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WEAK SOLUTIONS FOR A SIXTH-ORDER THIN FILM EQUATION

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ABSTRACT. In this paper, the authors investigate the initial boundary value problem for a sixth-order thin film equation. By using the method of continuity, we establish the existence of weak solutions. The uniqueness of solutions is also discussed by means of a regularizing technique based on elliptic operators.

1. Introduction. In this paper, we consider the following equation

$$\frac{\partial u}{\partial t} = \frac{1}{n+1} D^6(|u|^n u), \quad \text{in } Q_T,\ n > 0, \tag{1.1}$$

where $Q_T = (0,1) \times (0,T)$, $D = \partial/\partial x$.

Equation (1.1) is a typical higher order equation, which has a sharp physical background and a rich theoretical connotation. It is relevant to capillary driven flows of thin films of power-law fluids, where u denotes the height from the surface of the oil to the surface of the solid. Galaktionov [**9**] studied equation (1.1). A countable set of self-similar solutions of equation (1.1) is described.

King [**10**] who first derived the equation

$$\begin{aligned}\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\bigg(& u^n \frac{\partial^5 u}{\partial x^5} + \alpha_1 u^{n-1} \frac{\partial u}{\partial x}\frac{\partial^4 u}{\partial x^4} + \alpha_2 u^{n-1}\frac{\partial^2 u}{\partial x^2}\frac{\partial^3 u}{\partial x^3} \\ & + \beta_1 u^{n-2}\left(\frac{\partial u}{\partial x}\right)^2 \frac{\partial^3 u}{\partial x^3} + \beta_2 u^{n-2}\frac{\partial u}{\partial x}\left(\frac{\partial^2 u}{\partial x^2}\right)^2 \\ & + \gamma u^{n-3}\left(\frac{\partial u}{\partial x}\right)^3 \frac{\partial^2 u}{\partial x^2} + \mu u^{n-4}\left(\frac{\partial u}{\partial x}\right)^5\bigg),\end{aligned}$$

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where n, α_i, β_i, γ and μ are constants. Equation (1.1) is a special case of this equation.

The pure sixth-order thin film equation was first introduced in [**11, 12**] in the case $n = 3$

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(u^3 \frac{\partial^5 u}{\partial x^5}\right).$$

It describes the spreading of a thin viscous fluid under the driving force of an elastica (or light plate). In [**6, 7, 11, 12**], a more general form of this equation (now allowing for a reaction at the underlying solid interface) is shown to arise in the industrial application of the isolation oxidation of silicon.

During the past years, only a few works have been devoted to the sixth-order thin film equation [**2, 8**]. Bernis and Friedman [**2**] have studied the initial boundary value problems to the thin film equation

$$\frac{\partial u}{\partial t} + (-1)^{m-1} \frac{\partial}{\partial x}\left(f(u) \frac{\partial^{2m+1} u}{\partial x^{2m+1}}\right) = 0,$$

where $f(u) = |u|^n f_0(u)$, $f_0(u) > 0$, $n \geq 1$ and proved existence of weak solutions preserving nonnegativity. Barrett, Langdon and Nuernberg [**1**] considered the above equation with $m = 2$. A finite element method is presented which proves to be well posed and convergent. Numerical experiments illustrate the theory.

Recently, Evans, Galaktionov and King [**4, 5**] considered the sixth-order thin film equation containing an unstable (backward parabolic) second-order term

$$\frac{\partial u}{\partial t} = \text{div } \left[|u|^n \nabla \Delta^2 u\right] - \Delta(|u|^{p-1} u)), \quad n > 0, \ p > 1.$$

By a formal matched expansion technique, they show that, for the first critical exponent $p = p_0 = n + 1 + (4/N)$ for $n \in (0, 5/4)$, where N is the space dimension, the free-boundary problem admits a countable set of continuous branches of radially symmetric self-similar blow-up solutions $u_k(x,t) = (T-t)^{-N/(nN+6)} f_k(y)$, $y = x/[(T-t)^{1/(nN+6)}]$, where $T > 0$ is the blow-up time.

On the basis of physical consideration, as usual the equation (1.1) is supplemented with the zero-contact-angle, zero-shearing force and zero-flux conditions

$$\begin{aligned} Du(0,t) &= Du(1,t) = D^3u(0,t) = D^3u(1,t) = D^5u(0,t) \\ &= D^5u(1,t) = 0, \end{aligned} \tag{1.2}$$

and the initial value condition

$$u(x,0) = u_0(x). \tag{1.3}$$

For simplicity we set

$$A(u) = \int_0^u a(s)\,ds, \quad a(s) = |s|^n, \ n > 0.$$

Equation (1.1) is degenerate; therefore, it has no classical solution in general. We introduce weak solutions in the sense of the following

Definition. A function $u \in L^\infty(Q_T)$ is said to be a weak solution of the problem (1.1)–(1.3), if the following conditions are satisfied:

1) $Du, D^3u, D^5u \in L^2(0,T;H_0^1(I))$, $A(u) \in L^\infty(0,T;H^3(I))$, $\partial/\partial t A(u) \in L^2(Q_T)$,

2) For any $\varphi \in C^\infty(\overline{Q}_T)$, $\operatorname{supp}\varphi \subset (0,1) \times [0,T]$, $\varphi(x,T) = 0$,

the following integral equality holds:

$$-\int_0^1 u_0(x)\varphi(x,0)\,dx - \iint_{Q_T} u\varphi_t\,dx\,dt + \iint_{Q_T} D^3A(u)D^3\varphi\,dx\,dt = 0.$$

In this paper, we proved the existence of generalized solutions. The main difficulties for treating the problem (1.1)–(1.3) are caused by the nonlinearity of the principal part and the lack of maximum principle. Due to the nonlinearity of the principal part, our approach lies in the combination of the energy techniques with the method of continuity. The uniqueness of solutions is also discussed by means of a regularizing technique based on elliptic operators.

In addition, throughout this paper, we set $I = (0,1)$.

2. Preliminaries. In this section, we are going to prove the following theorems

Theorem 2.1. *Let* $u_0 \in H^3(I)$. *Then there is a function* $u \in L^\infty(0,T;H^3(I)) \cap L^2(0,T;H^6(I))$, $u_t \in L^2(Q_T)$ $Du, D^3u, D^5u \in L^2(0,T;H_0^1(I))$ *satisfying the equation*

$$\frac{\partial u}{\partial t} = D^6 u,$$

and the initial value condition (1.3).

Theorem 2.1 could be deduced from earlier works like [**13**]. Hence, we omit the details.

Theorem 2.2. *Let* $u_0 \in H^3(I)$, $b \in L^\infty(0,T;H^3(I))$, $\partial b/\partial t \in L^2(Q_T)$, $0 < b_0 \le b \le M_0$, $\partial b/\partial x|_{x=0,1} = 0$. *Then there exists a unique function* $u \in L^\infty(0,T;H^3(I)) \cap L^2(0,T;H^6(I))$, $Du, D^3u, D^5u \in L^2(0,T;H_0^1(I))$ *satisfying the equation*

$$\frac{\partial (bu)}{\partial t} = D^6 u, \tag{2.1}$$

and the initial value condition (1.3).

Proof. We shall use the method of continuity. For this purpose, we need some estimates.

Suppose that u satisfies (2.1), (1.2), (1.3). Multiplying equation (2.1) by bu and integrating the resulting relation over $(0,1)$ with respect to x, we have by integrating by parts

$$\frac{1}{2}\frac{d}{dt}\int_0^1 (bu)^2 dx + \int_0^1 D^3 u D^3(bu)\, dx = 0.$$

Observing that

$$\begin{aligned}\int_0^1 D^3uD^3(bu)\,dx \\ &= \int_0^1 b(D^3u)^2dx + 3\int_0^1 DbD^2uD^3u\,dx \\ &\quad + 3\int_0^1 D^2bDuD^3u\,dx + 3\int_0^1 D^3buD^3u\,dx \\ &\geq b_0\int_0^1 (D^3u)^2dx - C_1\sup|D^2u|\left(\int_0^1 (D^3u)^2dx\right)^{1/2} \\ &\quad - C_2\sup|Du|\left(\int_0^1 (D^3u)^2dx\right)^{1/2} \\ &\quad - C_3\sup|u|\left(\int_0^1 (D^3u)^2dx\right)^{1/2} \\ &\geq \frac{b_0}{2}\int_0^1 (D^3u)^2dx - C_4\int_0^1 (bu)^2\,dx,\end{aligned}$$

where C_i $(i = 1, 2, 3, 4)$ denote the constants dependent on $\|b\|_{L^\infty(0,T;H^3(I))}$ but independent of the solution u. Therefore, we derive

$$\frac{d}{dt}\int_0^1 (bu)^2dx + \frac{b_0}{2}\int_0^1 (D^3u)^2dx \leq C\int_0^1 (bu)^2dx.$$

Hence,

$$\int_0^1 (bu)^2dx \leq C\|u_0\|^2,$$
$$\int_0^1 u^2dx \leq C\|u_0\|^2,$$

where C denotes the constant dependent on T but independent of the solution u.

Multiply equation (2.1) by $\partial u/\partial t$ and integrate the resulting relation over $(0, 1)$, with respect to x. Integrating by parts, we obtain

$$\int_0^1 \frac{\partial(bu)}{\partial t}\frac{\partial u}{\partial t}\,dx + \frac{1}{2}\frac{d}{dt}\int_0^1 (D^3u)^2dx = 0. \tag{2.2}$$

Noticing that

$$\int_0^1 \frac{\partial(bu)}{\partial t}\frac{\partial u}{\partial t}dx = \int_0^1 b\left(\frac{\partial u}{\partial t}\right)^2 dx + \int_0^1 \frac{\partial b}{\partial t}u\frac{\partial u}{\partial t}dx,$$

we have

$$\begin{aligned}\iint_{Q_t} \frac{\partial(bu)}{\partial t}\frac{\partial u}{\partial t}\,dx &\geq b_0 \iint_{Q_t}\left(\frac{\partial u}{\partial t}\right)^2 dx\,ds \\ &\quad - C_1\left(\iint_{Q_t} u^2\left(\frac{\partial u}{\partial t}\right)^2 dx\,ds\right)^{1/2} \\ &\geq b_0 \iint_{Q_t}\left(\frac{\partial u}{\partial t}\right)^2 dx\,ds \\ &\quad - C_1 \sup|u|\left(\iint_{Q_t}\left(\frac{\partial u}{\partial t}\right)^2 dx\,ds\right)^{1/2} \\ &\geq \frac{3}{4}b_0 \iint_{Q_t}\left(\frac{\partial u}{\partial t}\right)^2 dx\,ds - C_2\sup|u|^2,\end{aligned}$$

where C_1, C_2 denote the constants depending on $\|\partial b/\partial t\|$ and b_0. We finally obtain

$$\iint_{Q_t}\left(\frac{\partial u}{\partial t}\right)^2 dx\,ds + \int_0^1 (D^3u)^2 dx \leq C\sup_{Q_t}|u|^2 + \int_0^1 (D^3u_0)^2 dx. \tag{2.3}$$

Setting $\sup_{Q_t}|u|^2 = \sup_I |u(x,t_0)|^2$ and using the embedding theorem, we have

$$\begin{aligned}\sup_I |u(x,t_0)|^2 &\leq \varepsilon\int_0^1 (D^3u(t_0,x))^2 dx \\ &\quad + C(\varepsilon)\int_0^1 u^2(x,t_0)\,dx \\ &\leq \varepsilon\int_0^1 (D^3u(t_0,x))^2 dx + C(\varepsilon)\|u_0\|^2,\end{aligned}$$

where $C(\varepsilon)$ denotes the constants depending on ε, but independent of u. From this and (2.3), we get

$$\sup_{0\leq t\leq T}\int_0^1 (D^3u)^2 dx \leq C, \tag{2.4}$$

$$\iint_{Q_t}\left(\frac{\partial u}{\partial t}\right)^2 dx \leq C. \tag{2.5}$$

Using equation (2.1) itself, we obtain another estimate

$$\iint_{Q_t} (D^6 u)^2 dx\, ds \le C, \tag{2.6}$$

where C denotes the constant dependent on T, but independent of the solution u.

Now we turn to the proof of the theorem. Set

$$\begin{aligned} L_0 u &= \frac{\partial u}{\partial t} - D^6 u, & L_1 u &= \frac{\partial (bu)}{\partial t} - D^6 u, \\ L_\lambda u &= \lambda L_1 u + (1-\lambda) L_0 u & &= \frac{\partial (b_\lambda u)}{\partial t} - D^6 u, \end{aligned}$$

where $b_\lambda = \lambda b + (1-\lambda)$.

We want to look for a solution in the space

$$\begin{aligned} X_1 = \Big\{ u; u \in L^2(0,T;H^6(I)) \cap L^\infty(0,T;H^3(I)), \\ Du, D^3 u, D^5 u \in L^2(0,T;H_0^1(I)), \\ \frac{\partial u}{\partial t} \in L^2(Q_T), u(0,x) = u_0(x) \Big\} \end{aligned}$$

where the norm in X_1 is defined by

$$\begin{aligned} \|u\|_{X_1} &= \|u\|_{L^2(0,T;H^6(I))} + \left\| \frac{\partial u}{\partial t} \right\|_{L^2(Q_T)} \\ &= \left(\int_0^T \|u\|_{H^6}^2 dt \right)^{1/2} + \left(\iint_{Q_t} \left| \frac{\partial u}{\partial t} \right|^2 dx\, dt \right)^{1/2}. \end{aligned}$$

Denote by $X_M = \{u \in X_1; \|u\|_{X_1} \le M\}$, the closed ball in X_1 with radius M, where M is a constant which is sufficiently large.

Let $\sum$ be the set of λ for which we can find a solution of the equation $L_\lambda u = f$ satisfying the conditions (1.2) and (1.3). We want to prove the following assertions:

1. $0 \in \sum$.
2. $\sum$ is open in I.

3. $\sum$ is closed in I.

Once the above assertions are proved, we immediately derive the existence of solutions of the problem (2.1), (1.2), (1.3).

By virtue of Theorem 2.1, we see that $0 \in \sum$. To prove that $\sum$ is open in I, we fix a point $\lambda_0 \in \sum$. It suffices to find a positive number $\varepsilon > 0$ such that

$$(\lambda_0 - \varepsilon, \lambda_0 + \varepsilon) \cap \sum \subset \sum .$$

For this purpose we first define an operator T_λ on X_M:

$$T_\lambda : X_M \longrightarrow X_M, u \longrightarrow w$$

where w is determined by the equation

$$L_{\lambda_0} w = L_{\lambda_0} u - L_\lambda u = (\lambda_0 - \lambda)\frac{\partial(bu)}{\partial t} + (\lambda - \lambda_0)\frac{\partial u}{\partial t}$$

and (1.2), (1.3). The operator T_λ is well defined, provided that M is sufficiently large and $|\lambda_0 - \lambda|$ sufficiently small by applying the estimates (2.4), (2.5).

The fact that w is uniquely determined by u is due to the linearity of L_{λ_0}.

Let $|\lambda_0 - \lambda|$ be sufficiently small. From

$$L_{\lambda_0}(w_1 - w_2) = (\lambda_0 - \lambda)\frac{\partial(bu_1 - bu_2)}{\partial t} + (\lambda - \lambda_0)\frac{\partial(u_1 - u_2)}{\partial t}$$

we obtain the contractiveness of the operator T_λ, that is, for every $u_1, u_2 \in X_M$ and $w_1 = T_\lambda u_1$, $w_2 = T_\lambda u_2$, the following holds

$$\|w_1 - w_2\|_{X_1} \leq \frac{1}{2}\|u_1 - u_2\|_{X_1}.$$

Hence the operator T_λ has a unique fixed point u in X_M, namely

$$L_{\lambda_0} u = L_{\lambda_0} u - L_\lambda u.$$

Thus we have proved that $\sum$ is open in I. Now we prove that $\sum$ is closed in I. It suffices to prove that, for any sequence $\{\lambda_n\}$ in $\sum$, if $\lambda_n \to \lambda_0 (n \to \infty)$, then $\lambda_0 \in \sum$. Since

$$L_{\lambda_n} u_n = \frac{\partial(b_{\lambda_n} u_n)}{\partial t} - D^6 u_n = 0,$$

we have $\|u_0\|_{X_1} \le C$ by the estimates (2.4)–(2.6). By extracting a subsequence from $\{u_n\}$, we get an element u in X_M such that $L_{\lambda_0}u = 0$ and this implies that $\lambda_0 \in \sum$.

Finally, we point out that the uniqueness of solutions of the problem (2.1), (1.2) (1.3) is evident. The proof is complete. $\square$

Remark 2.1. Theorem 2.2 could be deduced from the earlier works like [**13**]. The use here is the explicit estimates on these strong solutions.

3. Regularized problems. Now we state the main theorem of this section

Theorem 3.1. *Let* $A_\varepsilon(u_0) \in H^3(I)$, $D^i u_0(0) = D^i u_0(1) = 0$ $(i = 1, 3, 5)$

$$A_\varepsilon(u) = \int_0^u (a(s) + \varepsilon)\, ds, \quad \varepsilon > 0.$$

Then there exists a unique function u satisfying

$$\frac{\partial u}{\partial t} = D^6 A_\varepsilon(u) \tag{3.1}$$

and

$$\begin{gathered} A_\varepsilon(u) \in L^\infty(0, T; H^3(I)) \cap L^2(0, T; H^6(I)), \\ Du, D^3u, D^5u \in L^2(0, T; H_0^1(I)), \\ \frac{\partial u}{\partial t} \in L^2(Q_T), \quad u(0, x) = u_0(x). \end{gathered}$$

Proof. Set $v = A_\varepsilon(u), u = B(v)$. Then equation (3.1) changes to the form

$$F(v) = \frac{\partial B(v)}{\partial t} - D^6 v = 0; \tag{3.2}$$

without loss of generality, we may suppose that the function $A(u_0)$ is defined on the whole region Q_T and $F[A(u_0)] \in L^2(Q_T)$. Consider the following space

$$\begin{aligned} Y = \Big\{ & v; v \in L^\infty(0, T; H^3(I)) \cap L^2(0, T; H^6(I)), \frac{\partial v}{\partial t} \in L^2(Q_T), \\ & Dv, D^3v, D^5v \in L^2(0, T; H_0^1(I)), v(0, x) = A_\varepsilon(u_0(x)) \Big\} \end{aligned}$$

and denote

$$Y_M = \{v \in Y; \|v\|_Y \le M\}$$

the closed ball in Y with radius M, where

$$\|v\|_Y = \left(\int_0^T \|v(t)\|_{H^6}^2 dt\right)^{1/2} + \left(\int_0^T \left\|\frac{\partial v}{\partial t}\right\|_{L^2}^2 dt\right)^{1/2}$$

and the number M is sufficiently large.

Define a functional G on Y with parameter σ:

$$G[v,\sigma] = F[v] - \sigma F[A_\varepsilon(u_0)], \quad \sigma \in [0,1]. \tag{3.3}$$

Let $\sum$ be the set of σ for which we can find an element $v \in Y_M$ such that $G[v,\sigma] = 0$. The main purpose is to prove the following assertions.

1. $1 \in \sum$.
2. $\sum$ is open in I.
3. $\sum$ is closed in I.

The first conclusion follows from the definition of $\sum$. In order to prove the other two conclusions we need some estimates.

Suppose u is a solution of the problem (3.1), (1.2), (1.3). Multiply (3.1) by $(\partial/\partial t)A_\varepsilon(u)$ and integrate the resulting relation over $(0,1)$ with respect to x. Integrating by parts, we get

$$\int_0^1 (a(u)+\varepsilon)\left(\frac{\partial u}{\partial t}\right)^2 dx + \frac{1}{2}\frac{d}{dt}\int_0^1 (D^3 A_\varepsilon(u))^2 dx = 0.$$

Hence, by integrating over $(0,t)$, with respect to t, we get

$$\sup_{0\le t\le T}\int_0^1 (D^3 A_\varepsilon(u))^2 dx \le C, \tag{3.4}$$

$$\iint_{Q_t} \left(\frac{\partial u}{\partial t}\right)^2 dx\, dt \le C \tag{3.5}$$

where here and below C is independent of ε and t.

Using equations (3.1) and (3.4)–(3.5) yields

$$\iint_{Q_t} (D^6 A_\varepsilon(u))^2 dx \le C, \tag{3.6}$$

$$\iint_{Q_t} \left(\frac{\partial}{\partial t} A_\varepsilon(u)\right)^2 dx\, dt \le C. \tag{3.7}$$

Using these estimates, we can now prove 2 and 3.

For 2, we take $\sigma_0 \in \sum$. It suffices to find a neighborhood $N(\sigma_0)$ of σ_0 in I such that $N(\sigma_0) \subset \sum$. Let v_0 satisfy

$$G[v_0, \sigma_0] = 0.$$

Taking the Frechét derivatives at v_0 on both sides of (3.3), we find that

$$G'_v(v_0, \sigma_0)h = F'(v_0)h = \frac{\partial(b(v_0)h)}{\partial t} - D^6 h,$$

where

$$b(v_0) = B'(v_0) = \frac{1}{a(u_0) + \varepsilon} \in L^\infty(0, T; H^3(I)),$$
$$\frac{\partial}{\partial t} b(v_0) \in L^2(Q_T), \quad 0 < b_0 \le b(v_0) \le M_0.$$

Applying Theorem 2.2 to the equation

$$F'(v_0)h = \frac{\partial(b(v_0)h)}{\partial t} - D^6 h = 0,$$

we know that the map $F'(v_0) : Y \to L^2(Q_T)$ is invertible. Hence there exists a neighborhood $N(\sigma_0)$ of σ_0 in I in which v can be expressed by a function of σ, that is,

$$v = v(\sigma), \quad G[v(\sigma), \sigma] = 0, \quad \text{for all } \sigma \in N(\sigma_0).$$

This shows that $\sum$ is open in I.

Finally we prove 3. Let $\sigma_n \in \sum$, $\sigma_n \to \sigma_0 (n \to \infty)$. There exists $v_n \in Y$ such that

$$G[v_n, \sigma_n] = F[v_n] - \sigma_n F[A(u_0)] = 0.$$

By the estimates (3.4)–(3.7), we have

$$\|v_n\|_Y \le C.$$

So we can extract a subsequence of $\{v_n\}$, denoted also by $\{v_n\}$, such that

$$\begin{aligned} v_n &\longrightarrow v_0 \quad \text{in } L^2(0,T;H^6(I)),\\ \frac{\partial v_n}{\partial t} &\longrightarrow \frac{\partial v_0}{\partial t} \quad \text{in } L^2(Q_T),\\ v_n(x,t) &\longrightarrow v_0(x,t) \quad \text{a.e. in } Q_T, \end{aligned}$$

for some element $v_0 \in Y$. Change the corresponding equations to the form of integration

$$\begin{aligned} &-\iint_{Q_T} B(v_n)\frac{\partial \phi}{\partial t}\,dx\,dt - \iint_{Q_T} D^6 v_n \phi\,dx\,dt \\ &\qquad = \sigma_n \iint_{Q_T} F[A_\varepsilon(u_0)]\phi\,dx\,dt, \quad \text{for all } \phi \in C_0^\infty(Q_T), \end{aligned}$$

which implies that

$$\begin{aligned} &-\iint_{Q_T} B(v_0)\frac{\partial \phi}{\partial t}\,dx\,dt - \iint_{Q_T} D^6 v_0 \phi\,dx\,dt \\ &\qquad = \sigma_0 \iint_{Q_T} F[A_\varepsilon(u_0)]\phi\,dx\,dt, \quad \text{for all } \phi \in C_0^\infty(Q_T). \end{aligned}$$

This shows that $\sigma_0 \in \sum$, and hence 3 holds. We have thus proved the theorem. □

4. Existence.

Theorem 4.1. *Let $A(u_0) \in H^3(I)$, $D^i u_0(0) = D^i u_0(1) = 0$ $(i = 1,3,5)$. Then the problem* (1.1)–(1.3) *admits a solution u in the sense of Definition* 1.1.

Proof. Define

$$a_M(s) = \begin{cases} a(s) & 0 \le s \le M,\\ \text{smooth connected} & M < |s| < M+1,\\ a(M+1) & s \ge M+1,\\ a(-M-1) & s \le -M-1, \end{cases}$$

and consider the regularized problems

$$\frac{\partial u}{\partial t} = D^6 A_{\varepsilon,M}(u), \tag{4.1}$$

(4.2)
$$Du(0,t)=Du(1,t)=D^3u(0,t) = D^3u(1,t)=D^5u(0,t)=D^5u(1,t)=0,$$

$$u(x,0) = u_{0n}(x), \tag{4.3}$$

where $A_{\varepsilon,M}(u) = \int_0^u (a_M(s) + \varepsilon)\, ds$. From Theorem 3.1 the above problem admits solutions $u_{\varepsilon,M}$. We need some estimates on $u_{\varepsilon,M}$. Set

$$F_{\varepsilon,M} = \int_0^u A_{\varepsilon,M}(s)\, ds.$$

Multiplying (4.1) by $A_{\varepsilon,M}(u_{\varepsilon,M})$ and integrating the resulting relation over I with respect to x, we get

$$\frac{d}{dt}\left(\int_0^1 F_{\varepsilon,M}(u_{\varepsilon,M})\, dx\right) + \int_0^1 (D^3 A_{\varepsilon,M})^2 dx = 0.$$

It follows that

$$\sup_{0\le t\le T} \int_0^1 F_{\varepsilon,M}(u_{\varepsilon,M})\, dx \le C, \tag{4.4}$$

$$\iint_{Q_t} (D^3 A_{\varepsilon,M}(u_{\varepsilon,M})^2 dx\, ds \le C. \tag{4.5}$$

Multiplying (4.1) by $(\partial/\partial t) A_{\varepsilon,M}(u_{\varepsilon,M})$ and integrating the resulting relation over $(0,1)$ with respect to x. Integrating by parts, we get

$$\int_0^1 (a_M(u_{\varepsilon,M}) + \varepsilon)\left(\frac{\partial u_{\varepsilon,M}}{\partial t}\right)^2 dx + \frac{1}{2}\frac{d}{dt}\int_0^1 (D^3 A_{\varepsilon,M}(u_{\varepsilon,M}))^2 dx = 0.$$

Hence, by integrating over $(0,t)$, with respect to t, we get

$$\sup_{0\le t\le T} \int_0^1 (D^3 A_{\varepsilon,M}(u_{\varepsilon,M}))^2 dx \le C,$$

where C is independent of ε, M and t. It follows that

$$\sup_{0\le t\le T} |A_{\varepsilon,M}(u_{\varepsilon,M})| \le C.$$

This implies that

$$\sup_{0\le t\le T} |u_{\varepsilon,M}(x,t)| \le M_0, \tag{4.6}$$

with constant M_0 independent of ε, M.

If we take $M = M_0$ and denote u_{ε,M_0} by u_ε, then u_ε satisfies the equation

$$\frac{\partial u_\varepsilon}{\partial t} = D^6 A_\varepsilon(u_\varepsilon), \tag{4.7}$$

and the conditions (1.2), (1.3). Moveover, we have

$$\sup_{0\le t\le T} \int_0^1 \left(D^3 A_\varepsilon(u_\varepsilon)\right)^2 \le C, \quad \iint_{Q_t} \left(\frac{\partial}{\partial t} A_\varepsilon(u_\varepsilon)\right)^2 dx\, ds \le C, \tag{4.8}$$

where C is independent of ε. From (4.8), we can select a subsequence from $\{u_\varepsilon\}$, denoted also by $\{u_\varepsilon\}$, and a function $u \in C(\overline{Q}_T)$, such that

$$\begin{gathered} D^3 A_\varepsilon(u_\varepsilon) \to D^3 A(u), \quad \frac{\partial}{\partial t} A_\varepsilon(u_\varepsilon) \to \frac{\partial}{\partial t} A(u), \quad \text{in } L^2(Q_T), \\ A_\varepsilon(u_\varepsilon) \to A(u), \quad u_\varepsilon \to u, \quad \text{a.e. in } Q_T. \end{gathered}$$

Letting $\varepsilon \to 0$ in

$$\int_0^1 u_0(x)\varphi(x,0)dx + \iint_{Q_T} u_\varepsilon \frac{\partial \varphi}{\partial t}\, dx\, dt + \iint_{Q_T} A_\varepsilon(u_\varepsilon) D^6 \varphi\, dx\, dt = 0,$$

we get

$$\int_0^1 u_0(x)\varphi(x,0)\, dx + \iint_{Q_T} u \frac{\partial \varphi}{\partial t}\, dx\, dt + \iint_{Q_T} A(u) D^6 \varphi\, dx\, dt = 0.$$

The proof is complete. □

5. Uniqueness. In this section, we deal with the uniqueness of weak solutions to the initial boundary value problem of equation (1.1). The method we use for treating this problem was inspired by some ideas of Brézis and Crandall [**3**].

Theorem 5.1. *The initial boundary value problem* (1.1)–(1.3) *has at most one weak solution.*

Proof. Let $u_1, u_2 \in L^\infty(Q_T)$ be two weak solutions of problem (1.1)–(1.3). Denote by $w = u_1 - u_2$, $v = A(u_1) - A(u_2)$. From the definition of weak solutions, we have

$$\iint_{Q_T} w\partial\varphi t\,dx\,dt + \iint_{Q_T} vD^6\varphi\,dx\,dt = 0. \tag{5.1}$$

For small $\mu > 0$, we define the operator T_μ as the following

$$T_\mu : L^2(I) \longrightarrow H^6(I), \quad g \longrightarrow y,$$

where $y = T_\mu g$ is determined uniquely by the problem

$$-D^6 y + \mu y = g, \tag{5.2}$$

$$Dy = D^3 y = D^5 y = 0, \quad x \in \partial I. \tag{5.3}$$

It is easy to see that T_μ is a self-adjoint operator, i.e., for arbitrary $f, g \in L^2(\Omega)$,

$$\int_0^1 (T_\mu f)g\,dx = \int_0^1 f(T_\mu g)\,dx. \tag{5.4}$$

In fact, we get from the definition of T_μ

$$\begin{aligned}\int_0^1 (T_\mu f)g\,dx &= \int_0^1 (T_\mu f)(-D^6(T_\mu g) + \mu T_\mu g)\,dx\\ &= \int_0^1 D^3(T_\mu f)D^3(T_\mu g)\,dx + \mu\int_0^1 (T_\mu f)(T_\mu g)\,dx\\ &= \int_0^1 (-D^6(T_\mu f) + \mu T_\mu f)(T_\mu g)\,dx\\ &= \int_0^1 f(T_\mu g)\,dx.\end{aligned}$$

We also have the following properties

$$\mu\iint_{Q_T} (T_\mu g)^2 dx\,dt, \iint_{Q_T} (D^3 T_\mu g)^2 dx\,dt \le \iint_{Q_T} g^2 dx\,dt. \tag{5.5}$$

Let $k(x) \in C_0^\infty(I)$, $\psi(t) \in C_0^\infty(I)$. Replacing $\varphi(x,t)$ by $\psi(t)T_\mu k$ in (5.1), we have

$$\begin{aligned} 0 &= \iint_{Q_T} \psi'(t)(T_\mu k)w\,dx\,dt \\ &\quad + \iint_{Q_T} \psi(t)vD^6(T_\mu k)\,dx\,dt \\ &= \iint_{Q_T} \psi'(t)(T_\mu w)k(x)\,dx\,dt \\ &\quad - \iint_{Q_T} \psi(t)v(k(x) - \mu T_\mu k(x))\,dx\,dt \\ &= \iint_{Q_T} \psi'(t)(T_\mu w)k(x)\,dx\,dt \\ &\quad - \iint_{Q_T} \psi(t)k(x)(v - \mu T_\mu v)\,dx\,dt, \end{aligned}$$

and hence for any $\varphi \in C_0^\infty(Q_T)$,

$$\iint_{Q_T} T_\mu w \frac{\partial \varphi}{\partial t}\,dx\,dt = \iint_{Q_T} (v - \mu T_\mu v)\varphi\,dx\,dt,$$

which implies that $\partial/\partial t(T_\mu w) \in L^2(Q_T)$ and

$$\frac{\partial}{\partial t}[T_\mu w(\cdot,t)] = \mu T_\mu v(\cdot,t) - v(\cdot,t). \tag{5.6}$$

Set

$$g_\mu(t) = \int_0^1 (T_\mu w(x,t))w(x,t)\,dx;$$

clearly

$$\begin{aligned} g_\mu(t) &= \int_0^1 T_\mu w\big(-D^6(T_\mu w) + \mu T_\mu w\big)\,dx \\ &= \int_0^1 \big((D^3(T_\mu w))^2 + \mu(T_\mu w)^2\big)\,dx. \end{aligned} \tag{5.7}$$

For any $\psi(t) \in C_0^\infty(0,T)$, replacing $\varphi(x,t)$ by $\psi(t)T_\mu w$ in (5.1), we have

$$0 = \iint_{Q_T} \psi'(t)(T_\mu w)w\,dx\,dt$$

$$
\begin{aligned}
&+\iint_{Q_T} \psi(t) w \frac{\partial}{\partial t}(T_\mu w)\,dx\,dt \\
&+\iint_{Q_T} \psi(t) v D^6(T_\mu w)\,dx\,dt \\
=&\iint_{Q_T} \psi'(t)(T_\mu w) w\,k dx\,dt \\
&+\subset \int_{Q_T} \psi(t) w \frac{\partial}{\partial t}(T_\mu w)\,dx\,dt \\
&-\iint_{Q_T} \psi(t) v(w(x) - \mu T_\mu w(x))\,dx\,dt \\
=&\int_0^T \psi'(t)\,dt \int_0^1 w(T_\mu w)\,dx \\
&-2\int_0^T \psi(t)\,dt \int_0^1 w(v - \mu T_\mu v)\,dx,
\end{aligned}
$$

which implies that

$$
g'_\mu(t) = 2\int_0^1 (\mu T_\mu v(x,t) - v(x,t)) w(x,t)\,dx. \tag{5.8}
$$

Thus $g'_\mu(t) \in L^1(0,T)$, and hence $g_\mu(t)$ is absolutely continuous on $[0,T]$.

Denote $\alpha_\varepsilon(s)$ the kernel of modifier in one dimension and

$$
\psi_\varepsilon(t) = \int_t^\infty \alpha_\varepsilon(s-\varepsilon)\,ds.
$$

Replacing $\varphi(x,t)$ by $\psi_\varepsilon(t) T_\mu w$ in (5.1), we obtain

$$
0 = -\iint_{Q_T} \alpha_\varepsilon(t-\varepsilon) w(T_\mu w)\,dx\,dt + 2\iint_{Q_T} \psi_\varepsilon(t) v D^6(T_\mu w)\,dx\,dt.
$$

It follows that

$$
\begin{aligned}
g_\mu(0) &= \lim_{\varepsilon\to 0}\int_0^{2\varepsilon} \alpha_\varepsilon(t-\varepsilon) g_\mu(t)\,dt \\
&= \lim_{\varepsilon\to 0}\iint_{Q_T} \alpha_\varepsilon(t-\varepsilon) w(T_\mu w)\,dx\,dt \\
&= 2\lim_{\varepsilon\to 0}\iint_{Q_T} \psi_\varepsilon(t) v D^6(T_\mu w)\,dx\,dt \\
&= 0.
\end{aligned} \tag{5.9}
$$

Combining (5.5), (5.8), (5.9), and noticing that monotonicity of $A(s)$ implies that w and v have the same sign, we have

$$\begin{aligned}
0 \le g_\mu(t) &= g_\mu(t) - g_\mu(0) = \int_0^t g_\mu'(s)\,ds \\
&= 2\int_0^t ds \int_0^1 (\mu T_\mu v - v) w\,dx \\
&\le 2\int_0^t ds \int_0^1 (\mu T_\mu v) w\,dx \\
&\le \sqrt{\mu}\iint_{Q_T} \mu (T_\mu v)^2 dxdt + \sqrt{\mu}\iint_{Q_T} w^2\,dx\,dt \\
&\le \sqrt{\mu}\iint_{Q_T} (v^2 + w^2)\,dx\,dt \to 0, (\mu \to 0).
\end{aligned}$$

Finally, using (5.7), for any $\varphi \in C_0^\infty(Q_T)$, we have

$$\begin{aligned}
\left|\iint_{Q_T} w\varphi\,dx\,dt\right|^2 &= \left|\iint_{Q_T} (-D^6(T_\mu w) + \mu T_\mu w)\varphi\,dx\,dt\right|^2 \\
&= \left|\iint_{Q_T} (D^3(T_\mu w) D^3\varphi + \mu T_\mu w\varphi)\,dx\,dt\right|^2 \\
&\le C_\varphi \iint_{Q_T} (D^3(T_\mu w)^2 dx\,dt \\
&\quad + C_\varphi \mu \iint_{Q_T} (\mu T_\mu w\varphi)^2 dx\,dt \to 0.
\end{aligned}$$

The proof is complete. □

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