ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 41, Number 5, 2011

PHASE PORTRAITS AND INVARIANT STRAIGHT LINES OF CUBIC POLYNOMIAL VECTOR FIELDS HAVING A QUADRATIC RATIONAL FIRST INTEGRAL

JAUME LLIBRE, ADAM MAHDI AND NICOLAE VULPE

ABSTRACT. In this paper we classify all cubic polynomial differential systems having a rational first integral of degree two. In other words we characterize all the global phase portraits of the cubic polynomial differential systems having all their orbits contained in conics. We also determine their configurations of invariant straight lines. We show that there are exactly 38 topologically different phase portraits in the Poincaré disc associated with this family of cubic polynomial differential systems up to a reversed sense of their orbits.

1. Introduction and statement of the main results. Nonlinear ordinary differential equations appear in many branches of applied mathematics, physics and, in general, in applied sciences. For a differential system or a vector field defined on the plane $\mathbf{R}^2$, the existence of a first integral determines completely its phase portrait. Since for such vector fields the notion of integrability is based on the existence of a first integral the following natural question arises: *Given a vector field on* $\mathbf{R}^2$*, how to recognize if this vector field has a first integral*? One of the easiest planar vector fields having a first integral are the Hamiltonian ones. The integrable planar vector fields which are not Hamiltonian are, in general, very difficult to detect. In this paper we will characterize the cubic polynomial vectors fields having a rational first integral of degree 2.

2010 AMS *Mathematics subject classification.* Primary 34C05, 34A34, 34C14.
Keywords and phrases. Quadratic vector fields, integrability, rational first integral, phase portraits.
The first author is partially supported by a MICIIN/FEDER grant number MTM2008-03437, by an AGAUR grant number 2009SGR 410, and by ICREA Academia. The second author is partially supported by a MICIIN/FEDER grant number MTM2008-03437. The third author is partially supported by CRDF–MRDA CERIM-1006-06.
Received by the editors on February 6, 2009.
DOI:10.1216/RMJ-2011-41-5-1585

We study *cubic polynomial vector fields* in $\mathbf{R}^2$ defined by the systems

$$
\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{d}t} &= p_0 + p_1(x,y) + p_2(x,y) + p_3(x,y) = P(x,y),\\
\frac{\mathrm{d}y}{\mathrm{d}t} &= q_0 + q_1(x,y) + q_2(x,y) + q_3(x,y) = Q(x,y),
\end{aligned} \tag{1}
$$

where $p_0, q_0 \in \mathbf{R}$ and $p_i(x,y)$, $q_i(x,y)$ are homogenous polynomials of degree i $(i = 1,2,3)$ in x and y and $\big(p_3(x,y)\big)^2 + \big(q_3(x,y)\big)^2 \not\equiv 0$. We say that a polynomial differential system (1) is *degenerate* if P and Q have a common factor of degree ≥ 1.

Our goal is to determine all phase portraits of systems (1) having a quadratic rational first integral H that is

$$
H = \frac{c_{00} + c_{10}x + c_{01}y + c_{20}x^2 + c_{11}xy + c_{02}y^2}{d_{00} + d_{10}x + d_{01}y + d_{20}x^2 + d_{11}xy + d_{02}y^2} = \frac{H_N}{H_D}, \tag{2}
$$

with $c_{20}^2 + c_{11}^2 + c_{02}^2 + d_{20}^2 + d_{11}^2 + d_{02}^2 \neq 0$ and with the numerator and the denominator different from a constant, i.e., in this paper we do not allow that H or $1/H$ be a polynomial because in these cases the differential systems are essentially linear.

We remark that the quadratic vector fields having a rational first integral of degree 2 and their phase portraits have been characterized in [**1, 3**].

We note that the cubic polynomial differential systems having a rational first integral of degree 2 have all their orbits contained in conics. So, their orbits are very simple curves but this does not prevent their phase portraits from exhibiting a rich variety of dynamics as it is shown in our main result.

Theorem 1. *The phase portrait of a non-degenerate planar cubic polynomial differential system with a rational first integral of degree* 2 *is topologically equivalent to one of the* 27 *phase portraits described in Figure* 1.

Further, we show that a real cubic system having a rational first integral of degree 2 either has a finite number of invariant straight lines (real or complex) of total multiplicity 6 or it has infinitely many of them,

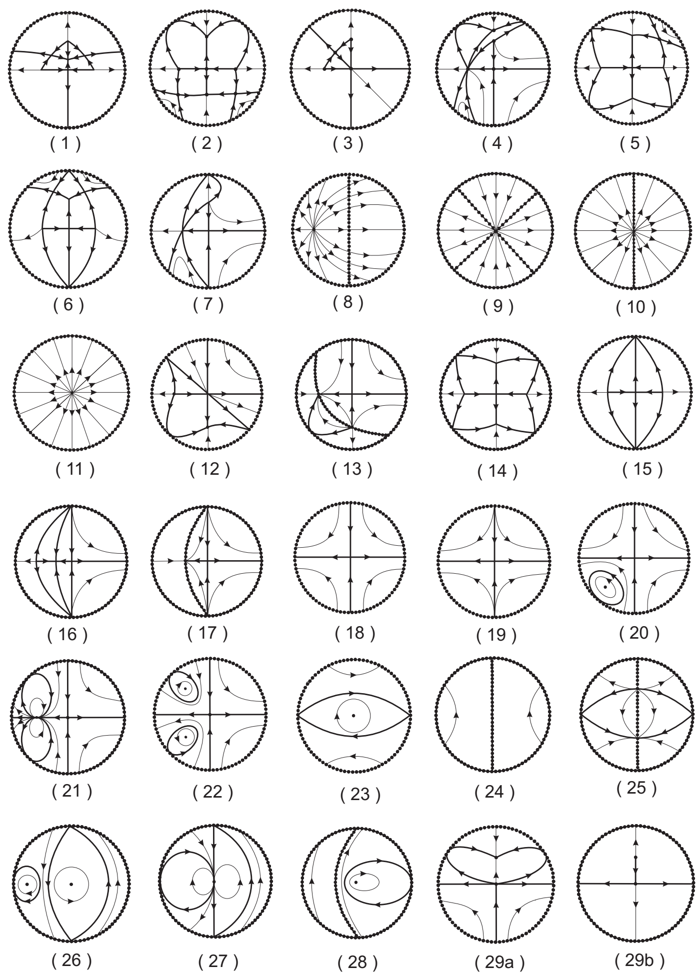


FIGURE 1. Phase portraits of non-degenerate cubic systems having a rational first integral of degree two.

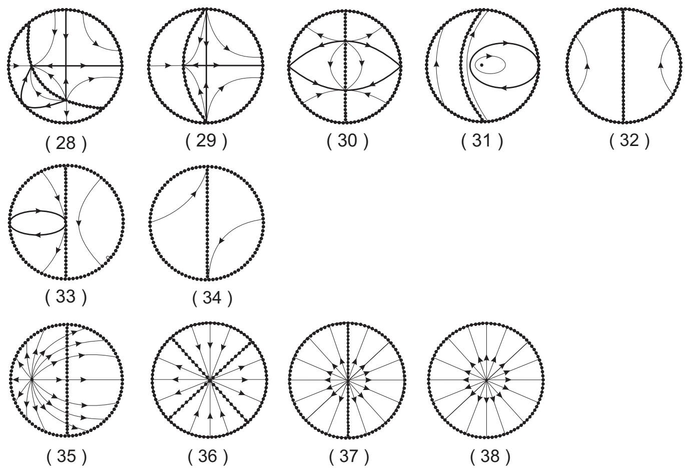


FIGURE 2. Phase portraits of degenerate cubic systems having a rational first integral of degree two.

see Section 3. We also study the configuration of the invariant straight lines (for the definition see Section 3) that these kinds of systems exhibit. If at least four invariant straight lines coincide, system (1) becomes degenerate. In Figure 2 we gather all the degenerate phase portraits of the vector field 3. These are in one sense limit cases of the non-degenerate ones. Phase portraits in Figure 2 (28–32) correspond to the cases when four invariant straight lines coincide and after removing the common factor from P and Q we get a quadratic system. In Figure 2 (33–34) we have two phase portraits that we get when six invariant straight lines coincide. Finally, we have four non-equivalent phase portraits in Figure 2 (35–38) when system (1) admits infinitely many invariant straight lines.

We remark that this paper includes as a subcase the results of [**2**].

The paper is organized as follows. In Section 2 we present some basic results of the systems studied in Theorem 1. Playing with the different configurations of the invariant straight lines we organize the proof of Theorem 1 in eight subsections inside Section 3.

2. Preliminaries. We note that the most general polynomial vector fields having a rational first integral (2) are $X = (P, Q)$ where

$$P(x,y) = -\frac{\partial H}{\partial y}(H_D)^2, \quad Q(x,y) = \frac{\partial H}{\partial x}(H_D)^2. \tag{3}$$

We denote by

$$\mathcal{M} = \begin{pmatrix} c_{00} & c_{10} & c_{01} & c_{20} & c_{11} & c_{02} \\ d_{00} & d_{10} & d_{01} & d_{20} & d_{11} & d_{02} \end{pmatrix}$$

the matrix of the coefficients of the polynomials H_N and H_D and by δ_{ij}, $1 \le i < j \le 6$, the minor of the matrix $\mathcal{M}$ constructed with columns i and j. Then the differential systems corresponding to the vector fields (3) take the form:

$$\begin{aligned} \dot{x} &= \delta_{13} + (\delta_{15}+\delta_{23})\,x + 2\delta_{16}\,y + (\delta_{25}-\delta_{34})\,x^2 + 2\delta_{26}xy + \delta_{36}\,y^2 \\ &\quad + x\,R_2(x,y), \\ \dot{y} &= -\delta_{12} - 2\delta_{14}\,x + (\delta_{23}-\delta_{15})\,y - \delta_{24}\,x^2 - 2\delta_{34}\,xy + (\delta_{26}-\delta_{35})y^2 \\ &\quad + y\,R_2(x,y), \end{aligned} \tag{4}$$

where $R_2(x,y) = \delta_{45}\,x^2 + 2\delta_{46}\,xy + \delta_{56}\,y^2$.

We say that the *infinity is degenerate* if it is full of singular points. In what follows sometimes instead of cubic polynomial differential system we will simply say *cubic system*.

Lemma 2. *If a cubic system* (1) *possesses a rational first integral of the form* (2) *then this system*:

(a) *has a line of singularities at infinity*;

(b) *becomes a quadratic system if and only if the homogeneous quadratic part of the polynomials* H_N *and* H_D *from* (2) *are proportional*;

(c) *becomes homogenous cubic degenerate of the form* $\dot{x} = x\,R_2(x,y)$, $\dot{y} = y\,R_2(x,y)$ *if* H_N *and* H_D *are homogeneous quadratic forms.*

Proof. As we said the most general form of the cubic vector fields having a rational first integral of form (2) takes the form (4). The cubic homogeneous part of the vector field is denoted by $(\overline{P}, \overline{Q})$. It is clear that $(\overline{P},\overline{Q}) = (xR_2, yR_2)$. Since $x\overline{Q} - y\overline{P} \equiv 0$, statement (a) follows. Homogeneous quadratic parts of the polynomials H_N and H_D are proportional if and only if $\delta_{45} = \delta_{46} = \delta_{56} = 0$. This condition is fulfilled if and only if $R_2 \equiv 0$, and this shows statement (b). If we assume that

$$\mathcal{M} = \begin{pmatrix} 0 & 0 & 0 & c_{20} & c_{11} & c_{02} \\ 0 & 0 & 0 & d_{20} & d_{11} & d_{02} \end{pmatrix},$$

then it is clear that (4) is as in statement (c). $\square$

The vector field $\mathcal{X}$ *associated* with system (1) is defined by

$$\mathcal{X} = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y}.$$

Let $f \in \mathbf{C}[x,y]$, i.e., f is a polynomial with complex coefficients in the variables x and y. The complex algebraic curve $f(x,y) = 0$ is an *invariant algebraic curve* of the real vector field $\mathcal{X}$ if, for some polynomial $K \in \mathbf{C}[x,y]$, we have

$$\mathcal{X}f := P\frac{\partial f}{\partial x} + Q\frac{\partial f}{\partial y} = Kf.$$

The polynomial K is called the *cofactor* of the invariant algebraic curve $f = 0$. We note that if a polynomial system has degree m then every cofactor has at most degree $m-1$.

Lemma 3. *Let $\mathcal{X}$ be a polynomial vector field in $\mathbf{R}^2$. If the polynomial functions f and g are relatively prime, then f/g is a rational first integral of $\mathcal{X}$ if and only if f and g are both invariant algebraic curves with the same cofactor.*

Proof. Let $H = f/g$ be a first integral with f and g two non-constant coprime polynomials. So $\mathcal{X}(f/g) = 0$, or equivalently $f(\mathcal{X}g) = g(\mathcal{X}f)$. Since f does not divide g, we have that $\mathcal{X}f = Kf$ for some polynomial $K \in \mathbf{C}[x,y]$. Therefore, we also have $\mathcal{X}g = Kg$. Conversely if we take two algebraic curves f and g with the same cofactor K, then we have $K = (\mathcal{X}f)/f$ and $K = (\mathcal{X}g)/g$. Then $(\mathcal{X}f)g - f(\mathcal{X}g) = 0$. Since

$$\mathcal{X}\left(\frac{f}{g}\right) = \frac{(\mathcal{X}f)g - f(\mathcal{X}g)}{g^2} = 0,$$

the lemma follows. $\square$

Corollary 4. *If f/g is a rational first integral of $\mathcal{X}$, then $(\alpha f + \beta g)/(\gamma f + \delta g)$ is also a rational first integral of $\mathcal{X}$ for all $\alpha, \beta, \gamma, \delta \in \mathbf{R}$ verifying the condition $\alpha\delta - \beta\gamma \neq 0$.*

Proof. Suppose that f/g is the first integral of $\mathcal{X}$ and $a, b, c, d \in \mathbf{R}$. Then $(af)/g + b = (af + bg)/g$ is another first integral of $\mathcal{X}$ and so is $cg/(af + bg) + d = (adf + (c + bd)g)/(af + bg)$. Putting $\alpha = ad$, $\beta = c + bd$, $\gamma = a$ and $\delta = b$ and by the assumption $\alpha\delta - \beta\gamma \neq 0$ we prove the corollary. $\square$

3. Classification of the configurations of the invariant straight lines. We say that the invariant straight line $\mathrm{L}(x, y) = ux + vy + w = 0$, where $u, v, w \in \mathbf{C}$ for the cubic vector field $\mathcal{X}$ has *multiplicity* m if there exists a sequence of real cubic vector fields $\mathcal{X}_k$ tending to $\mathcal{X}$, such that each $\mathcal{X}_k$ has exactly m distinct (complex) invariant straight lines $\mathrm{L}_k^1 = 0, \ldots, \mathrm{L}_k^m = 0$, tending to $\mathrm{L} = 0$ as $k \to \infty$ (with the topology of their coefficients).

In what follows, we construct the necessary and sufficient conditions for a cubic system (4) to have an invariant straight line. We consider the cubic system (1) and the following associated four polynomials:

$$C_i(x, y) = yp_i(x, y) - xq_i(x, y) \in \mathbf{R}[a, x, y], \quad i = 0, 1, 2, 3.$$

Remark 1. For system (4) it follows immediately that $C_3(x, y) \equiv 0$.

We denote by $\mathrm{Res}_z(f(z), g(z))$ the resultant of the polynomials $f(z)$ and $g(z)$. Following [**9**] we shall prove the next results.

Proposition 5. *The straight line $\widetilde{L}(x, y) = ux + vy = 0$ is invariant for a cubic system* (4) *with $p_0^2 + q_0^2 \neq 0$ if and only if for $i = 1, 2$ the following relations hold*:

$$(5) \qquad \mathrm{Res}_\gamma(C_i, C_0) = 0 \quad \left(\gamma = \frac{y}{x} \quad \text{or} \quad \gamma = \frac{x}{y}\right).$$

Proof. The line $\widetilde{L}(x, y)$=0 is invariant for system (4) if and only if

$$u(p_0 + p_1 + p_2 + p_3) + v(q_0 + q_1 + q_2 + q_3) = (ux + vy)(S_0 + S_1 + S_2),$$

for some homogeneous polynomials $S_i(x,y)$ of degree i in x and y. The last equality is equivalent to

$$\begin{aligned}
&up_0 + vq_0 = 0,\\
&up_1(x,y) + vq_1(x,y) = (ux+vy)S_0,\\
&up_2(x,y) + vq_2(x,y) = (ux+vy)S_1(x,y),\\
&up_3(x,y) + vq_3(x,y) = (ux+vy)S_2(x,y).
\end{aligned}$$

If $x = -v, y = u$, then the left-hand sides of the previous equalities become $C_0(-v,u)$, $C_1(-v,u)$, $C_2(-v,u))$ and $C_3(-v,u))$, respectively, and the last polynomial vanishes (see Remark 1). At the same time the right-hand sides of these identities vanish. Thus we obtain equations $C_i(-v,u) = 0$ $(i = 0,1,2)$ in which C_0 (respectively, C_1 and C_2) is a homogeneous polynomial of degree 1 (respectively 2 and 3) in the parameters u and v, and $C_0(x,y) \neq 0$ because $p_0^2 + q_0^2 \neq 0$. Hence, by the properties of the resultant, the necessary and sufficient conditions for the existence of a common solution of this system of equations are conditions (5). $\square$

Let $(x_0,y_0) \in \mathbf{R}^2$ be an arbitrary point on the phase plane of systems (4). Consider a translation τ bringing the origin of coordinates to the point (x_0,y_0). We denote by (4^τ) the system obtained after applying the transformation τ, and by $\widetilde{\mathbf{a}} = \mathbf{a}(x_0,y_0) \in \mathbf{R}^{20}$ the 20-tuple of its coefficients. If $\gamma = y/x$ or $\gamma = x/y$ then, for $i = 1,2$ we denote

$$\begin{aligned}
\Omega_i(\mathbf{a},x_0,y_0) &= \mathrm{Res}_\gamma\Big(C_i\big(\widetilde{\mathbf{a}},x,y\big), C_0\big(\widetilde{\mathbf{a}},x,y\big)\Big) \in \mathbf{R}[\mathbf{a},x_0,y_0],\\
\mathcal{E}_i(\mathbf{a},x,y) &= \Omega_i(\mathbf{a},x_0,y_0)|_{\{x_0=x,\ y_0=y\}} \in \mathbf{R}[\mathbf{a},x,y].
\end{aligned} \tag{6}$$

Remark 2. For $j = 1,2$ the polynomials $\mathcal{E}_j(\mathbf{a},x,y)$ are affine comitants (for more details see [**9**]), homogeneous in the coefficients of system (4) and non-homogeneous in the variables x and y.

The geometrical meaning of these affine comitants is given by the following lemma.

Lemma 6. *The straight line $L(x,y) = ux + vy + w = 0$ is invariant for a cubic system* (4) *if and only if the polynomial $L(x,y)$ is a common factor of the polynomials $\mathcal{E}_1$ and $\mathcal{E}_2$ over* $\mathbf{C}$.

Proof. Let $(x_0, y_0) \in \mathbf{R}^2$ be a non-singular point of system (4) (i.e., $P(x_0, y_0)^2 + Q(x_0, y_0)^2 \neq 0$) which lies on the line $L(x, y) = 0$, i.e., $ux_0 + vy_0 + w = 0$. Denote by $\widetilde{L}(x, y) = (L \circ \tau)\ (x, y) = ux + vy$ (τ is a translation) and consider the line $ux + vy = 0$. By Proposition 5 the straight line $\widetilde{L}(x, y) = 0$ will be an invariant line of systems (4^τ) if and only if the conditions (5) are satisfied for these systems, i.e., for $i = 1, 2$, $\Omega_i(\mathbf{a}, x_0, y_0) = 0$ for each point (x_0, y_0) on the line $L(x, y) = ux + vy + w = 0$. Thus from Nullstellensatz we have $\Omega_i(\mathbf{a}, x_0, y_0) = (ux_0 + vy_0 + w)\widetilde{\Omega}_i(\mathbf{a}, x_0, y_0)$. Taking into account relations (6) the lemma follows. $\square$

Proposition 7. *Every non-degenerate cubic system having a rational first integral of the form* (2) *possesses invariant affine straight lines* (*real and/or complex*) *of total multiplicity six.*

Proof. Calculating for systems (4) the affine invariant polynomials $\mathcal{E}_1$ and $\mathcal{E}_2$, we obtain

$$\mathcal{E}_1 = W(c_{ij}, d_{ij}, x, y), \quad \mathcal{E}_2 = W(c_{ij}, d_{ij}, x, y)\widetilde{W}(c_{ij}, d_{ij}, x, y),$$

where $W(c_{ij}, d_{ij}, x, y)$ (respectively $\widetilde{W}(c_{ij}, d_{ij}, x, y)$) is a homogenous polynomial of degree 6 (respectively, of degree 2) in the parameters c_{ij}, d_{ij} and a non-homogenous polynomial of degree 6 (respectively of degree 2) in the variables x and y. As the polynomial $W(c_{ij}, d_{ij}, x, y)$ is a common factor of the affine comitants $\mathcal{E}_1$ and $\mathcal{E}_2$ by Lemma 6 the polynomial $W(c_{ij}, d_{ij}, x, y)$ is a product of six invariant affine straight lines, which could be real and/or complex, distinct and/or coinciding, see [**7**, page 205]. This completes the proof of the proposition. $\square$

Note that the multiplicity of an invariant straight line $ux+vy+w = 0$ is given by the number of times that $ux + vy + w$ divides $\mathcal{E}_1$, for more details see [**4**].

From the proof the above proposition the next result follows immediately.

Corollary 8. *For a non-degenerate cubic system having a rational first integral of the form* (2) *the affine invariant polynomial* $\mathcal{E}_1(x, y)$ *gives six invariant straight lines taking into account their multiplicity.*

Information about the existence of invariant straight lines of total multiplicity six will be crucial in determining all the phase portraits of non-degenerate systems (4). Before we describe the way to achieve it we recall some notions about algebraic curves and define a configuration of invariant straight lines.

If $F = F(x, y) \in \mathbf{R}[x, y]$ is a real polynomial of degree two then it is known that this polynomial can be brought to one of the nine normal forms (see Proposition 20 of the Appendix). In the Appendix we recall two main invariants, Δ and δ of a conic. In this article we will often use the following terminology. We say that a conic $F = 0$ is of *hyperbolic type* if $\delta < 0$, of *parabolic type* if $\delta = 0$, and of *elliptic type* if $\delta > 0$.

We say that a real conic $F = 0$ is *reducible* if it factorizes in the complex domain, i.e., if $F = (ax + by)(cx + dy)$ for some $a, b, c, d \in \mathbf{C}$. If $F = 0$ is not reducible we say that it is *irreducible*. Thus a reducible conic of hyperbolic type is a real conic that factorizes in two real intersecting straight lines; a reducible conic of parabolic type is a real conic that factorizes in either two distinct real or complex parallel straight lines or one real straight line of multiplicity two; finally, a reducible conic of elliptic type is a real conic that factorizes in two complex conjugate straight lines (which intersect in a real point). Therefore, a reducible conic of hyperbolic, elliptic and parabolic types via an affine transformation can be brought respectively to $xy = 0$, $x^2 + y^2 = 0$; $x^2 + \alpha = 0$, where $\alpha \in \{0, \pm 1\}$.

Assume that the differential system (4) admits a rational first integral $H = H_N/H_D$. The pencil of conics generated by H_N and H_D has the conic $F = 0$ if there exist $\alpha, \beta \in \mathbf{R}$ such that

$$F = \alpha H_N + \beta H_D. \tag{7}$$

In this case we will also say that system (4) *has* or *possesses* the conic $F = 0$. Notice that the conic $F = 0$ coincides with $H = -\beta/\alpha$ for $\alpha \neq 0$, and $F = H_D$ for $\alpha = 0$.

Lemma 9. *If system* (4) *has at least two non-proportional reducible conics* (*say* $H_1 = 0$ *and* $H_2 = 0$), *then the following statements hold.*

(a) *The rational function* H_1/H_2 *also is a first integral for this system.*

(b) *The system becomes quadratic if and only if the homogeneous quadratic parts of the quadratic polynomials* H_1 *and* H_2 *are proportional.*

(c) *The system becomes homogenous cubic degenerate if H_1 and H_2 are homogeneous polynomials.*

Proof. From our assumptions and the fact that system (4) has the first integral H_N/H_D, we have

$$H_1 = \alpha_1 H_N + \beta_1 H_D, \qquad H_2 = \alpha_2 H_N + \beta_2 H_D,$$

so we have

$$\frac{H_1}{H_2} = \frac{\alpha_1 H_N + \beta_1 H_D}{\alpha_2 H_N + \beta_2 H_D}.$$

Since H_1 and H_2 are not proportional, we have $\alpha_1\beta_2 - \alpha_2\beta_1 \neq 0$, and now (a) follows from Corollary 4.

Calculating $\mathcal{X} = (P, Q)$ as in (3) for $H = H_N/H_D$ and choosing H_N and H_D with a proportional quadratic homogeneous parts we obtain (b), and for H_N and H_D homogeneous we have (c). □

Lemma 10. *In a real linear system of conics, there is at least one conic which contains a real line.*

Proof. For the proof see [**8**, page 260]. □

We call the *configuration of invariant straight lines* (or simply the *configuration*) of system (1) the set of all its invariant straight lines (real or complex), each endowed with its multiplicity and together with all the real isolated singular points of (1) located on these lines endowed with their multiplicities. We recall that the *multiplicity of a singular point* (x_0, y_0) of system (1) is defined as the intersection number of algebraic curves $P(x, y) = 0$ and $Q(x, y) = 0$ at the point (x_0, y_0), for more details see [**6**]. We indicate the multiplicity of an invariant straight line only when it is greater than one by a number near it. If the multiplicity of the singular point is not indicated it means that the singular point is *simple*, i.e., it has multiplicity one. When the multiplicity of the singular point is $k > 1$, then it is indicated by (k) near the singular point.

Using this information we divide the problem of finding all topologically nonequivalent phase portraits of systems (4) into 5 cases. We denote by $\mathbf{h}$, $\mathbf{p}$ and $\mathbf{e}$ the reducible conic of hyperbolic, parabolic and elliptic type respectively. By $\mathcal{S}(\mathbf{m},\mathbf{n})$ we denote the class of systems (4) having at least two different reducible conics of the type $\mathbf{m}$ and $\mathbf{n}$, where $\mathbf{m},\mathbf{n}\in\{\mathbf{h},\mathbf{e},\mathbf{p}\}$.

First we consider the family $\mathcal{S}(\mathbf{h},\mathbf{h})$ that consists of all system (3) having two non-proportional reducible conics of hyperbolic type, say $H_1=0$ and $H_2=0$. By Lemma 9 (a) H_1/H_2 is a first integral of the system and by Corollary 8 calculating $\mathcal{E}_1=H_1H_2\widehat{H}$, we find another reducible invariant conic $\widehat{H}=0$.

Then we study the family $\mathcal{S}(\mathbf{h},\mathbf{p})$ of all systems having two reducible conics: one of hyperbolic type $H_1=0$ and the other of parabolic type $H_2=0$. Calculating $\mathcal{E}_1=H_1H_2\widehat{H}$ we determine the third reducible conic $\widehat{H}=0$. Of course if $\widehat{H}=0$ is of hyperbolic type then the system belongs to $\mathcal{S}(\mathbf{h},\mathbf{h})$ since there are two reducible conics of hyperbolic type, namely $H_2=0$ and $\widehat{H}=0$. Thus we exclude this case when studying family $\mathcal{S}(\mathbf{h},\mathbf{p})$.

We are going to consider the following families $\mathcal{S}(\mathbf{h},\mathbf{h})$, $\mathcal{S}(\mathbf{h},\mathbf{p})$, $\mathcal{S}(\mathbf{h},\mathbf{e})$, $\mathcal{S}(\mathbf{p},\mathbf{p})$, $\mathcal{S}(\mathbf{p},\mathbf{e})$ in this order. We notice that according to Lemma 10 the family $\mathcal{S}(\mathbf{e},\mathbf{e})$ is included in one of the families mentioned before. At each step we exclude the cases that have been studied before. As a first step we shall construct all topologically distinct configurations of invariant straight lines occurring for each of the mentioned classes of systems (4). Then we consider systems having a reducible conic either of hyperbolic or parabolic (excluding the case of two parallel non-real lines) and no reducible conic of other type. At each step we again exclude the cases that lead to phase portraits that have been studied before. By Lemma 10 systems having only one reducible conic of elliptic type or parallel non-real lines do not exist.

3.1. Systems of type $\mathcal{S}(\mathbf{h},\mathbf{h})$. Assume that a system (4) possesses two distinct reducible conics of hyperbolic type, say $H_{\mathbf{h}}^{(i)}=L_1^{(i)}L_2^{(i)}$ $(i=1,2)$. We shall consider two geometrical distinct possibilities:

(i) either the centers of the conics $H_{\mathbf{h}}^{(1)}=0$ and $H_{\mathbf{h}}^{(2)}=0$ (i.e., the intersection points of the lines $L_1^{(i)}=0$ and $L_2^{(i)}=0$ $(i=1,2)$) are distinct,

(ii) or they coincide.

Proposition 11. *Assume that a cubic system has a rational first integral of degree* 2 *of the form* $H_{\mathbf{h}}^{(1)}/H_{\mathbf{h}}^{(2)}$ *where* $H_{\mathbf{h}}^{(i)}$ *for* $i = 1, 2$ *is a reducible conic of hyperbolic type and the centers of these conics are distinct. Then this system can be written in the form*

$$\begin{aligned} \dot{x} &= x(b + (1+b)x + x^2 + ay^2), \\ \dot{y} &= y(-b + (b-a)y + x^2 + ay^2), \end{aligned} \tag{8}$$

where $a, b \in \mathbf{R}$ *and* $a \neq -1$ *having the first integral* $H = (x-y+1)(x+ay+b)/(xy)$. *Moreover, the configurations of invariant straight lines of this system are*

$$\begin{aligned} &\textit{Config. } 1 \Longleftrightarrow && ab(b-1)(a+b)(a+b^2) \neq 0,\ a < 0; \\ &\textit{Config. } 2 \Longleftrightarrow && ab(b-1)(a+b)(a+b^2) \neq 0,\ a > 0; \\ &\textit{Config. } 3 \Longleftrightarrow && b(a+b)(b-1) = 0,\ a > 0; \\ &\textit{Config. } 4 \Longleftrightarrow && b(a+b)(b-1) = 0,\ a < 0; \\ &\textit{Config. } 5 \Longleftrightarrow && a = -b^2 \neq 0; \\ &\textit{Config. } 6 \Longleftrightarrow && a = 0,\ b(b-1) \neq 0; \\ &\textit{Config. } 7 \Longleftrightarrow && a = 0,\ b = 1; \\ &\textit{Config. } 8 \Longleftrightarrow && a = 0,\ b = 0. \end{aligned}$$

Proof. In order to have a cubic system, according to Lemma 9, we shall consider that the quadratic homogeneous parts of $H_{\mathbf{h}}^{(1)}$ and $H_{\mathbf{h}}^{(2)}$ are not proportional. Therefore, since the centers of these conics are distinct, there exists a component of a conic which intersects both components of the other one in two distinct points. So without loss of generality we can assume that $H_{\mathbf{h}}^{(2)} = xy$ (due to the affine transformation $x_1 = L_1^{(2)}$ and $y_1 = L_2^{(2)}$), and that the line $L_1^{(1)}$ intersects both lines $x = 0$ and $y = 0$ in two distinct points, say $(0, \alpha)$ and $(\beta, 0)$. Then we can assume that this line is $x - y + 1 = 0$ due to the rescaling $(x, y) \mapsto (x/\beta, -y/\alpha)$.

In short, we obtain the first integral $H = (x-y+1)(cx+ay+b)/(xy)$ and since $a^2 + c^2 \neq 0$ we can consider $c \neq 0$ (due to the change $(x, y) \mapsto (-y, -x)$). Finally, without loss of generality, we can assume $c = 1$ (multiplying by $1/c$ this being equivalent to the time rescaling $t \to t/c$ for systems (4)).

Thus the first integral of systems (4) can be written in the form (8) and, according to Corollary 8, they have the following real invariant straight lines

$$\begin{aligned} \mathcal{E}_1 &= xy(x-y+1)(x+ay+b)(bx+ay+b)(x-by+b) \\ &= L_1L_2L_3L_4L_5L_6 = 0, \end{aligned} \tag{9}$$

where $a, b \in \mathbf{R}$ and $a \neq -1$. It would be convenient to represent these six lines in the matrix form

$$(L_1, L_2, L_3, L_4, L_5, L_6) = (x, y)M + (0, 0, 1, b, b, b),$$

where

$$M = \begin{pmatrix} 1 & 0 & 1 & 1 & b & 1 \\ 0 & 1 & -1 & a & a & -b \end{pmatrix}.$$

It is known that two lines $A_1x+B_1y+C_1 = 0$ and $A_2x+B_2y+C_2 = 0$ are parallel if and only $A_1B_2 - A_2B_1 = 0$. Thus, in order to have six invariant straight lines in six different directions, all 2×2 minors of matrix M have to be different from zero. We denote by d_{ij}, $1 \leq i < j \leq 6$, the minors of the matrix M constructed using the columns i and j. Then we have

$$\begin{array}{lllll} d_{12} = 1, & d_{13} = -1, & d_{14} = a, & d_{15} = a, & d_{16} = -b, \\ d_{23} = -1, & d_{24} = -1, & d_{25} = -b, & d_{26} = -1, & d_{34} = a+1, \\ d_{35} = a+b, & d_{36} = -b+1, & d_{45} = a(1-b), & d_{46} = -b^2-a, & d_{56} = -b^2-a. \end{array}$$

Taking into consideration that $a \neq -1$ (which must be fulfilled for systems (8)) we conclude that the minors $d_{ij} \neq 0$ for all $1 \leq i < j \leq 6$ if and only if the condition

$$ab(b-1)(a+b)(a+b^2) \neq 0 \tag{10}$$

holds. So we conclude that in this generic case there are six different straight lines in six different directions, and we shall show that there exist two distinct configurations depending on the sign of parameter a.

Indeed since all the invariant straight lines $L_i = 0$, $i = 1, \dots, 6$ of system (8) are real, then clearly their intersection points are finite singularities of this system. In the generic case (10) we shall see that the system has four star points, which are intersections of three invariant straight lines.

Denoting by $\text{Int}\,(L_i, L_j, L_k)$ the intersection point of the straight lines L_i, L_j and L_k, it is easy to determine the coordinates of the four star points:

$$M_1 := \text{Int}\,(L_1, L_3, L_6) = (0, 1), \qquad M_2 := \text{Int}\,(L_2, L_3, L_5) = (-1, 0),$$
$$M_3 := \text{Int}\,(L_2, L_4, L_6) = (-b, 0), \quad M_4 := \text{Int}\,(L_1, L_4, L_5) = (0, -b/a).$$

We observe that two star points have fixed coordinates ($M_1 = (0,1)$ and $M_2 = (-1,0)$) as well as the singular point $M_0(0,0)$. On the other hand, the star points $M_3(x_0, 0)$ and $M_4(0, y_0)$ (where $x_0 = -b, y_0 = -b/a$) are moving on the axes when the parameters a and b vary.

It is easy to see that the quadrilateral formed by the points M_1, M_2, M_3 and M_4 is convex if $x_0 y_0 < 0$, and concave if $x_0 y_0 > 0$. Since $\text{sign}\,(x_0 y_0) = \text{sign}\,(b^2/a)$, then we get **Config.** 1 if $a < 0$ and **Config.** 2 if $a > 0$.

In what follows we assume that the condition $ab(b-1)(a+b)(a+b^2) = 0$ is fulfilled. Then, considering all the different possibilities in order that this expression be zero, we obtain the remaining six configurations of the statement of the proposition. $\square$

Phase portraits of systems of type $\mathcal{S}(\mathbf{h}, \mathbf{h})$. The goal of this subsection is to determine the phase portraits of the systems of type $\mathcal{S}(\mathbf{h}, \mathbf{h})$. In other words to determine all topologically non-equivalent phase portraits for this family of systems. Thus, first we introduce among other things the definition of the topological equivalence. Then we enunciate the Markus-Neumann-Peixoto theorem that allows us to determine all topologically equivalent systems by restricting ourselves mainly to studying the flow of the system on the set of their separatrices.

Let φ be a C^k local flow with $k \geq 0$ on the two-dimensional manifold M. Of course, for $k = 0$, the flow is continuous. We say that (M_1, φ_1) and (M_2, φ_2) are C^k*-equivalent* if there is a C^k diffeomorphism of M_1 onto M_2 which takes orbits of φ_1 onto orbits φ_2 preserving or reversing sense (but not necessarily the parametrization). Of course, a C^0 diffeomorphism is a homeomorphism.

We say that (M, φ) is C^k-*parallel* if it is C^k-equivalent to one of the following flows:

(i) $\mathbf{R}^2$ with the flow defined by $\dot{x} = 1$, $\dot{y} = 0$ (*strip flow*);

(ii) $\mathbf{R}^2 \setminus \{0\}$ with the flow defined by $\dot{r} = 0$, $\dot{\theta} = 1$ (*annular flow*);

(iii) $\mathbf{R}^2 \setminus \{0\}$ with the flow defined by $\dot{r} = r$, $\dot{\theta} = 0$ (*spiral flow*);

(iv) $\mathbf{S}^1 \times \mathbf{S}^1$ with rational flow (*toral flow*).

Let $p \in M$; we denote by $\gamma(p)$ the *orbit* of the flow φ on M through p, more precisely $\gamma(p) := \{\varphi_p(t) : t \in I_p\}$, where I_p is the maximal open interval of the solution of φ_p. The *positive semiorbit* of $p \in M$ is $\gamma^+(p) = \{t \in I_p, t \geq 0\}$. In a similar way we define the *negative semiorbit* $\gamma^-(p)$ of $p \in M$. We define the α-limit and ω-limit of $p \in M$ as

$$\alpha(p) = \overline{\gamma^-(p)} - \gamma^-(p), \qquad \omega(p) = \overline{\gamma^+(p)} - \gamma^+(p).$$

Let $\gamma(p)$ be an orbit of the flow φ defined on M. A *parallel neighborhood* of the orbit $\gamma(p)$ is an open neighborhood N of γ such that (N, φ) is C^k-equivalent to a parallel flow for some $k \geq 0$.

We say that $\gamma(p)$ is a *separatrix* of φ if $\gamma(p)$ is not contained in a parallel neighborhood N satisfying the following two assumptions:

(1) for every $q \in N$, $\alpha(q) = \alpha(p)$ and $\omega(q) = \omega(p)$,

(2) $\overline{N} \setminus N$ consists of $\alpha(p)$, $\omega(p)$ and exactly two orbits $\gamma(a)$, $\gamma(b)$ of φ, with $\alpha(a) = \alpha(p) = \alpha(b)$ and $\omega(a) = \omega(p) = \omega(b)$.

We denote by Σ the union of all separatrices of φ. Then Σ is a closed invariant subset of M. A component of the complement of Σ in M, with the restricted flow, is called a *canonical region* of φ.

Let (φ, M) be a continuous flow on the 2-manifold M, and let Σ be the set of all separatrices of (φ, M). In every canonical region U of (φ, M) we choose an orbit γ_U. Then a *separatrix configuration* of (φ, M) is formed by the union of the set Σ and the set of all orbits γ_U. In this work we do not draw the orbit γ_U in a canonical region unless it is necessary.

TABLE 1. The values of the determinant Δ and of the trace T at the singular points of system (8).

Singular point	Δ	T
$M_1 = (-1, 0)$	$(b-1)^2$	$2(1-b)$
$M_2 = (0, 0)$	$-b^2$	0
$M_3 = (0, 1)$	$(a+b)^2$	2(a+b)
$M_4 = (0, -b/a)$	$b^2(a+b)^2/a^2$	$2b(a+b)/a$
$M_5 = (-\frac{a+b}{a+1}, \frac{1-b}{a+1})$	$-(b-1)^2(a+b)^2/(a+1)^2$	0
$M_6 = (-b, 0)$	$b^2(b-1)^2$	$2b(b-1)$
$M_7 = (-\frac{b(a+b)}{a+b^2}, \frac{b(b-1)}{a+b^2})$	$-b^2(b-1)^2(a+b)^2/(a+b^2)^2$	$0.$

TABLE 2. The stability of the singular points of system (8).

Singular point	Type	Stable	Unstable
$M_1 = (-1, 0)$	Node	$b > 1$	$b < 1$
$M_2 = (0, 0)$	Saddle	———	———
$M_3 = (0, 1)$	Node	$a+b < 0$	$a+b > 0$
$M_4 = (0, -b/a)$	Node	$(a+b)b/a < 0$	$(a+b)b/a > 0$
$M_5 = (-\frac{a+b}{a+1}, \frac{1-b}{a+1})$	Saddle	———	———
$M_6 = (-b, 0)$	Node	$b \in (0, 1)$	$b \in \mathbf{R} \setminus [0, 1]$
$M_7 = (-\frac{b(a+b)}{a+b^2}, \frac{b(b-1)}{a+b^2})$	Saddle	———	———

Theorem 12 (Markus-Neumann-Peixoto). *Let (φ_1, M_1) and (φ_2, M_2) be two continuous flows on the 2-manifolds M_1 and M_2. Then two flows are topologically equivalent if and only if there exists a homeomorphism $h : M_1 \to M_2$, which takes the orbits of the separatrix configuration of (φ_1, M_1) into the orbits of the separatrix configuration of (φ_2, M_2).*

Now we shall determine the phase portraits of system (8).

First we focus on the generic case, i.e., when the condition (10) is fulfilled. In this case we have seven singular points M_i, $i = 1, \dots, 7$. In Table 1 we show the coordinates of M_i, and the determinant Δ and the trace T of the Jacobian matrix at M_i.

We always have three saddles: M_2, M_5 and M_7 since Δ is negative at these three singular points. The rest of the singular points M_1, M_3, M_4 and M_6 are nodes since the equation $T^2 = 4\Delta$ holds for all of them. Now we consider the stability of the nodes. The conditions for

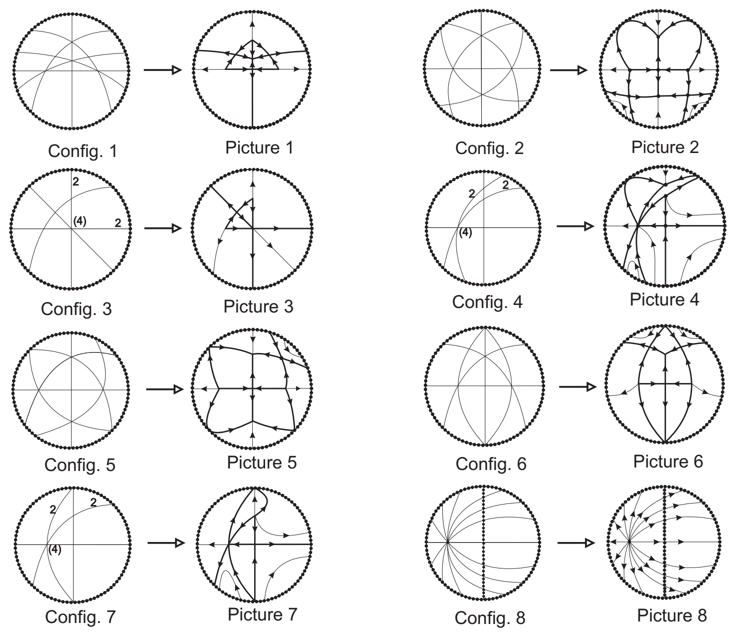


FIGURE 3. Configurations of the invariant straight lines of system (8) and their corresponding phase portraits.

the stability and instability of the nodes are given in Table 2. Thus when the condition (10) is fulfilled we get two topologically distinct phase portraits. For $a < 0$ we get a phase portrait that is topologically equivalent to **Picture** 1 and for $a > 0$ we have a phase portrait that is topologically equivalent to **Picture** 2, see Figure 3.

Now we analyze the phase portraits of system (8) in a non-generic case, i.e., when condition (10) is not fulfilled.

First we determine the phase portraits of system (8) having the configurations of the invariant straight lines **Config.** 3 and 4. Thus we consider system (8) when $b(a+b)(b-1) = 0$ for $a > 0$ and $a < 0$.

Assume that $b = 0$ and $a(a+1) \neq 0$ then system (8) takes the form

$$\dot{x} = x(x + x^2 + ay^2), \qquad \dot{y} = y(-ay + x^2 + ay^2). \tag{11}$$

The system has four singular points: two nodes $M_1 = (-1, 0)$ and $M_2 = (0, 1)$ one saddle $(-\frac{a}{a+1}, \frac{1}{a+1})$, and one degenerate singularity (see [**5**] for the definition of different types of singularities) at the origin. The trace T of the Jacobian matrix of system (11) at M_1, denoted by $T(M_1)$, is equal to 2 and at M_2 we have $T(M_2) = 2a$. So the first node M_1 is unstable and the second is stable for $a < 0$ and unstable for $a > 0$. Thus, we get respectively pictures that are topologically equivalent to **Pictures** 3 and 4 of Figure 3.

Now assume that $b = 1$ and $a(a-1) \neq 0$; then system (8) becomes

$$\dot{x} = x(1 + 2x + x^2 + ay^2), \qquad \dot{y} = y(-1 + (1-a)y + x^2 + ay^2). \tag{12}$$

The system has four singular points: two nodes $M_1 = (0,1)$, $M_2 = (0, -1/a)$, one saddle at the origin, and one degenerate singularity at $(-1,0)$. Similarly, as in the previous case for system (12), we have $T(M_1) = 2(a+1)$ and $T(M_2) = 2(a+1)/a$. We consider the product $T(M_1)T(M_2)) = 4(a+1)^2/a$. So the stability of the two nodes M_1 and M_2 are distinct for $a < 0$, and we get the phase portrait that is topologically equivalent to **Picture** 3. The stability of the nodes coincide for $a > 0$, and we get a phase portrait that is topologically equivalent to **Picture** 4.

Now assume that $b = -a$ and $a(a+1) \neq 0$; then system (8) becomes

$$\dot{x} = x(-a + (1-a)x + x^2 + ay^2), \qquad \dot{y} = y(a - 2ay + x^2 + ay^2). \tag{13}$$

System (13) has four singular points: two nodes $M_1 = (-1,0)$ and $M_2 = (a,0)$, one saddle at the origin and one degenerate singularity $(0,1)$. In this case we also get **Picture** 3 for $a > 0$ and **Picture** 4 for $a < 0$, see Figure 3. We showed that if $b(b+1)(a+b) = 0$, $a > 0$ then system (8) has the phase portrait of **Picture** 3, and if $b(b+1)(a+b) = 0$ and $a < 0$ then **Picture** 4, see Figure 3.

Now we determine the phase portrait of system (8) having the configuration **Config.** 5. Thus we consider system (8) when $a = -b^2$ and $b \neq 0, 1$, i.e.,

$$\dot{x} = x(b + (1+b)x + x^2 - b^2y^2), \qquad \dot{y} = y(-b + b(1+b)y + x^2 - b^2y^2).$$

The system has six singular points: four nodes $M_1 = (-1,0)$, $M_2 = (0,1)$, $M_3 = (0,1/b)$ and $M_4 = (-b,0)$, and two saddles one at the origin and the other at $(-\frac{b}{b+1}, \frac{b-1}{b})$. Since we have $T(M_1) = 2(1-b)$,

$T(M_2) = 2b(1-b)$, $T(M_3) = -2(1-b)$ and $T(M_4) = 2b(b-1)$. We determine the stability of the nodes and we get the phase portraits that are all topologically equivalent to the one of **Picture** 5 in Figure 3.

To determine the phase portrait of system (8) having the configuration **Config.** 6 we consider (8) when $a = 0$ and $b(b-1) \neq 0$, i.e.,

$$\dot{x} = x(b + (1+b)x + x^2), \qquad \dot{y} = y(-b + by + x^2).$$

Our system has six singular points: three nodes $M_1 = (-1, 0)$, $M_2 = (0, 1)$ and $M_3 = (-b, 0)$ and three saddles, one at the origin, $(-b, -b+1)$ and $(-1, (b-1)/b)$. We also have $T(M_1) = 2(1-b)$, $T(M_2) = 2b$ and $T(M_3) = 2b(b-1)$. We obtain the unique phase portrait that is topologically equivalent to the one of **Picture** 6, see Figure 3.

We determine the phase portrait of system (8) having the configuration **Config.** 7. Thus we consider (8) when $a = 0$ and $b = 1$, i.e.,

$$\dot{x} = x(1 + 2x + x^2), \qquad \dot{y} = y(-1 + y + x^2).$$

Then system (8) has three singular points: a degenerate one $(-1, 0)$, a saddle $(0, 0)$ and a node $(0, 1)$. We get the phase portrait **Picture** 7 in Figure 3.

Finally we determine the phase portrait of system (8) having the configuration **Config.** 8. Thus we consider system (8) when $a = 0$ and $b = 0$, i.e.,

$$\dot{x} = x^2(1+x), \qquad \dot{y} = yx^2. \tag{14}$$

This system is degenerate since it has the common factor x^2. Thus it has the line of singularities $x = 0$. We have the unique phase portrait, see **Picture** 8 in Figure 3.

Proposition 13. *Assume that a cubic system has a rational first integral of degree two of the form $H_{\mathbf{h}}^{(1)}/H_{\mathbf{h}}^{(2)}$ where $H_{\mathbf{h}}^{(i)} = 0$ is a reducible conic of hyperbolic type for $i = 1, 2$, and that the centers of the conics $H_{\mathbf{h}}^{(1)} = 0$ and $H_{\mathbf{h}}^{(2)} = 0$ coincide. Then this system can be written in the form*

$$\dot{x} = x(x^2 + ay^2), \qquad \dot{y} = y(x^2 + ay^2), \tag{15}$$

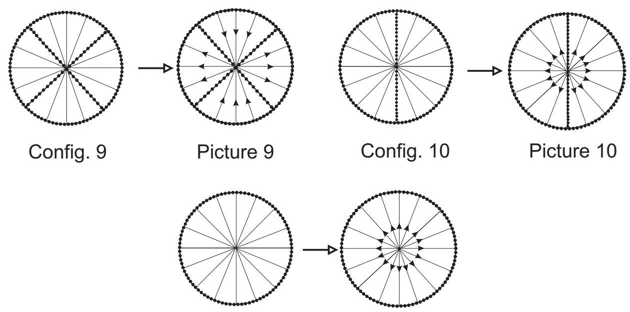


FIGURE 4. Configurations of invariant straight lines of system (15) and their corresponding phase portraits.

where $a \in \mathbf{R} \setminus \{-1\}$ *having the first integral* $H = (x-y)(x+ay)/(xy)$. *Moreover, the configurations of invariant straight lines of this system are*

$$\begin{aligned} \textit{Config. } 9 &\Longleftrightarrow \quad a < 0, \\ \textit{Config. } 10 &\Longleftrightarrow \quad a = 0, \\ \textit{Config. } 11 &\Longleftrightarrow \quad a > 0. \end{aligned}$$

Proof. In this case providing that $H_{\mathbf{h}}^{(2)} = xy$ we obtain that all four lines pass through the origin. By Lemma 9 there exist at least three directions. Then, without loss of generality, we can assume $H_{\mathbf{h}}^{(1)} = (x-y)(bx+ay)$, where $a^2+b^2 \neq 0$, and we can consider $b = 1$ due to the change $(x, y) \mapsto (-y, -x)$ and $a \neq -1$ otherwise $H_{\mathbf{h}}^1$ would not be of hyperbolic type. Thus we get one-parameter family of systems (15). Then studying all the possible configurations of the system varying the parameter a, we obtain **Config.** 9 if $a < 0$, **Config.** 10 if $a = 0$ and **Config.** 11 if $a > 0$.

Evidently the configurations determined above lead to the respective phase portraits of Figure 4. □

3.2. Systems of type $\mathcal{S}(\mathbf{h}, \mathbf{p})$. Assume that system (4) possesses only one reducible conic of hyperbolic type, say $H_{\mathbf{h}} = 0$, and at least one reducible conic of parabolic type, $H_{\mathbf{p}} = 0$. In this subsection we show all the configurations of the invariant straight lines that the system can have and their corresponding phase portraits.

Proposition 14. *Assume that a cubic system has a rational first integral of degree two of the form* $H_{\mathbf{p}}/H_{\mathbf{h}}$ *where* $H_{\mathbf{h}}=0$ *and* $H_{\mathbf{p}}=0$ *are reducible conics of hyperbolic and parabolic type respectively. Then this system can be written in the form*

$$\begin{aligned}\dot{x}&=x(cd+(c+d)x+x^2-b^2y^2),\\ \dot{y}&=y(-cd-b(c+d)y+x^2-b^2y^2),\end{aligned}\tag{16}$$

where $b\in\mathbf{R}$, *and either* $c,d\in\mathbf{R}$ *or* $d=\overline{c}\in\mathbf{C}\setminus\mathbf{R}$, *having the first integral* $H=(x+by+c)(x+by+d)/(xy)$. *Moreover, all the possible configurations of the invariant straight lines of this system which have not appeared in Propositions* 11 *and* 13 *are given in Figure* 5.

Proof. We can consider $H_{\mathbf{h}}=xy$ and the factorization over $\mathbf{C}$ of the parabolic conic will be $H_{\mathbf{p}}=(ax+by+c)(ax+by+d)=0$. As the conic $H_{\mathbf{p}}=0$ must be real we have $a=\overline{a}$ and $b=\overline{b}$, i.e., $a,b\in\mathbf{R}$. Moreover, since $a^2+b^2\neq 0$ we can consider $a\neq 0$ due to the change $(x,y)\mapsto(y,x)$, and then via the rescaling $x\to x/a$ we can assume $a=1$.

In short, we obtain the first integral $H=(x+by+c)(x+by+d)/(xy)$ and hence system (4) becomes of the form (16). To determine the type of the third reducible conic (say $\widehat{H}=0$) of this system, by Corollary 8, we calculate

$$\mathcal{E}_1=-2xy(x+by+c)(x+by+d)(dx+bcy+cd)(cx+bdy+cd).\tag{17}$$

Therefore $\widehat{H}=(dx+bcy+cd)(cx+bdy+cd)=0$. Since all systems with two different hyperbolic conics were considered in the previous section, we assume that $\widehat{H}$ is either non-hyperbolic or it is exactly $xy=0$. We are in the latter case if and only if $cd=0$. Suppose that $c=0$; then we have the first integral $H=(x+by)(x+by+d)/(xy)$. We assume that $bd\neq 0$; otherwise we get degenerate systems that were already considered. So by rescaling we get the first integral $H=(x+y)(x+y+1)/(xy)$ of the system

$$\dot{x}=x(x+x^2-y^2),\qquad \dot{y}=y(x^2-y-y^2).\tag{18}$$

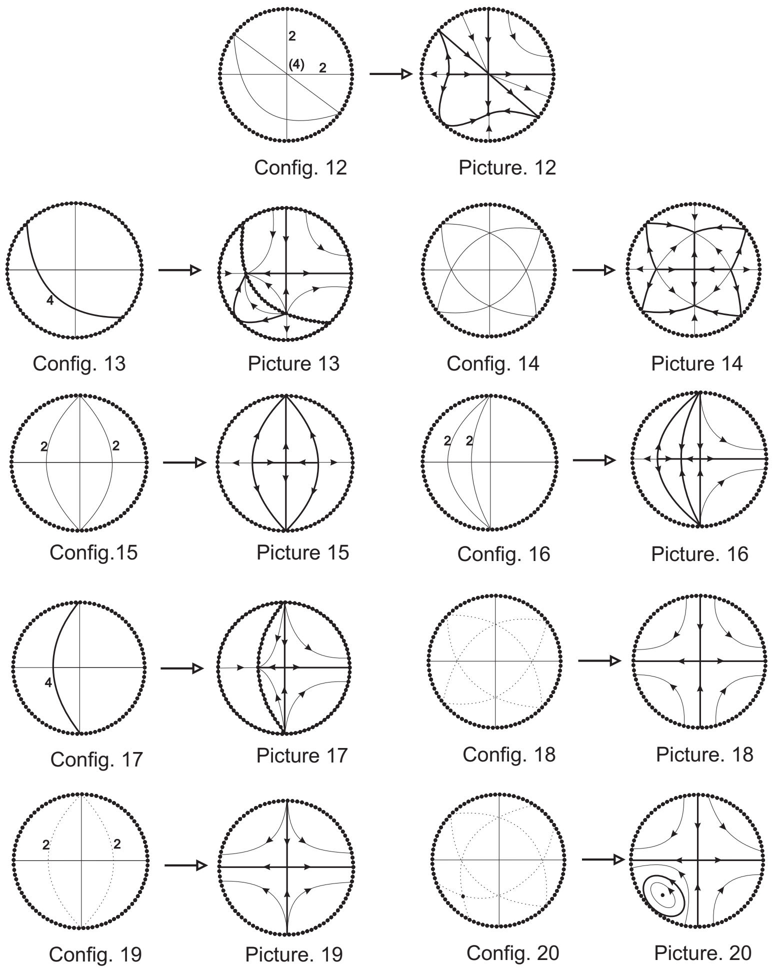


FIGURE 5. Configurations of invariant straight lines and corresponding phase portraits of system (16).

Calculating

$$\mathcal{E}_1 = x^2y^2(x+y)(1+x+y),$$

we get the configuration of invariant straight lines **Config.** 12.

We assume now that the third reducible conic $\widehat{H} = 0$ is not of hyperbolic type. Then the condition $\delta \geq 0$ (for the definition see the

Appendix) must hold. That is,

$$b^2(c-d)^2(c+d)^2 \le 0. \tag{19}$$

Without loss of generality, we can assume $b \in \{0,1\}$ due to the rescaling $y \to y/b$ if $b \neq 0$, and we shall consider these two cases.

Case: $b^2(c-d)^2(c+d)^2 < 0$. Then $c, d \in \mathbf{C} \setminus \mathbf{R}$ and $b \neq 0$. More precisely, $c = r + is$ and $d = r - is$ such that $rs \neq 0$ and $b = 1$. So system (16) can be written as

$$\begin{aligned}\dot{x} &= x[(r^2+s^2) + 2rx + x^2 - y^2],\\ \dot{y} &= y[-(r^2+s^2) - 2ry + x^2 - y^2].\end{aligned}$$

Since $rs \neq 0$ we can assume that $r = 1$ due to the rescaling $(x, y, t) \mapsto (rx, ry, t/r^2)$. Finally, we arrive at the one-parameter family of systems

$$\begin{aligned}\dot{x} &= x[(1+s^2) + 2x + x^2 - y^2],\\ \dot{y} &= y[-(1+s^2) - 2y + x^2 - y^2].\end{aligned}$$

having the configuration $\mathcal{E}_1$ of the invariant straight lines

$$\begin{aligned}&2xy(x+y+1+si)(x+y+1-si)\\ &\qquad [(s-i)x-(s+i)y-i(1+s^2)][(s+i)x-(s-i)y+i(1+s^2)],\end{aligned}$$

that correspond to **Config.** 20, see Figure 5.

Case: $b^2(c-d)^2(c+d)^2 = 0$. In this case the third conic is of parabolic type.

1) Assuming $b = 1$ we have $(c-d)(c+d) = 0$.

a) If $d = c$ then this parameter must be real and we get the following degenerate systems

$$\dot{x} = x(x-y+c)(x+y+c), \qquad \dot{y} = y(x-y-c)(x+y+c), \tag{20}$$

where $c \in \{0,1\}$ due to the rescaling $(x, y, t) \mapsto (cx, cy, t/c^2)$ if $c \neq 0$. So if $c = 1$ we get **Config.** 13 and if $c = 0$ we get a configuration topologically equivalent to **Config.** 9.

b) If $d = -c$, then we obtain the systems

$$\dot{x} = x(-c^2 + x^2 - y^2), \qquad \dot{y} = y(c^2 + x^2 - y^2), \tag{21}$$

where either $c \in \mathbf{R}$, or $0 \neq c = ir \in \mathbf{R}$. In the first case we can assume $c \in \{0, 1\}$ due to the rescaling $(x, y, t) \mapsto (cx, cy, t/c^2)$ if $c \neq 0$, whereas in the second case we can assume $c = i$ due to the rescaling $(x, y, t) \mapsto (rx, ry, t/r^2)$. System (21) possesses the following invariant straight lines

$$xy(x+y+c)(x+y-c)(x-y+c)(x-y-c) = 0.$$

We skip the case $c = 0$; we get the degenerate system belonging to the $\mathcal{S}(\mathbf{h}, \mathbf{h})$ family. We then obtain **Config.** 14 if $c = 1$, and **Config.** 18 if $c = i$, see Figure 5.

2) Assume now that $b = 0$. Then we get the family of systems

$$\dot{x} = x(x+c)(x+d), \qquad \dot{y} = y(-cd + x^2), \tag{22}$$

where either $c, d \in \mathbf{R}$ or $d = \overline{c} \in \mathbf{C} \setminus \mathbf{R}$. Using (17) we obtain the following invariant straight lines

$$xy(x+c)^2(x+d)^2 = 0.$$

a) Assume first that $c, d \in \mathbf{R}$. Due to rescaling $(x, y, t) \mapsto (cx, y, t/c^2)$ if $c \neq 0$ we may assume $c \in \{0, 1\}$. If $c = 1$, then it is easy to observe that the configurations of invariant straight lines of systems (22) are given by **Config.** 15 if $d < 0$, by **Config.** 16 if $d > 0$, $d \neq 1$, and by **Config.** 17 if $d = 1$. In the case $c = 0$, by the same reasons as above, we can assume that $d \in \{0, 1\}$. For $d = 0$ we get the configuration that is topologically equivalent to **Config.** 10. Finally, when $c = 1$, $d = 0$, and $c = 0$, $d = 1$ we get exactly system (14) that was considered in the previous section.

b) Suppose now that $d = \overline{c} \in \mathbf{C} \setminus \mathbf{R}$ and assume $c = r + is$ ($s \neq 0$ and we can consider $s = 1$ due to the rescaling $(x, y, t) \mapsto (sx, y, t/s^2)$). So we obtain the following one-parameter family of systems

$$\dot{x} = x[1 + (x+r)^2], \qquad \dot{y} = y[-1 - r^2 + x^2],$$

having the configuration of invariant straight lines corresponding to **Config.** 19, see Figure 5. $\square$

Phase portraits of systems of type $\mathcal{S}(\mathbf{h}, \mathbf{p})$. In this subsection we determine the phase portraits for each of the configurations of invariant straight lines of system (16).

First we determine the phase portrait of system (16) having the configuration of invariant straight lines **Config.** 12; thus, we consider system (18). The system has three singular points: one degenerate at the origin, one stable node at $(0,-1)$ and one unstable node at $(-1,0)$.

We determine the phase portrait of system (16) having the configuration **Config.** 13 of the invariant straight lines. Thus we consider system (16) with $b=1$ and $c=d=1$, i.e.,

$$\dot{x}=x(x-y+1)(x+y+1), \qquad \dot{y}=y(x-y-1)(x+y+1).$$

The system has a common factor $x+y+1$; hence, the phase portraits will contain the line of singular points $x+y+1=0$. There is one isolated singular point at the origin which is a saddle. For the phase portrait see **Picture** 13 in Figure 5.

To determine the phase portrait of system (16) having the configuration **Config.** 14 we consider system (16) with $b=1$ and $c=-d=1$, i.e.,

$$\dot{x}=x(-1+x^2-y^2), \qquad \dot{y}=y(1+x^2-y^2).$$

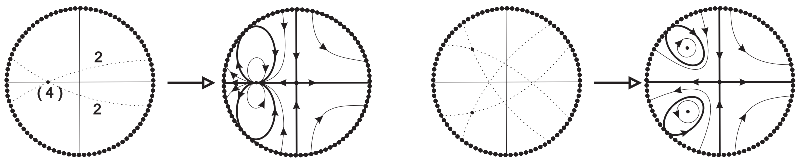
The system has five singular points which are two stable nodes $(0,1)$ and $(0,-1)$, two unstable nodes $(1,0)$ and $(-1,0)$, and one saddle at the origin, see **Picture** 14.

We determine the phase portrait of system (16) having the configuration **Config.** k, for $k=15,16,17$. So we consider system (16) with $b=0$, $c=1$, i.e.,

$$\dot{x}=x(x+1)(x+d), \qquad \dot{y}=y(-d+x^2),$$

and respectively $d<0$, $1\neq d>0$ and $d=1$. Suppose that $d\in\mathbf{R}\setminus\{0,1\}$. Then the system has three singular points: a saddle $M_1=(0,0)$, and two nodes $M_2=(-1,0)$ and $M_3=(-d,0)$. M_2 is stable for $d>1$ and unstable for $d<1$. Finally, M_3 is stable for $d\in(0,1)$, and unstable for $d\in(-\infty,0)\cup(1,+\infty)$. This leads to two topologically distinct phase portraits see **Picture** 15 for $d<0$, and see **Picture** 16 for $d>0$.

If $d=1$, then the above system has a line of singular points $x+1=0$. The only isolated singularity is the origin which is a saddle, see **Picture** 17.



Config. 21 Picture 21 Config. 22 Picture 22

FIGURE 6. Configurations of the invariant straight lines and corresponding phase portraits of system (23).

We determine the phase portrait of system (16) having the configuration **Config.** 20. Thus we consider system (16) with $b = 1$, $d = \overline{c} = 1 + si \in \mathbf{C}$ and $s \neq 0$, i.e.,

$$\dot{x} = x[(1 + s^2) + 2x + x^2 - y^2], \qquad \dot{y} = y[-(1 + s^2) - 2y + x^2 - y^2].$$

The system has two singular points: a saddle at the origin and a center at $(-1/2[1 + s^2], -1/2[1 + s^2])$, see **Picture** 20.

We determine the phase portrait of system (16) having the configuration **Config.** 18. Thus we consider system (16) for $b = 1$ and $c = i$, i.e.,

$$\dot{x} = x(1 + x^2 - y^2), \qquad \dot{y} = y(-1 + x^2 - y^2).$$

The system has only one singular point at the origin which is a saddle, see **Picture** 18.

We determine the phase portrait of system (16) having the configuration **Config.** 19. Thus we consider system (16) for $b = 1$ and $d = \overline{c} = r + i \in \mathbf{C}$, i.e.,

$$\dot{x} = x[1 + (x + r)^2], \qquad \dot{y} = y[-1 - r^2 + x^2].$$

The system has only one singular point at the origin which is saddle, see **Picture** 19.

3.3. Systems of type $\mathcal{S}(\mathbf{h}, \mathbf{e})$. Assume that system (4) possesses only one reducible conic of hyperbolic type, say $H_{\mathbf{h}} = 0$ and at least one reducible conic of elliptic type $H_{\mathbf{e}} = 0$.

Proposition 15. *Assume that a cubic system has a rational first integral of degree two of the form $H_{\mathbf{e}}/H_{\mathbf{h}}$ where $H_{\mathbf{h}} = 0$ and $H_{\mathbf{e}} = 0$*

are reducible conics of hyperbolic and elliptic type respectively. Then this system can be written in the form

$$\begin{aligned} \dot{x} &= x\big[b^2+d^2+2(b+cd)x+(c^2+1)x^2-y^2\big],\\ \dot{y} &= y\big[-b^2-d^2-2dy+(c^2+1)x^2-y^2\big], \end{aligned} \tag{23}$$

where $d \in \{0,1\}$, *having the first integral* $H = \big((x+b)^2+(cx+y+d)^2\big)/(xy)$. *Moreover, all the possible configurations of invariant straight lines of this system which have not appeared in Propositions* 11, 13 *and* 14 *are given in Figure* 6.

Proof. Since the conic $H_{\mathbf{e}}=0$ is reducible we may assume $H_{\mathbf{e}} = x^2+y^2$ due to an affine transformation. On the other hand, we have $H_{\mathbf{h}} = L_1L_2$, and since a rotation keeps the form of $H_{\mathbf{e}}$, we may consider $L_1 = ax+b$ and $L_2 = cx+ey+d$, where $ae \neq 0$ (as $H_{\mathbf{h}}=0$ is of hyperbolic type). Then, via the affine transformation $x_1 = L_1$, $y_1 = L_2$ we obtain $H_{\mathbf{h}} = x_1y_1$ and $H_{\mathbf{e}} = (x_1-b)^2+(cx_1/e-ay_1/e+(ad-bc)/e)^2$. Since $ae \neq 0$, applying the change $(x_1,y_1)\mapsto(x, ey/a)$ and renaming the parameters we arrive to the first integral $H = \big((x+b)^2+(cx+y+d)^2\big)/(xy)$.

In short, systems (4) become of the form (23) where we may assume $d\in\{0,1\}$ due to the rescaling $(x,y,t)\mapsto(dx,dy,t/d^2)$ if $d\neq 0$. To determine the type of the third reducible conic (say $\widehat{H}=0$) of these systems according to Corollary 8 we calculate

$$\begin{aligned} \mathcal{E}_1 = -2xy\big[(x+b)^2+(cx+y+d)^2\big]&\\ \big[(bcx-dx-by)^2+(b^2+d^2+bx+cdx+dy)^2\big]&. \end{aligned}$$

Hence the third reducible conic $\widehat{H} = (bcx-dx-by)^2+(b^2+d^2+bx+cdx+dy)^2 = 0$ is of elliptic type if

$$b^2+2bcd-d^2 \neq 0; \tag{24}$$

for the details, see Appendix 1. Otherwise, $\widehat{H}=0$ are two complex parallel lines so we do not analyze this case since we have already considered these types of systems in the previous section. Calculating the resultant of the quadratic homogenous parts of the conics $H_{\mathbf{e}}=0$ and $\widehat{H}=0$ with respect to the variable y, we obtain

$$\mathrm{Res}_y[H_{\mathbf{e}}^{(2)}, \widehat{H}^{(2)}] = 16b^2(bc-d)^2(c^2+1)x^4.$$

Hence, the components (i.e., complex lines) of these conics are parallel if and only if the condition $b(bc-d)=0$ holds. In fact, we show that they coincide when this condition holds.

To show this, we notice that $(b^2+d^2)H_e - \widehat{H} = 4b(bc-d)xy$. This means that if $b(bc-d)=0$ then the following equality holds $(b^2+d^2)H_e = \widehat{H}$.

First we consider system (23) when $d=0$. We exclude $b=0$ because of condition(24). As we showed before the two conics $H_{\mathbf{e}}=0$ and $\widehat{H}=0$ in this case coincide if and only if $c=0$ and we get the configuration **Config.** 21, see Figure 6. If $c\neq 0$, then all reducible conics are different and we get the unique configuration **Config.** 22.

Now we consider system (23) for $d=1$. By the previous comment the two conics $H_{\mathbf{e}}=0$ and $\widehat{H}=0$ coincide if and only if $b(bc-1)=0$. If $b=0$, then the two elliptic conics $H_{\mathbf{e}}=0$ and $\widehat{H}=0$ coincide and this gives us a real point $(0,-d)$ of multiplicity four and we have the configuration **Config.** 21. If $bc-1=0$, then again the two conics coincide, and this gives a point $(-b,0)$ of multiplicity four and we get the configuration homeomorphically equivalent to the previous one. Assume now that $b(bc-1)\neq 0$. Then all three conics $xy=0$, $H_{\mathbf{e}}=0$ and $\widehat{H}=0$ are distinct and their centers are real singular points. We get **Config.** 22. □

Thus we have completed the study of the invariant straight lines of system (4) in the case when, among the three reducible conics, there is one of hyperbolic type and another one of elliptic type.

Phase portraits of systems of type $\mathcal{S}(\mathbf{h},\mathbf{e})$. In this subsection we determine the phase portraits for each configuration of invariant straight lines of system (23).

First consider system (23) when $d=0$, i.e.,

$$\begin{aligned}\dot{x} &= x\big[b^2+2bx+(c^2+1)x^2-y^2\big],\\ \dot{y} &= y\big[-b^2+(c^2+1)x^2-y^2\big].\end{aligned}\tag{25}$$

Because of condition (24) we have $b\neq 0$.

Assume that $c\neq 0$. Then the system has three singular points: one saddle at the origin and two centers $(-b,-bc)$ and $(-b,bc)$, see **Picture** 22 in Figure 6.

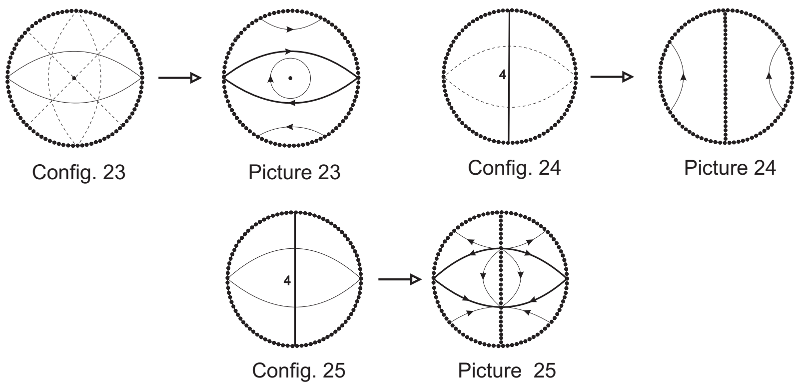


FIGURE 7. Configurations of invariant straight lines and their corresponding phase portraits of system (27).

Now assume that $c = 0$; then system (25) has two singular points: one saddle at the origin and one degenerate singular point $(-b, 0)$, see **Picture** 21.

Consider now system (23) when $d = 1$, i.e.,

$$\begin{aligned} \dot{x} &= x\big[b^2 + 1 + 2(b+c)x + (c^2+1)x^2 - y^2\big], \\ \dot{y} &= y\big[-b^2 - 1 - 2y + (c^2+1)x^2 - y^2\big]. \end{aligned} \tag{26}$$

Because of the condition (24) we have $b^2 + 2bc - 1 \neq 0$.

If $b(bc - 1) \neq 0$, then system (26) has three singular points: one saddle at the origin and two centers $(-b, bc-1)$ and $((b+b^3)/(1-b^2-2bc)), -((1+b^2)(bc-1))/(-1+b^2+2bc)$, see **Picture** 22.

If $b(bc-1) = 0$, then system (26) has two singular points: one saddle at the origin and one degenerate singular point $(-(b+c)/(1+c^2), 0)$, see **Picture** 21.

3.4. Systems of type $\mathcal{S}(\mathbf{p}, \mathbf{p})$. Assume that system (4) possesses two different reducible conics of parabolic type, say $H_{\mathbf{p}}^{(1)} = 0$ and $H_{\mathbf{p}}^{(2)} = 0$. In this subsection we determine all the configurations of invariant straight lines that the system can have and their corresponding phase portraits.

Proposition 16. *Assume that a cubic system has a rational first integral of degree two of the form $H_{\mathbf{p}}^{(1)}/H_{\mathbf{p}}^{(2)}$ where $H_{\mathbf{p}}^{(1)} = 0$ and*

$H_{\mathbf{p}}^{(2)} = 0$ *are reducible conics of parabolic type. Then this system can be written in the form*

$$\dot{x} = y(x^2 + a), \qquad \dot{y} = x(y^2 + b), \tag{27}$$

where $a, b \in \mathbf{R}$*, having the first integral* $H = (x^2 + a)/(y^2 + b)$*. Moreover, all the possible configurations of invariant straight lines of this system which have not appeared in Propositions* 11, 13, 14 *and* 15 *are given in Figure* 7.

Proof. Assume that system (4) possesses two distinct reducible conics of parabolic type $H_{\mathbf{p}}^{(i)} = L_1^{(i)} L_2^{(i)}$ $(i = 1, 2)$. In order to have a cubic system, according to Lemma 9 we will consider the situation when the quadratic homogeneous parts of $H_{\mathbf{p}}^{(1)}$ and $H_{\mathbf{p}}^{(2)}$ are not proportional. This means that we have two couples of parallel lines crossing in two distinct directions, say the direction of the line $L_1 = ax + by = 0$ and $L_2 = cx + dy = 0$, with $ad - bc \neq 0$. Then, via the linear transformation $x_1 = L_1$ and $y_1 = L_2$, we get the following first integral $H = (x^2 + a)/(y^2 + b)$. Therefore, applying the time rescaling $t \to t/2$ we arrive at the family of systems (27) where $a, b \in \mathbf{R}$. Moreover, due to the rescaling $(x, y, t) \mapsto (|a|^{1/2} x, y, |a|^{-1} t)$ if $a \neq 0$ we may assume $a \in \{0, \pm 1\}$.

To determine the type of the third reducible conic (say $\widehat{H} = 0$) of this system according to Corollary 8 we calculate

$$\mathcal{E}_1 = (a + x^2)(b + y^2)(ay^2 - bx^2). \tag{28}$$

So $\widehat{H} = ay^2 - bx^2$. If $ab > 0$, then the conic $\widehat{H} = 0$ is of hyperbolic type and system (27) is included in the family $\mathcal{S}(\mathbf{h}, \mathbf{p})$ that has been studied before. Thus we assume $ab \leq 0$.

If $ab < 0$ $\widehat{H}$ is of elliptic type and we get the configuration **Config.** 23 of Figure 7.

If $ab = 0$ we my assume $a = 0$ (due to the change $(x, y) \mapsto (y, x)$). So if $b > 0$ we get **Config.** 24 and if $b < 0$ we get **Config.** 25. If $b = 0$, then we get a degenerate system that was already examined (**Config.** 9). □

Phase portraits of systems of type $\mathcal{S}(\mathbf{p}, \mathbf{p})$. In this subsection we determine the phase portraits for each configuration of invariant straight lines of system (27) given in Figure 7.

Consider system (27) having **Config.** 23 of invariant straight lines, i.e., when $ab < 0$. The system has only one singular point at the origin which is a center. There are also two parallel lines surrounding the origin either $x^2 + a = 0$ when $a < 0$ or $y^2 + b =$ when $b < 0$. Thus the phase portrait corresponding to **Config.** 23 is **Picture** 23, see Figure 7.

Now consider system (27) having the configuration **Config.** 24, i.e.,

$$\dot{x} = x^2 y, \qquad \dot{y} = x(y^2 + b), \tag{29}$$

where $b > 0$. There is a line of singular points $x = 0$ and no isolated singular points. There are no real invariant straight lines because $\mathcal{E}_1 = -bx^4(b + y^2)$. We get the phase portrait **Picture** 24.

Finally, we determine the phase portrait of system (27) having **Config.** 25, i.e., system (29) when $b < 0$. There are two straight invariant lines $y^2 + b = 0$ and a line of singular points $x = 0$. We get the phase portrait **Picture** 25.

3.5. Systems of type $\mathcal{S}(\mathbf{p}, \mathbf{e})$. Assume that system (4) possesses two different reducible conics, one of parabolic type, say $H_{\mathbf{p}} = 0$, and another one of elliptic type $H_{\mathbf{e}} = 0$. In this subsection we determine all the configurations of invariant straight lines that these kind of systems can have and their corresponding phase portraits.

Proposition 17. *Assume that a cubic system has a rational first integral of degree two of the form $H_{\mathbf{p}}/H_{\mathbf{e}}$ where $H_{\mathbf{p}} = 0$ and $H_{\mathbf{e}} = 0$ are reducible conics of parabolic and elliptic type respectively. Then this system can be written in the form*

$$\begin{aligned} \dot{x} &= 2y\big(a^2 + b + 2ax + x^2\big), \\ \dot{y} &= -2\big((a^2 + b)x + ax^2 - ay^2 - xy^2\big), \end{aligned} \tag{30}$$

where $a, b \in \mathbf{R}$, having the first integral $H = ((x + a)^2 + b)/(x^2 + y^2)$. Moreover, all the possible configurations of invariant straight lines of this system which have not appeared in Propositions 11, 13–16 *are given in Figure* 8.

Proof. Assume that system (4) possesses one reducible conic of parabolic type, say $H_{\mathbf{p}} = 0$, and one reducible conic of elliptic type $H_{\mathbf{e}} = 0$. Then by Proposition 20 we may assume $H_{\mathbf{e}} = x^2 + y^2$ due to an affine transformation and $H_{\mathbf{p}} = (cx + dy + a)^2 + b$ where $a, b, c, d \in \mathbf{R}$. Moreover, due to a rotation (which keeps the form of

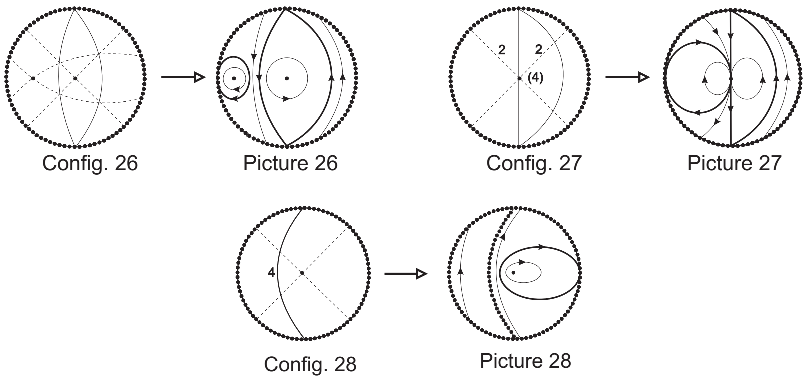


FIGURE 8. Configurations of invariant straight lines and their corresponding phase portraits of system (30).

$H_{\mathbf{e}}$) we may consider that the couple of parallel lines $H_{\mathbf{p}}$ is of the form $(x+a)^2+b$. Thus we can assume that the system has the first integral $H = ((x+a)^2+b)/(x^2+y^2)$.

If $a = 0$, then the system belongs (up to a time rescaling) to the family (27) (when $a = -b$). So we do not obtain new configurations of invariant straight lines. Thus we assume that $a \neq 0$.

To determine the type of the third reducible conic (say $\widehat{H} = 0$) of this system according to Corollary 8 we calculate

$$\mathcal{E}_1 = 8(x^2+y^2)((x+a)^2+b)\widehat{H}, \tag{31}$$

where $\widehat{H} = a^2(a^2+2b)+b^2+2a(a^2+b)x+a^2x^2-by^2$. For the conic $\widehat{H} = 0$ we have the invariants $\Delta = 0$ and $\delta = -ba^2$ (see the Appendix). So for $b > 0$ we have $\delta < 0$ and the third reducible conic is of hyperbolic type. We skip this case since systems of $\mathcal{S}(\mathbf{h},\mathbf{p})$ type have already been considered. If $b < 0$, then $\delta > 0$ and the $\widehat{H}$ is of elliptic type. If $b \neq -a^2$, i.e., when the reducible conics $x^2+y^2 = 0$ and $(x+a)^2+b = 0$ have no points in common in the real plane we have **Config.** 26 of Figure 8. If $b = -a^2$, then we have **Config.** 27. Finally, if $b = 0$, we get the degenerate system

$$\dot{x} = 2(x+a)^2y, \qquad \dot{y} = -2(x+a)(ax-y^2), \tag{32}$$

and we have **Config.** 28. $\square$

Phase portraits of systems of type $\mathcal{S}(\mathbf{p}, \mathbf{e})$. In this subsection we determine the phase portraits for each of the configurations of invariant straight lines of system (30) given in Figure 8.

We determine the phase portraits of system (30) having the configuration of invariant straight lines **Config.** 26, i.e., when $b < 0$ and $b \neq -a^2$. We have two centers: one at the origin and one at $\big((-a^2-b)/a, 0\big)$. Moreover, we have two real invariant straight lines given by $(x+a)^2+b=0$. We get the phase portrait **Picture** 26 of Figure 8.

Now we determine the phase portrait of system (30) having **Config.** 27, i.e.,

$$\dot{x} = 2y\big(2ax + x^2\big), \qquad \dot{y} = -2\big(ax^2 - ay^2 - xy^2\big),$$

where $a \neq 0$. There is only one singular point at the origin (which is of multiplicity four) and two real straight parallel invariant lines given by $x(x+2a)=0$, see **Picture** 27.

Finally, we consider the phase portrait of system (30) having **Config.** 28, i.e., system (32). There is a line of singular points $x+a=0$ and one center at the origin. For the phase portrait see **Picture** 28.

3.6. The subfamily of system (4) which possesses a single reducible conic–which is of hyperbolic type. Assume that system (4) possesses only one reducible conic of hyperbolic type. In this subsection we determine all the configurations of invariant straight lines that these kinds of system can have and their corresponding phase portraits.

Proposition 18. *Assume that a cubic system possesses a single reducible conic of hyperbolic type and does not have other reducible conics. Then this system can be written in the form*

$$(33) \quad \dot{x} = x\big(e + cx + ax^2 - by^2\big), \qquad \dot{y} = y\big(-e - dy + ax^2 - by^2\big),$$

where $a^2+b^2 \neq 0$ *having the first integral* $H = (ax^2+by^2+cx+dy+e)/(xy)$. *Moreover, the configurations of invariant straight lines of this system which have not appeared in Propositions* 11, 13–17 *are given in Figure* 9.

Proof. Assume that system (4) possesses a single reducible conic of hyperbolic type. Then, without loss of generality, we may assume that

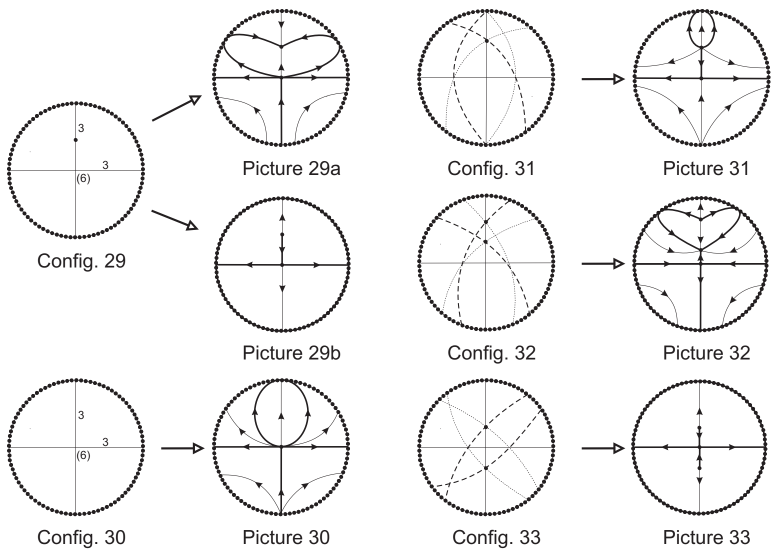


FIGURE 9. Configuration of invariant straight lines of system (33) and their corresponding phase portraits.

the system has a first integral of the form $H = w(x,y)/(xy)$, where $w(x,y) = ax^2 + by^2 + cx + dy + e$. We notice that the polynomial $w(x,y)$ must be irreducible otherwise $w(x,y) = 0$ would consist of two real or complex lines, and these kinds of systems were studied before.

Moreover, we can assume that $w(x,y) = 0$ does not intersect the conic $xy = 0$ in two points $(x_0, 0)$ and $(0, y_0)$ such that $x_0y_0 \neq 0$; otherwise, the straight line passing through these two points would be invariant and again we would be in a case studied before. We consider two cases: $w(0,0) = 0$ and $w(0,0) \neq 0$.

Case: $w(0,0) = 0$**.** This implies that $e = 0$. Moreover, without loss of generality, we can assume that $w(x,y) = 0$ is tangent to $y = 0$ (otherwise $w(x,y)$ intersects $xy = 0$ in two points) so we have $c = 0$. Since $c = e = 0$ we get that $d \neq 0$; otherwise, the conic $w(x,y) = 0$ would be reducible. By a time rescaling we can assume $d = -1$. Now we consider two possibilities: either $w(x,y)$ intersects the line $x = 0$ at two points or only at one.

Consider that $w(x,y) = ax^2 + by^2 - y = 0$ intersects the line $x = 0$ in two different points, one the origin and the other say $(0, \widetilde{y})$. By a time rescaling $y \to \alpha y$ we can assume that $\widetilde{y} = 1$. In short, we get the first integral $H = (ax^2 + y^2 - y)/(xy)$ of the system

$$\dot{x} = x(ax^2 - y^2), \qquad \dot{y} = y(ax^2 + y - y^2). \tag{34}$$

Finally, since $a \neq 0$, otherwise $ax^2 + y^2 - y$ would not be irreducible, we can assume that $a = \pm 1$. There are two singular points, the origin and $(0,1)$. Calculating $\mathcal{E}_1 = ax^3y^3$ we see that the only invariant reducible conic of system (34) is $xy = 0$ having multiplicity 3. We get **Config.** 29, see Figure 9.

Now we assume that $w(x,y) = ax^2 + by^2 - y = 0$ intersects the line $x = 0$ only at the origin. This implies that $b = 0$ and by $(x,y) \to (x/a, y/a)$ and the time rescaling we can assume that $a = 1$. We get the first integral $H = (x^2 - y)/(xy)$ of the system

$$\dot{x} = x^3, \qquad \dot{y} = y(x^2 + y).$$

Similarly, calculating $\mathcal{E}_1 = x^3y^3$ we conclude that $xy = 0$ is the only reducible conic having multiplicity 3, and we get **Config.** 30.

Case: $w(0,0) \neq 0$**.** So $e \neq 0$ (by rescaling $e = 1$); thus, we can assume that $w(x,y) = ax^2 + by^2 + cx + dy + 1$. Now we consider two cases: $b = 0$ and $b \neq 0$.

If $b = 0$, then $d \neq 0$; otherwise, $w(x,y)$ would factorize. By rescaling $y \to -y/d$ we get $d = -1$. We assume that $a > 0$ to avoid having a new invariant straight line different than $xy = 0$. Now by a rescaling of the x-axis we can assume that $a = 1$. Thus we get the first integral $H = (x^2 + cx - y + 1)/(xy)$ of the system

$$\dot{x} = x\big(1 + cx + x^2\big), \qquad \dot{y} = y\big(-1 + y + x^2\big). \tag{35}$$

Since $x^2 + cx - y + 1 = 0$ must not have points in common with $y = 0$ we have $|c| < 2$. To determine all six invariant straight lines we calculate

$$\mathcal{E}_1 = xy(1 + cx + x^2)(1 + cx - 2y + x^2 - cxy + y^2).$$

We show that, even though we have six invariant straight lines, the only real reducible conic that system (35) possesses, according to (7), is $xy = 0$. Consider the conic $1+cx+x^2 = 0$. It is clear that there do not exist α and β (as in (7)) such that $1+cx+x^2 = \alpha(x^2+cx-y+1)+\beta(xy)$, so system (35) does not possess the conic $1 + cx + x^2 = 0$. This shows that, even though this reducible conic is invariant for system (35), its two imaginary invariant straight lines lie on different level sets (complex conjugate) of the first integral H. The same can be shown for the conic $1 + cx - 2y + x^2 - cxy + y^2 = 0$. For this reason we cannot introduce more then one reducible invariant conic into the expression of the first integral H. We get **Config.** 31.

Assume now that $b \neq 0$. We can assume that $a \neq 0$ since, otherwise, doing the change of variables $(x, y) \to (y, x)$ we would arrive at the system considered previously (system (35)). We consider two cases: $ab > 0$ and $ab < 0$. If $ab > 0$ it is not restrictive to assume that $a > 0$ and $b > 0$. By a rescaling of both axes $(x, y) \to (x/\sqrt{a}, y/\sqrt{b})$ and the time rescaling we get $w(x, y) = x^2 + y^2 + cx + dy + 1$. So we have the first integral $H = (x^2 + y^2 + cx + dy + 1)/(xy)$ of the system

$$(36) \qquad \dot{x} = x\big(1 + cx + x^2 - y^2\big), \qquad \dot{y} = y\big(-1 - dy + x^2 - y^2\big).$$

Since $w(x, y) = 0$ must not have points in common with $y = 0$ we get $|c| < 2$. The oval $w(x, y)$ has to cross $x = 0$ in two distinct points so $|d| > 2$; otherwise, we get a system that belongs to the family $\mathcal{S}(\mathbf{h}, \mathbf{e})$. To determine all the invariant straight lines we calculate

$$\begin{aligned}\mathcal{E}_1 = xy\big(&1 + 2cx + (2 + c^2)x^2 + 2cx^3 + x^4 + 2dy + 3cdxy \\ &+ (2d + c^2d)x^2y + cdx^3y + (-2 + d^2)y^2 + (-2c + cd^2)xy^2 \\ &+ (2 - c^2 + d^2)x^2y^2 - 2dy^3 - cdxy^3 + y^4\big).\end{aligned}$$

For the fixed real values of c, d the polynomial $\mathcal{E}_1$ always factorizes since according to Corollary 8 it describes six invariant straight lines. Each invariant conic (real or complex) of system(36) passes through four points: two imaginary $I_{1,2} = (c/2 \pm (c^2 - 4)^{1/2}, 0)$ and two real $R_{1,2} = (0, d/2 \pm (d^2 - 4)^{1/2})$ defined by the system of equations $H_N := x^2 + y^2 + cx + dy + 1 = 0$, $H_D := xy = 0$. We denote again by $\langle AB \rangle$ a line passing through a point A and B. Then we have six invariant lines: two real xy=0, and four imaginary $\langle I_1R_1 \rangle$, $\langle I_1R_2 \rangle$, $\langle I_2R_1 \rangle$ and $\langle I_2R_2 \rangle$. Analogously to the previous case, we can show that the only reducible conic that system (36) possesses is $xy = 0$, and we get **Config.** 32.

Finally consider now $ab < 0$. Without loss of generality, we can assume that $a > 0$ and $b < 0$. By rescaling $(x, y) \to (x/\sqrt{a}, y/\sqrt{-b})$ and time rescaling we get $w(x, y) = x^2 - y^2 + cx + dy + 1$. So we have the first integral $H = (x^2 - y^2 + cx + dy + 1)/(xy)$ of the system

$$\dot{x} = x\big(1 + cx + x^2 + y^2\big), \qquad \dot{y} = y\big(-1 - dy + x^2 + y^2\big). \tag{37}$$

Again, since $w(x, y) = 0$ must not have points in common with $y = 0$, we get $|c| < 2$. So to determine the configuration we calculate

$$\begin{aligned}\mathcal{E}_1 = xy\big(&1 + 2cx + (2 + c^2)x^2 + 2cx^3 + x^4 + 2dy + 3cdxy \\ &+ (2d + c^2 d)x^2 y + cdx^3 y + (2 + d^2)y^2 + (2c + cd^2)xy^2 \\ &+ (-2 + c^2 + d^2)x^2y^2 + 2dy^3 + cdxy^3 + y^4\big).\end{aligned}$$

Here also the only reducible conic that system (37) possesses is $xy = 0$. The four other invariant straight lines are $\langle I_1 R_1\rangle$, $\langle I_1 R_2\rangle$, $\langle I_2 R_1\rangle$ and $\langle I_2 R_2\rangle$, where $I_{1,2} = (c/2 \pm (c^2 - 4)^{1/2}, 0)$ and $R_{1,2} = (0, d/2 \pm (d^2 + 4)^{1/2})$. Thus we have **Config.** 33. $\square$

Phase portraits of system (33). In this subsection we determine the phase portraits for each configuration of invariant straight lines of system (33).

First consider system (33) having the configuration of invariant straight lines **Config.** 29, i.e., the system

$$\dot{x} = x\big(ax^2 - y^2\big), \qquad \dot{y} = y\big(y + ax^2 - y^2\big),$$

where $a \neq 0$. We have two singular points: one at the origin and one at $(0, 1)$. For $a > 0$ we get **Picture** 29a, and for $a < 0$ we get **Picture** 29b.

We determine the phase portrait of system (33) having **Config.** 30. Thus we consider system

$$\dot{x} = ax^3, \qquad \dot{y} = y\big(y + ax^2\big),$$

where $a \neq 0$. We have only one degenerate singular point at the origin and no other singular points. We get the phase portrait **Picture** 30.

We now consider system (33) having **Config.** 31, i.e., system (35). The system has two singular points: one saddle at the origin and a node at $(0,1)$. The phase portrait is given in **Picture** 31.

To determine the phase portrait of system (33) having **Config.** 32 we consider system (36). There are three singular points: one saddle at the origin and two nodes $(0,-d/2\pm\sqrt{d^2-4}/2)$. We have the phase portrait **Picture** 32.

Finally, we study the phase portrait of system (33) having **Config.** 33. Thus we consider system (37). There are three singular points: one saddle at the origin and two nodes $(0,-d/2\pm\sqrt{d^2+4}/2)$. We get **Picture** 33.

3.7. The subfamily of systems (4) which possesses a single reducible conic–which is of parabolic type. Assume that system (4) possesses only one reducible conic of parabolic type. In this subsection we shall determine all configurations of invariant straight lines that these kinds of systems can have and their corresponding phase portraits.

As we know (see the Appendix) there are three normal forms of the reducible conic of parabolic type, namely, real parallel lines, complex parallel lines and double line.

Proposition 19. *Assume that a cubic system possesses a single reducible conic of parabolic type and does not have other reducible conics. Then this system can be written in the form*

$$\begin{aligned}\dot{x} &= -(p+x^2)(e+bx+2cy),\\ \dot{y} &= dp-2fx-dx^2+bpy-2exy-bx^2y-2cxy^2,\end{aligned}\tag{38}$$

having the first integral $H=(bxy+cy^2+dx+ey+f)/(x^2+p)$. *Moreover, all configurations of invariant straight lines of this system which have not appeared in Propositions* 11, 13–18 *are given in Figure* 10.

Proof. We assume that system (3) has a single reducible conic of parabolic type $H_{\mathbf{p}}=0$. By an affine change of variables we assume that $H_{\mathbf{p}}=x^2+p$ where $p\in\{0,\pm1\}$. Thus, the first integral of the systems possessing only one reducible conic of parabolic type can be written as

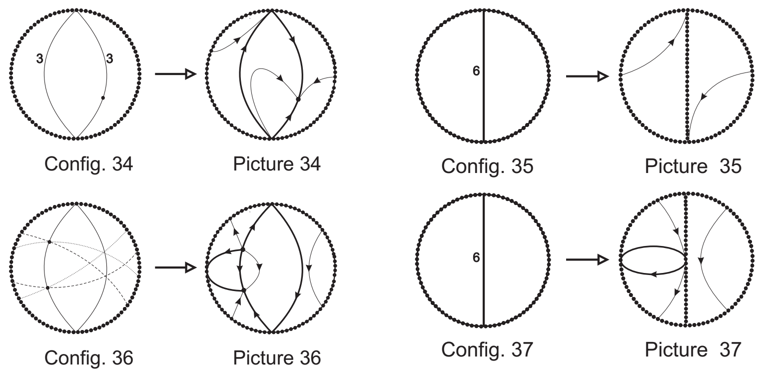


FIGURE 10. Configurations of invariant straight lines of system (33) and their corresponding phase portraits.

$H = w(x, y)/(x^2 + p)$, where $w(x, y) = ax^2 + bxy + cy^2 + dx + ey + f$. We notice that $w(x, y)$ must be irreducible otherwise $w(x, y) = 0$ would consist of two straight lines (complex or real) which would contradict our assumption of having only one reducible conic. Since by Lemma 10 in a pencil of conics there is at least one conic which contains a real line, and recall in our case the infinity is always degenerate, we can assume that $p \in \{0, -1\}$. Without loss of generality, we can also assume that $a = 0$. Now we consider two cases: $c = 0$ and $c \neq 0$.

Case $c = 0$**.** We have the first integral $H = (bxy + dx + ey + f)/(x^2 + p)$, and we have to assume that $b \neq 0$; otherwise, system (3) would be quadratic. So by a time rescaling we have $b = 1$ as well as $d = 0$ by the change of variables $y \to y - d$. Clearly $f \neq 0$; otherwise, the numerator factorizes so by the time rescaling, $f = 1$. We end up with the first integral $H = (xy + ey + 1)/(x^2 + p)$.

If $p = -1$, then we have to assume that $e = \pm 1$; otherwise, we would have another invariant straight line passing through points $(1, -1/(e + 1))$ and $(-1, -1/(e - 1))$. We can assume that $e = 1$ since if it is negative we change the variables $(x, y) \to (-x, -y)$ and we are done. Finally, we get the following system

$$\dot{x} = 1 + x - x^2 - x^3, \qquad \dot{y} = -2x - y - 2xy - x^2y,$$

having the first integral $H = (xy + y + 1)/(x^2)$. To determine the configuration of invariant straight lines according to Corollary 8 we calculate $\mathcal{E}_1 = 2(x-1)^3(x+1)^3$, and we get **Config.** 34.

Now, if $p = 0$ we have the first integral $H = (xy + ey + 1)/x^2$. If $e = 0$, then we have a degenerate system

$$\dot{x} = -x^2, \qquad \dot{y} = -x(2 + xy).$$

To determine the configuration of invariant straight lines we calculate $\mathcal{E}_1 = 2x^6$, and we get **Config.** 35. If $e \neq 0$, then by an appropriate time and axes rescaling we arrive at $H = (xy + y + 1)/x^2$; thus, system (3) is

$$\dot{x} = -x^2(1 + x), \qquad \dot{y} = -x(2 + 2y + xy),$$

and we have a configuration that is topologically equivalent to **Config.** 20.

Case $c \neq 0$**.** Here by a time rescaling we have $c = 1$. So we get the first integral $H = (bxy + y^2 + dx + ey + f)/(x^2 + p)$. Now we can assume that $b = 0$, first by the change of variables $y \to -b/2x + y$ and then by canceling the coefficient of x^2 in the numerator of the first integral which appears after the change of variables. We have the first integral $H = (y^2 + dx + ey + f)/(x^2 + p)$. Again by the change $y \to y - e/2$ we have $e = 0$. We can also assume that $d = 1$ by rescaling $x \to x/d$ and the time rescaling. We end up with the first integral $H = (y^2 + x + f)/(x^2 + p)$. Let $p = -1$; then, we get

$$\dot{x} = 2y - 2x^2y, \qquad \dot{y} = -1 - 2fx - x^2 - 2xy^2, \tag{39}$$

having the first integral $H = (y^2 + x + f)/(x^2 - 1)$, where $|f| < 1$. Calculating $\mathcal{E}_1 = 8(x-1)(x+1)\widetilde{H}$, where $\widetilde{H} = 1 + 4fx + (2 + 4f^2)x^2 + 4fx^3 + x^4 + 4fy^2 + 8xy^2 + 4fx^2y^2 + 4y^4$, we get the configuration of invariant straight lines. These lines pass through four points: two imaginary $I_{1,2} = (\pm i(-f-1)^{1/2}, 0)$ and two real $R_{1,2} = (0, \pm(1-f)^{1/2})$ defined by the system of equations $y^2 + x + f = 0$, $(x^2 - 1) = 0$. We have two real invariant straight lines $xy = 0$ and four imaginary: $\langle I_1R_1\rangle$, $\langle I_1R_2\rangle$, $\langle I_2R_1\rangle$ and $\langle I_2R_2\rangle$. The only reducible conic that system (39) possesses is $xy = 0$. We get **Config.** 36.

If $p = 0$ we have the first integral $H = (y^2 + x + f)/x^2$. We consider two possibilities: $f = 0$ or $f \neq 0$. If $f = 0$, then we get the system

$$\dot{x} = -2x^2y, \qquad \dot{y} = -x^2 - 2xy^2, \tag{40}$$

having the first integral $H = (y^2 + x)/x^2$, and we get the configuration of the invariant straight lines **Config.** 37.

If $f \neq 0$, then by the change of variables $x \to fx$, $y \to |f|^{1/2}y$, and the time rescaling we get $H = (\pm y^2 + x + 1)/x^2$. We skip the system

$$\dot{x} = -2x^2y, \qquad \dot{y} = -x(2 + x + 2y^2),$$

having the first integral $H = (y^2 + x + 1)/x^2$, since it belongs to family $\mathcal{S}(\mathbf{p}, \mathbf{e})$. We also skip the system

$$\dot{x} = 2x^2y, \qquad \dot{y} = -x(2 + x - 2y^2),$$

having the first integral $H = (-y^2 + x + 1)/x^2$, because it has been studied before (**Config.** 13). $\square$

Phase portraits of system (38). In this subsection we determine the phase portraits for each configuration of invariant straight lines of system (38).

Consider system (38) having **Config.** 34, i.e.,

$$\dot{x} = -(x^2 - 1)(1 + x), \qquad \dot{y} = -2x - y - 2xy - x^2y.$$

There is only one singular point, a node, at $(1, -1/2)$. We have the phase portrait **Picture** 34.

Now we consider (38) having **Config.** 35. Thus we study the system

$$\dot{x} = -x^3, \qquad \dot{y} = -x(2 + xy).$$

There is a line of singular points $x = 0$ and no other singular point. We get the phase portrait **Picture** 35.

We consider the phase portrait of system (38) having the configuration of invariant straight lines **Config.** 36. Thus, we study

$$\dot{x} = -2y(x^2 - 1), \qquad \dot{y} = -1 - 2fx - x^2 - 2xy^2,$$

where $|f| < 1$. The only singular points of the system are two nodes $(0, \pm\sqrt{1-f})$, and we get the phase portrait **Picture** 36.

Finally, we determine the phase portrait of system (40). We notice that there is a line of singular points $x = 0$ and no other singular point. We get the phase portrait **Picture** 37.

Acknowledgments. The second author thanks Joaquim Roé for his good comments on pencil of conics.

APPENDIX

1. Affine conics. Here we recall basic information about the conics. The starting point is to note that every conic

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \tag{A.1}$$

can be written in matrix form as $vAv^T = 0$ where

$$A = \begin{pmatrix} a & h & g \\ h & b & f \\ g & f & c \end{pmatrix}, \qquad v^T = \begin{pmatrix} x \\ y \\ 1 \end{pmatrix},$$

and A is called the *matrix* of the conic (41). By B we denote the *leading matrix* of A, i.e.,

$$B = \begin{pmatrix} a & h \\ h & b \end{pmatrix}.$$

We define two numbers $\Delta = \det A$ and $\delta = \det B$.

After an affine change of coordinates, any conic can be represented as one of the nine canonical forms shown in Proposition 20. By calculating the Δ and δ invariants for a given conic, we can tell to which class it belongs as is shown in the next proposition. We do not distinguish between real parallel lines, complex parallel lines and a double line. We also do not distinguish between real and complex ellipses.

Proposition 20. *Let $g = 0$ be a conic in $\mathbf{R}^2$. Then g is affinely equivalent to one of the nine normal forms shown in Table* 3.

Proof. For the proof, see [**7**]. □

TABLE 3.

Normal Forms	Conic	Δ invariant	δ invariant
$x^2+y^2-1=0$	real ellipse	$\Delta \neq 0$	$\delta > 0$
$x^2+y^2+1=0$	complex ellipse	$\Delta \neq 0$	$\delta > 0$
$x^2-y^2-1=0$	hyperbola	$\Delta \neq 0$	$\delta < 0$
$x^2-y=0$	parabola	$\Delta \neq 0$	$\delta = 0$
$x^2-y^2=0$	real line-pair	$\Delta = 0$	$\delta < 0$
$x^2+y^2=0$	complex line-pair	$\Delta = 0$	$\delta > 0$
$x^2-1=0$	real parallel lines	$\Delta = 0$	$\delta = 0$
$x^2+1=0$	complex parallel lines	$\Delta = 0$	$\delta = 0$
$x^2=0$	double line	$\Delta = 0$	$\delta = 0.$

REFERENCES

1. J.C. Artés, J. Llibre and N. Vulpe, *Quadratic systems with a rational first integral of degree* 2: *A complete classification in the coefficient space* R12, Rend. Circ, Mat. Palermo **56** (2002), 417–444.

2. L. Cairó and J. Llibre, *Phase portraits of cubic polynomial vector fields of Lotka-Volterra type having a rational first integral of degree* 2, J. Phys. A **40** (2007), 6329–6348.

3. ———, *Phase portraits of quadratic polynomial vector fields having a rational first integral of degree* 2, Nonlinear Anal.: Theory Methods **67** (2007), 327–348.

4. C. Christopher, J. Llibre and J.V. Pereira, *Multiplicity of invariant algebraic curves in polynomial vector fields*, Pacific J. Math. **229** (2007), 63–117.

5. F. Dumortier, J. Llibre and J.C. Artés, *Qualitative theory of planar differential systems*, Universitext, Springer-Verlag, Berlin, 2006.

6. W. Fulton, *Algebraic curves. An introduction to algebraic geometry*, W.A. Benjamin, Inc., New York, 1969 (notes written with the collaboration of Richard Weiss, Mathematics Lecture Notes Series).

7. C.G. Gibson, *Elementary geometry of algebraic curves*: *An undergraduate introduction*, Cambridge University Press, Cambridge, 1998.

8. H. Levy, *Projective and related geometries*, second ed., The Macmillan Co., New York, 1967.

9. J. Llibre and N. Vulpe, *Planar cubic polynomial differential systems with the maximum number of invariant straight lines*, Rocky Mountain J. Math. **36** (2006), 1301–1373.

Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Spain
Email address: jllibre@mat.uab.cat

Mathematics Department, University of North Carolina at Charlotte, Charlotte, NC 28223 and Faculty of Applied Mathematics, AGH University of Science and Technology, al. Mickiewicza 30, 30-059 Kraków, Poland
Email address: adam.mahdi@uncc.edu, mahdi@mat.uab.cat

Institute of Mathematics and Computer Science, Academy of Science of Moldova, 5 Academiei str, ChiCsinău, MD-2028, Moldova
Email address: nvulpe@mail.md