists an $x_0\in|f_0|^{-1}(\{1\})\cap\mathrm{Ch}(A)$. Let $u\in P_{\mathcal{G}_A}(x_0)$. Applying (2.2), we have $\|T(f_0)/T(u^{-1})\|_\infty=$

$\|f_0/u^{-1}\|_\infty = \|f_0 u\|_\infty = 1$. So, there exists a $y_0 \in \mathrm{Ch}\,(B)$ such that $|T(f_0)(y_0)/T(u^{-1})(y_0)| = 1$. On one hand, since $T(1) = 1$, (2.2) implies $1/|T(u^{-1})(y_0)| \le \|T(1)/T(u^{-1})\|_\infty = \|1/u^{-1}\|_\infty = \|u\|_\infty = 1$, and thus $1 \le |T(u^{-1})(y_0)|$. On the other hand, $|T(f_0)(y_0)| \le 1$ since $\|T(f_0)\|_\infty = 1$. Therefore, $1 \le |T(u^{-1})(y_0)| = |T(f_0)(y_0)| \le 1$, and consequently $|T(f_0)(y_0)| = 1$. Since $\|f_j u\|_\infty \le 1$ and $|T(f_0)|^{-1}(\{1\}) \subset |T(f_j)|^{-1}(\{1\})$, we have $\|f_j u\|_\infty = \|f_j/u^{-1}\|_\infty = \|T(f_j)/T(u^{-1})\|_\infty = 1$, and so $|f_j(x_0)| = 1$ by Lemma 2.2. Hence, $x_0 \in \cap_{j=1}^n |f_j|^{-1}(\{1\})$.

Finally, we show that there exists a unique $x \in G_y \cap \mathrm{Ch}\,(A)$ with $T^{-1}(W_y) = V_x$. Let $x_1 \in G_y$. For each $f \in T^{-1}(W_y)$, we define $\widetilde{f} \in A$ by $\widetilde{f} = (\overline{f(x_1)}f + 1)/2$. Then $f^{-1}(\{f(x_1)\}) = \widetilde{f}^{-1}(\{1\})$ is a peak set for A with $\widetilde{f}^{-1}(\{1\}) \subset |f|^{-1}(\{1\})$ for every $f \in T^{-1}(W_y)$. Thus $\cap_{f \in T^{-1}(W_y)} \widetilde{f}^{-1}(\{1\})$ is not empty. According to [**1**, Corollary 2.4.5], there exists an $x \in \cap_{f \in T^{-1}(W_y)} \widetilde{f}^{-1}(\{1\}) \cap \mathrm{Ch}\,(A) \subset G_y \cap \mathrm{Ch}\,(A)$. By (2.3), we have $\|f\|_\infty = \|T(f)\|_\infty = 1$ with $|f(x)| = 1$ for all $f \in T^{-1}(W_y)$. Thus $T^{-1}(W_y) \subset V_x$. A similar argument applied to T^{-1} shows that $T(V_x) \subset W_{y'}$ for some $y' \in \mathrm{Ch}\,(B)$. Therefore, $W_y \subset T(V_x) \subset W_{y'}$. Lemma 2.3 implies that $y = y'$, and consequently $T(V_x) = W_y$. For any $x' \in \mathrm{Ch}\,(A)$ with $T(V_{x'}) = W_y$, we conclude that $T(V_x) = W_y = T(V_{x'})$, that is, $V_x = V_{x'}$. It follows from Lemma 2.3 that the x is unique. □

We define the mapping ϕ from $\mathrm{Ch}\,(B)$ into $\mathrm{Ch}\,(A)$ by $T(V_{\phi(y)}) = W_y$ for $y \in \mathrm{Ch}\,(B)$. Then ϕ is well-defined. Applying Lemma 2.4 to T^{-1}, we may define the mapping ψ from $\mathrm{Ch}\,(A)$ into $\mathrm{Ch}\,(B)$ by $T^{-1}(W_{\psi(x)}) = V_x$ for $x \in \mathrm{Ch}\,(A)$. The definitions of ϕ and ψ with Lemma 2.3 imply that $\psi(\phi(y)) = y$ for any $y \in \mathrm{Ch}\,(B)$ and $\phi(\psi(x)) = x$ for any $x \in \mathrm{Ch}\,(A)$. Thus ϕ is bijective and $\psi = \phi^{-1}$.

Lemma 2.5. *The equation*

$$|T(f)(y)| = |f(\phi(y))| \tag{2.4}$$

holds for every $f \in \mathcal{G}_A$ and $y \in Ch\,(B)$.

Proof. Let $f \in \mathcal{G}_A$ and $y \in \mathrm{Ch}\,(B)$. By Lemma 2.1, there exists a $u_0 \in P_{\mathcal{G}_A}(\phi(y))$ with $\sigma_\pi(fu_0) = \{f(\phi(y))\}$. The equation (2.2) implies that

$$|f(\phi(y))| = \|fu_0\|_\infty = \left\| \frac{T(f)}{T({u_0}^{-1})} \right\|_\infty \geq \left| \frac{T(f)(y)}{T({u_0}^{-1})(y)} \right|$$

and $\|1/T(u_0^{-1})\|_\infty = \|u_0\|_\infty = 1$. Since $|u(\phi(y))| = 1 = \|u\|_\infty$ for every $u \in V_{\phi(y)}$, the equation (2.2) shows that

$$\left\| \frac{T(u)}{T({u_0}^{-1})} \right\|_\infty = \|u_0 u\|_\infty = 1$$

for all $u \in V_{\phi(y)}$. Lemma 2.2 with $P_{\mathcal{G}_B}(y) \subset W_y = T(V_{\phi(y)})$ implies that $|1/T({u_0}^{-1})(y)| = 1$. Hence $|T(f)(y)| \leq |f(\phi(y))|$. Applying similar arguments to T^{-1} and $\phi(y)$, we have $|T^{-1}(T(f))(\phi(y))| \leq |T(f)(\phi^{-1}(\phi(y)))|$; thus, $|f(\phi(y))| \leq |T(f)(y)|$. Consequently, the equation $|T(f)(y)| = |f(\phi(y))|$ holds. $\square$

Remark 2.3. By the Alexandroff theorem [**6**, Theorem 8, Chapter 5], the weak topology on X induced by $\{|f| : f \in \mathcal{G}_A\}$ coincides with the original topology on X. Also, the weak topology on Y induced by $\{|\mathfrak{f}| : \mathfrak{f} \in \mathcal{G}_B\}$ coincides with the original topology on Y. Thus, by (2.4), the bijection $\phi : \mathrm{Ch}\,(B) \to \mathrm{Ch}\,(A)$ is continuous. Similar arguments show that ϕ^{-1} is also continuous. We conclude that ϕ is a homeomorphism from $\mathrm{Ch}\,(B)$ onto $\mathrm{Ch}\,(A)$.

Note that Lemma 2.5 implies that, if $\beta \in \mathbf{C}$, then

$$|T(\beta)| = |\beta|. \tag{2.5}$$

Lemma 2.6. *We have $T^{-1}(P_{\mathcal{G}_B}(y)) \subset P_{\mathcal{G}_A}(\phi(y))$ for any $y \in$ Ch$\,(B)$, and $T(-1) = -1$.*

Proof. For $y \in \mathrm{Ch}\,(B)$, let $u \in T^{-1}(P_{\mathcal{G}_B}(y))$ and $\beta \in \sigma_\pi(u)$. Then $|\beta| = \|u\|_\infty = \|T(u)\|_\infty = 1$. The equation (2.5) implies that

$$\left\| \frac{T(u)}{T(-\beta)} \right\|_\infty = 1.$$

Since $\beta \in \sigma_\pi(u)$, there exists an $x \in X$ with $u(x) = \beta$. Therefore,

$$2 = \left| \frac{u(x)}{-\beta} - 1 \right| \leq \left\| \frac{u}{-\beta} - 1 \right\|_\infty \leq 2.$$

By (2.1), we have

$$\left\| \frac{T(u)}{T(-\beta)} - 1 \right\|_\infty = \left\| \frac{u}{-\beta} - 1 \right\|_\infty = 2,$$

which implies that $(T(u)/T(-\beta))(y_0) = -1$ for some $y_0 \in \mathrm{Ch}\,(B)$ since $\mathrm{Ch}\,(B)$ is a boundary for B. By (2.5), $|T(u)(y_0)| = |T(-\beta)(y_0)| = 1$. Since $T(u) \in P_{\mathcal{G}_B}(y)$, we have $T(u)(y_0) = 1$, and so $T(-\beta)(y_0) = -1$. Thus, $2 \leq |-\beta - 1|$ since $\|T(-\beta) - 1\|_\infty = |-\beta - 1|$ by (2.1) and $T(1) = 1$. This shows $\beta = 1$, that is, $\sigma_\pi(u) = \{1\}$. By (2.4), we have $|u(\phi(y))| = |T(u)(y)| = 1$, and so $u(\phi(y)) = 1$. We conclude that $T^{-1}(P_{\mathcal{G}_B}(y)) \subset P_{\mathcal{G}_A}(\phi(y))$.

In order to prove $T(-1) = -1$, it is sufficient to show that $T(-1) = -1$ on $\mathrm{Ch}\,(B)$ since $\mathrm{Ch}\,(B)$ is a boundary for B. Let $y \in \mathrm{Ch}\,(B)$. By Lemma 2.1 with the surjectivity of T, there exists a $u \in \mathcal{G}_A$ such that $T(u) \in P_{\mathcal{G}_B}(y)$ and $\sigma_\pi(T(u)/T(-1)) = \{1/T(-1)(y)\}$. By (2.5), we get $\|T(u)/T(-1)\|_\infty = |1/T(-1)(y)| = 1$. On the other hand, equation (2.1) implies that

$$\left\| \frac{T(u)}{T(-1)} - 1 \right\|_\infty = \left\| \frac{u}{-1} - 1 \right\|_\infty = 2$$

since $T^{-1}(P_{\mathcal{G}_B}(y)) \subset P_{\mathcal{G}_A}(\phi(y))$. Hence there exists a $y_0 \in Y$ such that $(T(u)/T(-1))(y_0) = -1$. Since $\sigma_\pi(T(u)/T(-1)) = \{1/T(-1)(y)\}$, we have $-1 = (T(u)/T(-1))(y_0) = 1/T(-1)(y)$, that is, $T(-1)(y) = -1$. $\square$

Lemma 2.7. *For any $\lambda \in \mathbf{C}$ with $|\lambda| = 1$, the range $T(\lambda)(Ch\,(B))$ is contained in $\{\lambda, \overline{\lambda}\}$.*

Proof. Let $\lambda \in \mathbf{C}$ with $|\lambda| = 1$. Equation (2.1) and $T(1) = 1$ imply that

$$\|T(\lambda) - 1\|_\infty = \left\| \frac{T(\lambda)}{T(1)} - 1 \right\|_\infty = \|\lambda - 1\|_\infty = |\lambda - 1|.$$

Since $T(-1) = -1$ by Lemma 2.6, we also have $\|T(\lambda) + 1\|_\infty = |\lambda + 1|$. Thus the inequalities

$$|T(\lambda)(y) - 1| \leq |\lambda - 1| \quad \text{and} \quad |T(\lambda)(y) + 1| \leq |\lambda + 1|$$

hold for any $y \in \mathrm{Ch}\,(B)$. On the other hand, the equation (2.5) implies that $|T(\lambda)(y)| = |\lambda| = 1$. We thus conclude that $T(\lambda)(y) = \lambda$ or $\overline{\lambda}$ for each $y \in \mathrm{Ch}\,(B)$. □

Lemma 2.8. *Let $y \in Ch\,(B)$, $\mathfrak{u} \in P_{\mathcal{G}_B}(y)$ and $\lambda \in \mathbf{C}$ with $|\lambda| = 1$. For any $\beta \in \sigma_\pi(T^{-1}(T(\lambda)\mathfrak{u}))$, there exists a $y_0 \in Ch\,(B)$ such that $T(-\beta)(y_0) = -T(\lambda)(y_0)$.*

Proof. Let $y \in \mathrm{Ch}\,(B)$, $\mathfrak{u} \in P_{\mathcal{G}_B}(y)$ and $\lambda \in \mathbf{C}$ with $|\lambda| = 1$. Suppose that $\beta \in \sigma_\pi(T^{-1}(T(\lambda)\mathfrak{u}))$. We have $|\beta| = 1$ since

$$\|T^{-1}(T(\lambda)\mathfrak{u})\|_\infty = \|T(\lambda)\mathfrak{u}\| = \|\mathfrak{u}\| = 1$$

by (2.4) and Lemma 2.7. It follows from (2.2) that

$$\left\|\frac{T(\lambda)\mathfrak{u}}{T(-\beta)}\right\|_\infty = \left\|\frac{T^{-1}(T(\lambda)\mathfrak{u})}{-\beta}\right\|_\infty = 1.$$

Since $\beta \in \sigma_\pi(T^{-1}(T(\lambda)\mathfrak{u}))$, there exists an $x \in X$ such that $\beta = T^{-1}(T(\lambda)\mathfrak{u})(x)$. Therefore, by (2.1),

$$\left\|\frac{T(\lambda)\mathfrak{u}}{T(-\beta)} - 1\right\|_\infty = \left\|\frac{T^{-1}(T(\lambda)\mathfrak{u})}{-\beta} - 1\right\|_\infty = 2.$$

It follows that there exists a $y_0 \in \mathrm{Ch}\,(B)$ with

$$\frac{T(\lambda)\mathfrak{u}}{T(-\beta)}(y_0) = -1.$$

By Lemma 2.7, $|T(\lambda)| = |T(-\beta)| = 1$. Hence $|\mathfrak{u}(y_0)| = 1$. Since $\mathfrak{u} \in P_{\mathcal{G}_B}(y)$, we have $\mathfrak{u}(y_0) = 1$, and so $T(-\beta)(y_0) = -T(\lambda)(y_0)$. □

Lemma 2.9. *The inclusion $T(i)P_{\mathcal{G}_B}(y) \subset T(iP_{\mathcal{G}_A}(\phi(y)))$ holds for any $y \in Ch\,(B)$.*

Proof. Let $y \in \mathrm{Ch}\,(B)$ and $\mathfrak{u} \in P_{\mathcal{G}_B}(y)$. We show that $T^{-1}(T(i)\mathfrak{u})/i \in P_{\mathcal{G}_A}(\phi(y))$. Let $\beta \in \sigma_\pi(T^{-1}(T(i)\mathfrak{u}))$. Since

$$\|T^{-1}(T(i)\mathfrak{u})\|_\infty = \|T(i)\mathfrak{u}\|_\infty = \|\mathfrak{u}\|_\infty = 1,$$

we have $|\beta| = 1$. Lemma 2.8 yields the existence of $y_0 \in \mathrm{Ch}\,(B)$ with $T(-\beta)(y_0) = -T(i)(y_0)$. On the other hand, Lemma 2.7 shows that $T(-\beta)(y_0) = -\beta$ or $-\overline{\beta}$ and $T(i)(y_0) = i$ or $-i$. Thus $\beta = i$ or $-i$. Since

$$\left\| \frac{T^{-1}(T(i)\mathfrak{u})}{i} - 1 \right\|_\infty = \left\| \frac{T(i)\mathfrak{u}}{T(i)} - 1 \right\|_\infty = \|\mathfrak{u} - 1\|_\infty < 2,$$

we have $\beta \neq -i$. Hence $\beta = i$, and so $\sigma_\pi(T^{-1}(T(i)\mathfrak{u})) = \{i\}$. It follows from (2.4) that

$$|T^{-1}(T(i)\mathfrak{u})(\phi(y))| = |T(i)(y)\mathfrak{u}(y)| = 1.$$

Therefore, $T^{-1}(T(i)\mathfrak{u})(\phi(y)) = i$. We conclude that $T^{-1}(T(i)\mathfrak{u})/i \in P_{\mathcal{G}_A}(\phi(y))$. $\square$

Define a subset $K_T \subset \mathrm{Ch}\,(B)$ by

$$K_T = \{y \in \mathrm{Ch}\,(B) : T(i)(y) = i\}$$

for a surjection T from $\mathcal{G}_A$ onto $\mathcal{G}_B$. By Lemma 2.7, K_T is a clopen subset of $\mathrm{Ch}\,(B)$. By the continuity of $T(i)$, we see that $(K_T)^{\mathrm{cl}}$ and $(\mathrm{Ch}\,(B) \setminus K_T)^{\mathrm{cl}}$ are disjoint, where $(\cdot)^{\mathrm{cl}}$ denotes the closure in Y.

Lemma 2.10. *The equation* $T(-i) = -T(i)$ *holds on* $\mathrm{Ch}\,(B)$.

Proof. Let $y \in \mathrm{Ch}\,(B)$. We have two cases: If $y \in K_T$, then set $F = (\mathrm{Ch}\,(B) \setminus K_T)^{\mathrm{cl}}$; If $y \in \mathrm{Ch}\,(B) \setminus K_T$, then set $F = (K_T)^{\mathrm{cl}}$. By Lemma 2.1, there exists a $\mathfrak{u} \in P_{\mathcal{G}_B}(y)$ such that

$$\sigma_\pi\left(\frac{\mathfrak{u}}{T(-i)}\right) = \left\{\frac{1}{T(-i)(y)}\right\}$$

and that the inequality

$$\left|\frac{\mathfrak{u}}{T(-i)}\right| < \left|\frac{1}{T(-i)(y)}\right| = \frac{1}{|-i(\phi(y))|} = 1$$

holds on F. Lemma 2.9 implies that there exists a $u \in P_{\mathcal{G}_A}(\phi(y))$ with $T(i)\mathfrak{u} = T(iu)$. Hence we have, by (2.1),

$$\left\| \frac{T(i)\mathfrak{u}}{T(-i)} - 1 \right\|_\infty = \left\| \frac{T(iu)}{T(-i)} - 1 \right\|_\infty = \left\| \frac{iu}{-i} - 1 \right\|_\infty = \|-u-1\|_\infty = 2,$$

and, by (2.2),

$$\left\| \frac{T(i)\mathfrak{u}}{T(-i)} \right\|_\infty = \left\| \frac{T(iu)}{T(-i)} \right\|_\infty = \left\| \frac{iu}{-i} \right\|_\infty = \|-u\|_\infty = 1.$$

There exists a $y_0 \in \mathrm{Ch}\,(B)$ such that

$$\frac{T(i)\mathfrak{u}}{T(-i)}(y_0) = -1.$$

Note that $|\mathfrak{u}(y_0)| = 1$ by Lemma 2.7. It follows from $\mathfrak{u} \in P_{\mathcal{G}_B}(y)$ that $\mathfrak{u}(y_0) = 1$; hence, $T(i)(y_0) = -T(-i)(y_0)$. Since $|\mathfrak{u}/T(-i)| < 1$ on F, we have $y_0 \in \mathrm{Ch}\,(B) \setminus F$. This shows that $T(i)(y) = T(i)(y_0)$. Moreover, the assumption for $\mathfrak{u}$, $\sigma_\pi(\mathfrak{u}/T(-i)) = \{1/T(-i)(y)\}$, implies that

$$\frac{1}{T(-i)(y_0)} = \frac{1}{T(-i)(y)}.$$

Consequently, we obtain

$$T(i)(y) = T(i)(y_0) = -T(-i)(y_0) = -T(-i)(y),$$

and so $T(-i) = -T(i)$. □

Lemma 2.11. *For any $\lambda \in \mathbf{C}$ with $|\lambda| = 1$, the equality*

$$T(\lambda) = \begin{cases} \lambda & \text{on } K_T \\ \overline{\lambda} & \text{on } \mathrm{Ch}\,(B) \setminus K_T \end{cases}$$

holds.

Proof. Let $\lambda \in \mathbf{C}$ with $|\lambda| = 1$. Since $|T(i)| = |T(-i)| = 1$ on $\mathrm{Ch}\,(B)$, we have

$$|\lambda - i| = \left\| \frac{\lambda}{i} - 1 \right\|_\infty = \left\| \frac{T(\lambda)}{T(i)} - 1 \right\|_\infty = \|T(\lambda) - T(i)\|_\infty$$

and

$$|\lambda + i| = \left\| \frac{\lambda}{-i} - 1 \right\|_\infty = \left\| \frac{T(\lambda)}{T(-i)} - 1 \right\|_\infty = \|T(\lambda) - T(-i)\|_\infty.$$

If $y \in K_T$, then by Lemma 2.10

$$|T(\lambda)(y) - i| \leq |\lambda - i| \quad \text{and} \quad |T(\lambda)(y) + i| \leq |\lambda + i|.$$

It follows from Lemma 2.7 that $T(\lambda)(y) = \lambda$. Similar arguments show that $T(\lambda)(y) = \overline{\lambda}$ for any $y \in \mathrm{Ch}\,(B) \setminus K_T$. □

Lemma 2.12. *The inclusion $T(\lambda)P_{\mathcal{G}_B}(y) \subset T(\lambda P_{\mathcal{G}_A}(\phi(y)))$ holds for any $y \in \mathrm{Ch}\,(B)$ and $\lambda \in \mathbf{C}$ with $|\lambda| = 1$.*

Proof. Let $y \in \mathrm{Ch}\,(B)$, $\mathfrak{u} \in P_{\mathcal{G}_B}(y)$, and $\lambda \in \mathbf{C}$ with $|\lambda| = 1$. We show that $T^{-1}(T(\lambda)\mathfrak{u})/\lambda \in P_{\mathcal{G}_A}(\phi(y))$. Let $\beta \in \sigma_\pi(T^{-1}(T(\lambda)\mathfrak{u}))$. Lemma 2.8 implies that there is a $y_0 \in \mathrm{Ch}\,(B)$ with $T(-\beta)(y_0) = -T(\lambda)(y_0)$. By Lemma 2.11, we have $-\beta = -\lambda$ or $-\overline{\beta} = -\overline{\lambda}$, which implies that $\beta = \lambda$. Thus $\sigma_\pi(T^{-1}(T(\lambda)\mathfrak{u})) = \{\lambda\}$. It follows from (2.4) that $|T^{-1}(T(\lambda)\mathfrak{u})(\phi(y))| = |(T(\lambda)\mathfrak{u})(y)| = 1$. Thus $T^{-1}(T(\lambda)\mathfrak{u})(\phi(y)) = \lambda$, and hence $T^{-1}(T(\lambda)\mathfrak{u})/\lambda \in P_{\mathcal{G}_A}(\phi(y))$. □

Proposition 2.13. *The equality*

$$T(f)(y) = \begin{cases} f(\phi(y)) & y \in K_T \\ \overline{f(\phi(y))} & y \in \mathrm{Ch}\,(B) \setminus K_T \end{cases}$$

holds for every $f \in \mathcal{G}_A$ and $y \in \mathrm{Ch}\,(B)$.

Proof. For each $f \in \mathcal{G}_A$ and $y \in \mathrm{Ch}\,(B)$, set $\lambda = -f(\phi(y))/|T(f)(y)|$ and $\mu = 1/T(f)(y)$. Then $|\lambda| = 1$ by (2.4). Lemma 2.1 implies that there exists a $\mathfrak{u} \in P_{\mathcal{G}_B}(y)$ such that

$$\sigma_\pi\left(\frac{\mathfrak{u}}{T(f)}\right)_\infty = \{\mu\} \tag{2.6}$$

and that

$$\left|\frac{\mathfrak{u}}{T(f)}\right| < |\mu| \text{ on } F, \tag{2.7}$$

where $F = (K_T)^{\mathrm{cl}}$ if $y \in \mathrm{Ch}\,(B) \setminus K_T$; otherwise $F = (\mathrm{Ch}\,(B) \setminus K_T)^{\mathrm{cl}}$. By Lemma 2.12, there exists a $u \in P_{\mathcal{G}_A}(\phi(y))$ with $T(\lambda)\mathfrak{u} = T(\lambda u)$. Thus, by (2.1), we have

$$\begin{aligned}\left\| \frac{T(\lambda)\mathfrak{u}}{T(f)} - 1 \right\|_\infty &= \left\| \frac{T(\lambda u)}{T(f)} - 1 \right\|_\infty = \left\| \frac{\lambda u}{f} - 1 \right\|_\infty \\ &\geq \left| \frac{\lambda}{f(\phi(y))} - 1 \right| = \left| - \frac{1}{|T(f)(y)|} - 1 \right| \\ &= |\mu| + 1.\end{aligned}$$

By Lemma 2.11, equation (2.6) implies that

$$\left\| \frac{T(\lambda)\mathfrak{u}}{T(f)} \right\|_\infty = \left\| \frac{\mathfrak{u}}{T(f)} \right\|_\infty = |\mu|,$$

that is,

$$\left\| \frac{T(\lambda)\mathfrak{u}}{T(f)} - 1 \right\|_\infty \leq \left\| \frac{T(\lambda)\mathfrak{u}}{T(f)} \right\|_\infty + 1 = |\mu| + 1.$$

This yields that

$$\left\| \frac{T(\lambda)\mathfrak{u}}{T(f)} - 1 \right\|_\infty = |\mu| + 1,$$

so there is a $y_0 \in \mathrm{Ch}\,(B)$ such that

$$\frac{T(\lambda)(y_0)\mathfrak{u}(y_0)}{T(f)(y_0)} = -|\mu|.$$

By Lemma 2.7 with $|\lambda| = 1$, we have $|\mathfrak{u}(y_0)/T(f)(y_0)| = |\mu|$. It follows from (2.6) that $\mathfrak{u}(y_0)/T(f)(y_0) = \mu$. Note that $y_0 \in \mathrm{Ch}\,(B) \setminus F$ by (2.7). Hence we have $T(\lambda)(y_0) = T(\lambda)(y)$. Consequently,

$$-\frac{|\mu|}{\mu} = \frac{T(\lambda)(y_0)\mathfrak{u}(y_0)}{T(f)(y_0)} \frac{T(f)(y_0)}{\mathfrak{u}(y_0)} = T(\lambda)(y).$$

Lemma 2.11 implies that

$$T(\lambda)(y) = \begin{cases} -f(\phi(y))/|T(f)(y)| & y \in K_T \\ -\overline{f(\phi(y))}/|T(f)(y)| & y \in \mathrm{Ch}\,(B) \setminus K_T. \end{cases}$$

By this equation, we conclude that $T(f)(y)$ is equal to $f(\phi(y))$ if $y \in K_T$ or $\overline{f(\phi(y))}$ if $y \in \mathrm{Ch}\,(B) \setminus K_T$. $\square$

3. Main results.

Theorem 3.1. *Let A and B be uniform algebras on compact Hausdorff spaces X and Y, respectively. Let $\alpha \in \mathbf{C}\setminus\{0\}$ and $(\mathcal{G}_A, \mathcal{G}_B) = (\exp A, \exp B)$ or (A^{-1}, B^{-1}). Suppose that $\rho : \mathcal{G}_A \to \mathcal{G}_A$ and $\tau : \mathcal{G}_B \to \mathcal{G}_B$ are bijections. If $T : \mathcal{G}_A \to \mathcal{G}_B$ is a surjection such that*

$$\|T(f)\tau(T(g)) - \alpha\|_\infty = \|f\rho(g) - \alpha\|_\infty$$

for all $f, g \in \mathcal{G}_A$, then there exist a clopen subset K of $\mathrm{Ch}\,(B)$ and a homeomorphism ϕ from $\mathrm{Ch}\,(B)$ onto $\mathrm{Ch}\,(A)$ such that the equality

$$T(f)(y) = T(1)(y) \times \begin{cases} f(\phi(y)) & y \in K \\ \overline{f(\phi(y))} & y \in \mathrm{Ch}\,(B) \setminus K \end{cases}$$

holds for every $f \in \mathcal{G}_A$. If, in addition, $T(1) = 1$, then T can be extended to a real-algebra isomorphism. Moreover, T can be extended to a complex-algebra isomorphism if $T(i) = i$.

Here we denote by $\mathcal{A}^{-1}$ the group of invertible elements in a unital commutative Banach algebra $\mathcal{A}$. Let 1 be the unit element in $\mathcal{A}$ and $\sigma(f)$ the spectrum of $f \in \mathcal{A}$. For a subset $\mathcal{S}$ of $\mathcal{A}$ with the maximal ideal space $M_{\mathcal{A}}$, define $\widehat{\mathcal{S}} = \{\widehat{f} : f \in \mathcal{S}\}$, where $\widehat{\cdot}$ is the Gelfand transform. The spectral radius $\mathrm{r}(f)$ of $f \in \mathcal{A}$ is defined by $\mathrm{r}(f) = \sup\{|\widehat{f}(x)| : x \in M_{\mathcal{A}}\}$. Let $\mathrm{cl}\,(\widehat{\mathcal{S}})$ be the uniform closure of $\widehat{\mathcal{S}}$ in $C(M_{\mathcal{A}})$.

Theorem 3.2. *Let $\mathcal{A}$ and $\mathcal{B}$ be unital commutative Banach algebras and $\alpha \in \mathbf{C} \setminus \{0\}$. Let $\rho : \mathcal{A}^{-1} \to \mathcal{A}^{-1}$ and $\tau : \mathcal{B}^{-1} \to \mathcal{B}^{-1}$ be bijections. Suppose that $\mathcal{A}$ is semi-simple. If $T : \mathcal{A}^{-1} \to \mathcal{B}^{-1}$ is a surjection such that*

$$\mathrm{r}(T(f)\tau(T(g)) - \alpha) = \mathrm{r}(f\rho(g) - \alpha)$$

for all $f, g \in \mathcal{A}^{-1}$, then $\mathcal{B}$ is semi-simple and there exist a clopen subset K of $\mathrm{Ch}\,(\mathrm{cl}\,(\widehat{\mathcal{B}}))$ and a homeomorphism ϕ from $\mathrm{Ch}\,(\mathrm{cl}\,(\widehat{\mathcal{B}}))$ onto

$\mathrm{Ch}\,(\mathrm{cl}\,(\widehat{\mathcal{B}}))$ *with*

$$\widehat{T(f)}(y) = \widehat{T(1)}(y) \times \begin{cases} \widehat{f}(\phi(y)) & y \in K \\ \overline{\widehat{f}(\phi(y))} & y \in \mathrm{Ch}\,(\mathrm{cl}\,(\widehat{B})) \setminus K \end{cases}$$

for every $f \in \mathcal{A}^{-1}$*. If, in addition,* $T(1) = 1$*, then* T *can be extended to a real-algebra isomorphism from* $\mathcal{A}$ *onto* $\mathcal{B}$*. Moreover, if* $T(i) = i$*, then* T *can be extended to a complex-algebra isomorphism.*

By Theorem 3.2, we also obtain the following result: the case when ρ and τ are both identities was proven by Hatori, Miura and Takagi [**3**, Theorem 7.4]. Honma [**5**, Theorem 1.1] obtained the corresponding result for $\mathcal{A} = C(X)$, $\mathcal{B} = C(Y)$, and $\rho = \tau = \bar{\cdot}$, the complex conjugate.

Corollary 3.3. *Let* $\mathcal{A}$ *and* $\mathcal{B}$ *be unital commutative Banach algebras. Let* $\rho : \mathcal{A} \to \mathcal{A}$ *and* $\tau : \mathcal{B} \to \mathcal{B}$ *be bijections. Suppose that* $\mathcal{A}$ *is semi-simple. If* $T : \mathcal{A} \to \mathcal{B}$ *is a surjection such that, for some* $\alpha \in \mathbf{C} \setminus \{0\}$*,*

$$\mathrm{r}(T(f)\tau(T(g)) - \alpha) = \mathrm{r}(f\rho(g) - \alpha)$$

for all $f, g \in \mathcal{A}$*, then* $\mathcal{B}$ *is semi-simple, and there exist a clopen subset* K *of* $\mathrm{Ch}\,(\mathrm{cl}\,(\widehat{B}))$ *and a homeomorphism* ϕ *from* $\mathrm{Ch}\,(\mathrm{cl}\,(\widehat{B}))$ *onto* $\mathrm{Ch}\,(\mathrm{cl}\,(\widehat{A}))$ *with*

$$\widehat{T(f)}(y) = \widehat{T(1)}(y) \times \begin{cases} \widehat{f}(\phi(y)) & y \in K \\ \overline{\widehat{f}(\phi(y))} & y \in \mathrm{Ch}\,(\mathrm{cl}\,(\widehat{B})) \setminus K \end{cases}$$

for every $f \in \mathcal{A}$*. If, in addition,* $T(1) = 1$*, then* T *is a real-algebra isomorphism. Moreover, if* $T(i) = i$*, then* T *is a complex-algebra isomorphism.*

As a consequence of Corollary 3.3, we generalize [**4**, Theorem 3.2] and [**4**, Theorem 6.2]. In fact, Hatori, Miura and Takagi proved the corresponding results for the two cases: ρ and τ are both identities; and ρ and τ are symmetric involutions.

Corollary 3.4. *Let $\mathcal{A}$ and $\mathcal{B}$ be unital commutative Banach algebras. Let $\rho : \mathcal{A} \to \mathcal{A}$ and $\tau : \mathcal{B} \to \mathcal{B}$ be bijections. Suppose that $\mathcal{A}$ is semi-simple. If $T : \mathcal{A} \to \mathcal{B}$ is a surjection such that*

$$\sigma(T(f)\tau(T(g))) = \sigma(f\rho(g))$$

for all $f, g \in \mathcal{A}$, then $\mathcal{B}$ is semi-simple, and there exists a homeomorphism Φ from $M_{\mathcal{B}}$ onto $M_{\mathcal{A}}$ such that

$$\widehat{T(f)}(y) = \widehat{T(1)}(y)\widehat{f}(\Phi(y))$$

for every $f \in \mathcal{A}$ and $y \in M_{\mathcal{B}}$. If, in addition, $T(1) = 1$, then T is a complex-algebra isomorphism.

4. Proof of main results.

Proof of Theorem 3.1. Let $T : \mathcal{G}_A \to \mathcal{G}_B$ be as in Theorem 3.1. By the same arguments used in Remark 2.1, we have

$$\left\|\frac{T(f)}{T(g)} - 1\right\|_\infty = \left\|\frac{f}{g} - 1\right\|_\infty$$

for all $f, g \in \mathcal{G}_A$. Set $T'(f) = T(f)/T(1)$ for $f \in \mathcal{G}_A$. Then T' is a surjection from $\mathcal{G}_A$ onto $\mathcal{G}_B$ with $T'(1) = 1$ and

$$\left\|\frac{T'(f)}{T'(g)} - 1\right\|_\infty = \left\|\frac{f}{g} - 1\right\|_\infty$$

for all $f, g \in \mathcal{G}_A$. By Proposition 2.13 applied to T', there exist a clopen subset $K = K_{T'}$ of $\mathrm{Ch}\,(B)$ and a homeomorphism ϕ from $\mathrm{Ch}\,(B)$ onto $\mathrm{Ch}\,(A)$ such that

$$T'(f)(y) = \begin{cases} f(\phi(y)) & y \in K \\ \overline{f(\phi(y))} & y \in \mathrm{Ch}\,(B) \setminus K \end{cases}$$

for every $f \in \mathcal{G}_A$. The conclusion follows from $T' = T/T(1)$. Finally, we show that T' can be extended to a real-algebra isomorphism from A onto B. For any $f \in A$, there exist an $f_0 \in \exp A$ and a $\lambda \in \mathbf{C}\setminus\{0\}$ with

$f = f_0 + \lambda$. We define a map S from A into B by $S(f) = T'(f_0) + T'(\lambda)$. Since

$$T'(f_0) + T'(\lambda) = \begin{cases} (f_0 + \lambda) \circ \phi & \text{on } K \\ \overline{(f_0 + \lambda) \circ \phi} & \text{on Ch}\,(B) \setminus K, \end{cases}$$

the map S is a well-defined injective linear multiplicative map from A into B with $S|_{\mathcal{G}_A} = T'$. We show that S is surjective. Let $\mathfrak{f} \in B$; then there exist an $\mathfrak{f}_0 \in \exp B$ and a $\mu \in \mathbf{C} \setminus \{0\}$ with $\mathfrak{f} = \mathfrak{f}_0 + \mu$. Since T' is a surjection from $\mathcal{G}_A$ onto $\mathcal{G}_B$, there exist $g_1, g_2 \in \mathcal{G}_A$ such that $T'(g_1) = \mathfrak{f}_0$ and $T'(g_2) = \mu$. Set $g = g_1 + g_2$. Then we have

$$S(g) = \begin{cases} g \circ \phi & \text{on } K \\ \overline{g \circ \phi} & \text{on Ch}\,(B) \setminus K \end{cases} = T'(g_1) + T'(g_2) = \mathfrak{f}$$

on $\text{Ch}\,(B)$. Since $\text{Ch}\,(B)$ is a boundary for B, the map S is surjective. Consequently, S is a real-algebra isomorphism from A onto B. $\square$

Proof of Theorem 3.2. Let $T : \mathcal{A}^{-1} \to \mathcal{B}^{-1}$ be a surjection with

$$\mathrm{r}(T(f)\tau(T(g)) - \alpha) = \mathrm{r}(f\rho(g) - \alpha)$$

for all $f, g \in \mathcal{A}^{-1}$. First, we consider the case when $\mathcal{B}$ is semi-simple. Note that $\widehat{\mathcal{A}^{-1}} = \widehat{\mathcal{A}}^{-1}$ and $\widehat{\mathcal{B}^{-1}} = \widehat{\mathcal{B}}^{-1}$. By similar arguments as in Remark 2.1, we have

$$\mathrm{r}(T(f)T(g)^{-1} - 1) = \mathrm{r}(fg^{-1} - 1)$$

for all $f, g \in \mathcal{A}^{-1}$. Define a map $\widehat{T}$ from $\widehat{\mathcal{A}}^{-1}$ into $\widehat{B}^{-1}$ by $\widehat{T}(\widehat{f}) = \widehat{T(f)}/\widehat{T(1)}$ for $\widehat{f} \in \widehat{\mathcal{A}}^{-1}$. By hypotheses, $\widehat{T}$ is a well-defined surjection from $\widehat{\mathcal{A}}^{-1}$ into $\widehat{\mathcal{B}}^{-1}$ such that $\widehat{T}(1) = 1$ and

$$\left\| \frac{\widehat{T}(\widehat{f})}{\widehat{T}(\widehat{g})} - 1 \right\|_\infty = \left\| \frac{\widehat{f}}{\widehat{g}} - 1 \right\|_\infty$$

for all $\widehat{f}, \widehat{g} \in \widehat{\mathcal{A}}^{-1}$.

Set $A = \mathrm{cl}\,(\widehat{\mathcal{A}})$ and $B = \mathrm{cl}\,(\widehat{\mathcal{B}})$. Then A and B are uniform algebras on $M_{\mathcal{A}}$ and $M_{\mathcal{B}}$, respectively. We will construct a mapping $\widetilde{T} : A^{-1} \to B^{-1}$ with $\widetilde{T}|_{\widehat{\mathcal{A}}^{-1}} = \widehat{T}$. It should be mentioned that the proof below is essentially due to [**4**, Proof of Claim 1]. Let $f \in A^{-1}$;

then there exists a sequence $\{\widehat{f_n}\}_{n=1}^{\infty} \subset \widehat{\mathcal{A}}$ with $\|\widehat{f_n} - f\|_{\infty} \to 0$ as $n \to \infty$. It is well known that $M_A = M_{\mathcal{A}}$ (cf. [**4**, preface of Section 2]), thus $0 \notin f(M_{\mathcal{A}})$. We may assume that $\{\widehat{f_n}\}_{n=1}^{\infty} \in \widehat{\mathcal{A}}^{-1}$ and there exists a $k > 0$ such that

$$\|\widehat{f_n}\| \leq k \quad \text{and} \quad \left\|\frac{1}{\widehat{f_n}}\right\|_{\infty} \leq k$$

for all $n \in \mathbf{N}$. By the same argument as in Remark 2.2, we see that

$$\left\|\frac{\widehat{T}(\widehat{f})}{\widehat{T}(\widehat{g})}\right\|_{\infty} = \left\|\frac{\widehat{f}}{\widehat{g}}\right\|_{\infty}$$

for all $\widehat{f}, \widehat{g} \in \widehat{\mathcal{A}}^{-1}$. Since $\widehat{T}(1) = 1$, the inequalities $\|\widehat{T}(\widehat{f_n})\|_{\infty} = \|\widehat{f_n}\|_{\infty} \leq k$ and

$$\left\|\frac{\widehat{f_n}}{\widehat{f_m}} - 1\right\|_{\infty} \leq \left\|\frac{1}{\widehat{f_m}}\right\|_{\infty} \|\widehat{f_n} - \widehat{f_m}\|_{\infty} \leq k\|\widehat{f_n} - \widehat{f_m}\|_{\infty}$$

hold for all $n, m \in \mathbf{N}$. Thus we have

$$\begin{aligned}
\|\widehat{T}(\widehat{f_m}) - \widehat{T}(\widehat{f_m})\|_{\infty} &\leq \|\widehat{T}(\widehat{f_m})\|_{\infty}\left\|\frac{\widehat{T}(\widehat{f_n})}{\widehat{T}(\widehat{f_m})} - 1\right\|_{\infty} \\
&= \|\widehat{T}(\widehat{f_m})\|_{\infty}\left\|\frac{\widehat{f_n}}{\widehat{f_m}} - 1\right\|_{\infty} \\
&\leq k^2\|\widehat{f_n} - \widehat{f_m}\|_{\infty}
\end{aligned}$$

for all $n, m \in \mathbf{N}$. Hence $\{\widehat{T}(\widehat{f_n})\}_{n=1}^{\infty}$ is a Cauchy sequence in $\widehat{\mathcal{B}}$ with respect to the uniform norm on $M_{\mathcal{B}}$, and there exists $\lim_{n\to\infty} \widehat{T}(\widehat{f_n})$ in B. Since $\|1/\widehat{T}(\widehat{f_n})\| = \|1/\widehat{f_n}\|_{\infty} \leq k$, we have $1/k \leq |\widehat{T}(\widehat{f_n})|$ on M_B for all $n \in \mathbf{N}$. Consequently, $\lim_{n\to\infty} \widehat{T}(\widehat{f_n}) \in B^{-1}$. Note that $\lim_{n\to\infty} \widehat{T}(\widehat{f_n})$ does not depend on a particular choice of a sequence $\{\widehat{f_n}\}_{n=1}^{\infty}$ which converges to f: for if $\{\widehat{g_n}\}_{n=1}^{\infty} \subset \widehat{\mathcal{A}}^{-1}$ is another sequence with $\|\widehat{g_n} - f\|_{\infty} \to 0$ as $n \to \infty$, then

$$\left\|\frac{\widehat{T}(\widehat{f_n})}{\widehat{T}(\widehat{g_n})} - 1\right\|_{\infty} = \left\|\frac{\widehat{f_n}}{\widehat{g_n}} - 1\right\|_{\infty} \to 0$$

as $n \to \infty$, thus $\lim_{n\to\infty} \widehat{T}(\widehat{f_m}) = \lim_{n\to\infty} \widehat{T}(\widehat{g_m})$. Set

$$\widetilde{T}(f) = \lim_{n\to\infty} \widehat{T}(\widehat{f_n})$$

for every $f \in A^{-1}$. Then $\widetilde{T}$ is a well-defined mapping from A^{-1} into B^{-1} satisfying $\widetilde{T}|_{\widehat{\mathcal{A}}^{-1}} = \widehat{T}$ and

$$\left\| \frac{\widetilde{T}(f)}{\widetilde{T}(g)} - 1 \right\|_\infty = \left\| \frac{f}{g} - 1 \right\|_\infty$$

for all $f, g \in A^{-1}$.

We will show that $\widetilde{T}$ is surjective. Let $\mathfrak{f} \in B^{-1}$. Since $\widehat{T} : \widehat{\mathcal{A}}^{-1} \to \widehat{\mathcal{B}}^{-1}$ is surjective, there exists a sequence $\{\widehat{h_n}\}_{n=1}^\infty \subset \widehat{\mathcal{A}}^{-1}$ with $\|\widehat{T}(\widehat{h_n}) - \mathfrak{f}\|_\infty \to 0$ as $n \to \infty$. We may assume that there exists an $l > 0$ such that

$$\|\widehat{T}(\widehat{h_n})\|_\infty \le l \quad \text{and} \quad \left\| \frac{1}{\widehat{T}(\widehat{h_n})} \right\|_\infty \le l$$

for all $n \in \mathbf{N}$. By the argument used in the previous paragraph, we get

$$\|\widehat{h_n} - \widehat{h_m}\|_\infty \le l^2 \|\widehat{T}(\widehat{h_n}) - \widehat{T}(\widehat{h_m})\|_\infty.$$

Therefore, $\{\widehat{h_n}\}_{n=1}^\infty$ is a Cauchy sequence in $\widehat{\mathcal{A}}$ with respect to the uniform norm on $M_{\mathcal{A}}$. Let $h \in A$ be the limit of $\{\widehat{h_n}\}_{n=1}^\infty$. In the same way as the proof of $\lim_{n\to\infty} \widehat{T}(\widehat{f_n}) \in B^{-1}$, we have $h \in A^{-1}$. By the definition of $\widetilde{T}$,

$$\widetilde{T}(h) = \lim_{n\to\infty} \widehat{T}(\widehat{h_n}) = \mathfrak{f}.$$

We thus conclude that $\widetilde{T}$ is surjective. By Theorem 3.1, applied to $\widetilde{T}$, there exist a clopen subset K of $\mathrm{Ch}\,(B)$ and a homeomorphism ϕ from $\mathrm{Ch}\,(B)$ onto $\mathrm{Ch}\,(A)$ with

$$\widetilde{T}(f)(y) = \begin{cases} f(\phi(y)) & y \in K \\ \overline{f(\phi(y))} & y \in \mathrm{Ch}\,(B) \setminus K \end{cases}$$

for every $f \in A^{-1}$. The conclusion follows from $\widehat{T(f)}/\widehat{T(1)} = \widehat{T}(\widehat{f}) = \widetilde{T}(\widehat{f})$ for $f \in \mathcal{A}^{-1}$.

Finally, we consider the case when $\mathcal{B}$ is not assumed to be semi-simple. Let Γ be the Gelfand transformation on $\mathcal{B}$. Then $\Gamma \circ T$ is a surjection from $\mathcal{A}^{-1}$ onto $\widehat{\mathcal{B}}^{-1}$. Note that $\widehat{\mathcal{B}}$ is a unital commutative Banach algebra with respect to the quotient norm. Also, $M_{\widehat{\mathcal{B}}} = M_{\mathcal{B}}$, thus $\widehat{\mathcal{B}}$ is semi-simple. Now it is obvious that

$$\mathrm{r}(\widehat{T(f)}\tau\widehat{(T(g)})-\alpha)=\mathrm{r}(T(f)\tau(T(g))-\alpha)=\mathrm{r}(f\rho(g)-\alpha)$$

for all $f,g \in \mathcal{A}^{-1}$. Set $\mathfrak{h} = T(\rho^{-1}(\alpha g^{-1}))$ for each $g \in \mathcal{A}^{-1}$. By similar arguments used in Remark 2.1, we have $\alpha\widehat{T(g)}^{-1} = \widehat{\tau(\mathfrak{h})}$ for all $g \in \mathcal{A}^{-1}$. Hence $\Gamma \circ T$ satisfies

$$\mathrm{r}((\Gamma \circ T)(f)(\Gamma \circ T)(g)^{-1}-1)=\mathrm{r}(fg^{-1}-1)$$

for all $f,g \in \mathcal{A}^{-1}$, which implies that $\Gamma \circ T$ is injective. Since $T(\mathcal{A}^{-1}) = \mathcal{B}^{-1}$, $\Gamma|_{\mathcal{B}^{-1}}$ is one-to-one. We show that Γ is injective. Suppose that $\Gamma(\mathfrak{f}) = 0$ for $\mathfrak{f} \in \mathcal{B}$, that is, $\widehat{\mathfrak{f}} = 0$ on $M_{\mathcal{B}}$. Set $\mathfrak{f}_0 = 1 + \mathfrak{f}$. Then $\mathfrak{f}_0 \in \mathcal{B}^{-1}$ because $\widehat{\mathfrak{f}_0} = 1$ on $M_{\mathcal{B}}$. Since $\Gamma|_{\mathcal{B}^{-1}}$ is one-to-one, we have $\mathfrak{f}_0 = 1$, thus $f = 0$. We conclude that Γ is injective, and so $\mathcal{B}$ is semi-simple. $\square$

Proof of Corollary 3.3. Let $T : \mathcal{A} \to \mathcal{B}$ be a surjection such that, for some $\alpha \in \mathbf{C} \setminus \{0\}$,

$$\mathrm{r}(T(f)\tau(T(g))-\alpha)=\mathrm{r}(f\rho(g)-\alpha)$$

for all $f,g \in \mathcal{A}$. First, we prove that $\mathcal{B}$ is semi-simple. Let Γ be the Gelfand transformation on $\mathcal{B}$. Then $\Gamma \circ T$ is a surjection from $\mathcal{A}$ onto the unital semi-simple commutative Banach algebra $\widehat{\mathcal{B}}$ with respect to the quotient norm. Note that $\mathrm{r}(\widehat{T(f)}\tau\widehat{(T(g)})-\alpha) = \mathrm{r}(T(f)\tau(T(g))-\alpha) = \mathrm{r}(f\rho(g)-\alpha)$ for all $f,g \in \mathcal{A}$. By similar arguments used in Remark 2.1, we have $\widehat{T(\mathcal{A}^{-1})} \subset \widehat{\mathcal{B}}^{-1}$ and

$$\mathrm{r}(\widehat{T(f)}\widehat{T(g)}^{-1}-1)=\mathrm{r}(fg^{-1}-1)$$

for every $f \in \mathcal{A}$ and $g \in \mathcal{A}^{-1}$. We show that $\widehat{T(\mathcal{A}^{-1})} = \widehat{\mathcal{B}}^{-1}$. Let $\mathfrak{g} \in \mathcal{B}^{-1}$; then there exist a $g \in \mathcal{A}$ and an $h \in \mathcal{A}$ such that $T(g) = \mathfrak{g}$ and $T(h) = \tau^{-1}(\alpha\mathfrak{g}^{-1})$ since T is surjective. We obtain

$$0=\mathrm{r}(\widehat{T(g)}\tau\widehat{(T(h)})-\alpha)=\mathrm{r}(g\rho(h)-\alpha).$$

Hence $g \in \mathcal{A}^{-1}$. Thus $\mathfrak{g} \in T(\mathcal{A}^{-1})$, and consequently $\widehat{\mathcal{B}}^{-1} = \widehat{\mathcal{B}^{-1}} \subset \widehat{T(\mathcal{A}^{-1})}$. By a similar argument to the proof of Theorem 3.1, we have that Γ is injective, and so $\mathcal{B}$ is semi-simple.

Set $A = \mathrm{cl}\,(\widehat{\mathcal{A}})$ and $B = \mathrm{cl}\,(\widehat{\mathcal{B}})$. By the proof of Theorem 3.1, there exist a surjection $\widetilde{T} : A \to B$, a clopen subset K of $\mathrm{Ch}\,(B)$, and a homeomorphism $\phi : \mathrm{Ch}\,(B) \to \mathrm{Ch}\,(A)$ such that $\widetilde{T}(\widehat{f}) = \widehat{T(f)}/\widehat{T(1)}$ for every $f \in \mathcal{A}^{-1}$ and that

$$\widetilde{T}(f)(y) = \begin{cases} f(\phi(y)) & y \in K \\ \overline{f(\phi(y))} & y \in \mathrm{Ch}\,(B) \setminus K \end{cases}$$

for every $f \in A^{-1}$. By the proof of Theorem 3.1, we may define a map $S : A \to B$ by $S(f) = \widetilde{T}(f_0) + \widetilde{T}(\lambda)$ for $f \in A$, where $f_0 \in A^{-1}$ and $\lambda \in \mathbf{C} \setminus \{0\}$ with $f = f_0 + \lambda$. Since

$$\widetilde{T}(f_0) + \widetilde{T}(\lambda) = \begin{cases} (f_0 + \lambda) \circ \phi & \text{on } K \\ \overline{(f_0 + \lambda)} \circ \phi & \text{on } \mathrm{Ch}\,(B) \setminus K \end{cases},$$

we have

$$\left\| \frac{S(\widehat{f})}{\widetilde{T}(g)} - 1 \right\|_\infty = \left\| \frac{\widehat{f}}{g} - 1 \right\|_\infty$$

for every $f \in \mathcal{A}$ and $g \in A^{-1}$. Let $g \in A^{-1}$ and $\{g_n\}_{n=1}^\infty \subset \mathcal{A}^{-1}$ with $\|\widehat{g_n} - g\|_\infty \to 0$ as $n \to \infty$. By the definition of $\widetilde{T}$, $\widetilde{T}(g) = \lim_{n\to\infty} \widehat{T(g_n)}/\widehat{T(1)}$. Since

$$\mathrm{r}(T(f)T(g_n)^{-1} - 1) = \mathrm{r}(f{g_n}^{-1} - 1),$$

by letting $n \to \infty$, we have

$$\left\| \frac{\widehat{T(f)}}{\widehat{T(1)}\widetilde{T}(g)} - 1 \right\|_\infty = \left\| \frac{\widehat{f}}{g} - 1 \right\|_\infty = \left\| \frac{S(\widehat{f})}{\widetilde{T}(g)} - 1 \right\|_\infty$$

for every $f \in \mathcal{A}$ and $g \in A^{-1}$. Since $\widetilde{T}(A^{-1}) = B^{-1}$, we obtain

$$\left\| \frac{\widehat{T(f)}}{\widehat{T(1)}}\mathfrak{g} - 1 \right\|_\infty = \|S(\widehat{f})\mathfrak{g} - 1\|_\infty$$

for every $f \in \mathcal{A}$ and $\mathfrak{g} \in B^{-1}$. For each $f \in \mathcal{A}$, $\mathfrak{g} \in B^{-1}$, and $n \in \mathbf{N}$, we have

$$\left\| \frac{\widehat{T(f)}}{\widehat{T(1)}} \mathfrak{g} - \frac{1}{n} \right\|_\infty = \left\| S(\widehat{f})\mathfrak{g} - \frac{1}{n} \right\|_\infty$$

since $n\mathfrak{g} \in B^{-1}$. Taking $n \to \infty$, we have, for every $f \in \mathcal{A}$ and $\mathfrak{g} \in B^{-1}$, that

$$\left\| \frac{\widehat{T(f)}}{\widehat{T(1)}} \mathfrak{g} \right\|_\infty = \|S(\widehat{f})\mathfrak{g}\|_\infty.$$

We show that $\widehat{T(f)} = \widehat{T(1)} S(\widehat{f})$ on $M_{\mathcal{B}}$ for every $f \in \mathcal{A}$. Let $f \in \mathcal{A}$ and $y \in \mathrm{Ch}\, B$. Set $\mu = \widehat{T(f)}(y)/\widehat{T(1)}(y)$ and $\nu = S(\widehat{f})(y)$. If $\nu = 0$, then for each $n \in \mathbf{N}$ there exists a $\mathfrak{g}_n \in P_{B^{-1}}(y)$ such that $\|S(\widehat{f})\mathfrak{g}_n\|_\infty < 1/n$. Hence

$$|\mu| = |\mu \mathfrak{g}_n(y)| \le \left\| \frac{\widehat{T(f)}}{\widehat{T(1)}} \mathfrak{g}_n \right\|_\infty = \|S(\widehat{f})\mathfrak{g}_n\|_\infty < \frac{1}{n},$$

and consequently, $\mu = 0 = \nu$. By a quite similar argument, we see that $\mu = 0$ implies $\nu = 0$. If $\mu\nu \neq 0$, then Lemma 2.1 shows that there exists a $\mathfrak{g} \in P_{B^{-1}}(y)$ such that $\sigma_\pi(\widehat{T(f)}\mathfrak{g}/\widehat{T(1)}) = \{\mu\}$ and $\sigma_\pi(S(\widehat{f})\mathfrak{g}) = \{\nu\}$. Hence

$$|\mu| = \left\| \frac{\widehat{T(f)}}{\widehat{T(1)}} \mathfrak{g} \right\|_\infty = \|S(\widehat{f})\mathfrak{g}\|_\infty = |\nu|.$$

Since $-\mathfrak{g}/\nu \in B^{-1}$, we have

$$\left\| \frac{\widehat{T(f)}}{\widehat{T(1)}} \left(-\frac{\mathfrak{g}}{\nu} \right) - 1 \right\|_\infty = \left\| S(\widehat{f}) \left(-\frac{\mathfrak{g}}{\nu} \right) - 1 \right\|_\infty = 2.$$

Note that $\|\widehat{T(f)}\mathfrak{g}/(\nu \widehat{T(1)})\|_\infty = 1$. Thus, there exists a $y' \in \mathrm{Ch}\,(B)$ such that

$$\frac{\widehat{T(f)}(y')}{\widehat{T(1)}(y')} \left(-\frac{\mathfrak{g}(y')}{\nu} \right) = -1.$$

Therefore, $|\widehat{T(f)}(y')\mathfrak{g}(y')/\widehat{T(1)}(y')| = |\nu| = |\mu|$. Since $\sigma_\pi(\widehat{T(f)}\mathfrak{g}/\widehat{T(1)}) = \{\mu\}$, we have $\widehat{T(f)}(y')\mathfrak{g}(y')/\widehat{T(1)}(y') = \mu$. Consequently, we have

$\mu = \nu$. Hence $\widehat{T(f)}(y) = \widehat{T(1)}(y)S(\widehat{f})(y)$, as claimed. Now it is obvious by the definition of S that

$$\widehat{T(f)}(y) = \widehat{T(1)}(y) \times \begin{cases} \widehat{f}(\phi(y)) & y \in K \\ \overline{\widehat{f}(\phi(y))} & y \in \mathrm{Ch}\, B \setminus K \end{cases}$$

for every $f \in \mathcal{A}$. $\square$

Proof of Corollary 3.4. Let $T : \mathcal{A} \to \mathcal{B}$ be a surjection such that

$$\sigma(T(f)\tau(T(g))) = \sigma(f\rho(g)) \tag{4.1}$$

for all $f, g \in \mathcal{A}$. Then

$$\sigma(T(f)\tau(T(g)) - 1) = \sigma(f\rho(g) - 1),$$

which implies that

$$\mathrm{r}(T(f)\tau(T(g)) - 1) = \mathrm{r}(f\rho(g) - 1)$$

for all $f, g \in \mathcal{A}$. Applying Corollary 3.3, $\mathcal{B}$ is semi-simple, and there exists a clopen subset K of $\mathrm{Ch}\,(\mathrm{cl}\,(\widehat{\mathcal{B}}))$ and a homeomorphism ϕ from $\mathrm{Ch}\,(\mathrm{cl}\,(\widehat{\mathcal{B}}))$ onto $\mathrm{Ch}\,(\mathrm{cl}\,(\widehat{\mathcal{A}}))$ such that the equality

$$(\widehat{T(f)}/\widehat{T(1)})(y) = \begin{cases} \widehat{f}(\phi(y)) & y \in K \\ \overline{\widehat{f}(\phi(y))} & y \in \mathrm{Ch}\,(\mathrm{cl}\,(\widehat{\mathcal{B}})) \setminus K \end{cases}$$

holds for every $f \in \mathcal{A}$. Since ρ is surjective, there exists an $f_1 \in \mathcal{A}$ such that $\rho(f_1) = 1$. By (4.1), we have $\sigma(T(1)\tau(T(f_1))) = \sigma(\rho(f_1)) = \{1\}$. Thus $T(1)\tau(T(f_1)) = 1$, and consequently $T(1)^{-1} = \tau(T(f_1))$. Then $T(i)T(1)^{-1} = i$ since $\sigma(T(i)T(1)^{-1}) = \sigma(T(i)\tau(T(f_1))) = \sigma(i\rho(f_1)) = \{i\}$. This implies that $K = \mathrm{Ch}\,(\mathrm{cl}\,(\widehat{\mathcal{B}}))$. It follows that the equality $\widehat{T(f)} = \widehat{T(1)}(\widehat{f} \circ \phi)$ holds on $\mathrm{Ch}\,(\mathrm{cl}\,(\widehat{\mathcal{B}}))$ for every $f \in \mathcal{A}$. Set $S = T(1)^{-1}T$. Then S is an algebra isomorphism from $\mathcal{A}$ onto $\mathcal{B}$. For any $y \in M_{\mathcal{B}}$, a mapping S_y defined by $S_y(f) = \widehat{S(f)}(y)$ for $f \in \mathcal{A}$ is a multiplicative linear functional on $\mathcal{A}$ with $S_y(1) = 1$. Thus, we may define a map Φ from $M_{\mathcal{B}}$ into $M_{\mathcal{A}}$ by $\Phi(y) = S_y$ for $y \in M_{\mathcal{B}}$. Then we see that Φ is a continuous map from $M_{\mathcal{B}}$ onto $M_{\mathcal{A}}$ with $\Phi = \phi$

on $\mathrm{Ch}(\mathrm{cl}(\widehat{\mathcal{B}}))$. Since S is bijective, we may define a continuous map Ψ from $M_{\mathcal{A}}$ into $M_{\mathcal{B}}$ by $\Psi(y) = (S^{-1})_x$ for $x \in M_{\mathcal{A}}$ as in the same manner. We see that $\Psi = \Phi^{-1}$; thus, Φ is a homeomorphism. By the definition of S_y, we have

$$\widehat{T(f)}(y) = \widehat{T(1)}(y)\widehat{S(f)}(y) = \widehat{T(1)}(y)\widehat{f}(\Phi(y))$$

for every $f \in \mathcal{A}$ and $y \in M_{\mathcal{B}}$. $\square$

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