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NONNIL-NOETHERIAN RINGS AND THE SFT PROPERTY

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ABSTRACT. A commutative ring R is said to be nonnil-Noetherian if every ideal which is not contained in the nilradical of R is finitely generated. We show that many of the properties of Noetherian rings are true for nonnil-Noetherian rings. Then we study the rings of formal power series over a nonnil-Noetherian ring. We prove that if R is an SFT nonnil-Noetherian ring then $\dim R[[X_1, \dots, X_n]] = \dim R + n$ and that the ring $R[[X_1, \dots, X_n]]$ is also SFT. We provide an answer to an open question concerning the relationship between the nilradical of R and the nilradical of $R[[X]]$ [**6**, page 284]. We prove that, for a commutative ring R, $\mathrm{Nil}\,(R)[[X_1, \dots, X_n]] = \mathrm{Nil}\,(R[[X_1, \dots, X_n]])$ if and only if $\mathrm{Nil}\,(R)$ is an SFT ideal of R, and in that case $\mathrm{Nil}\,(R[[X_1, \dots, X_n]])$ is also an SFT ideal of $R[[X_1, \dots, X_n]]$.

1. Introduction. In this paper, all rings are commutative with identity; $\{X_1, \dots, X_n\}$ is a finite, nonempty set of analytically independent indeterminates over any ring. The dim(ension) of a ring means its Krull dimension.

Let R be a commutative ring with identity. An ideal I of R is said to be a nonnil ideal if it is not contained in $\mathrm{Nil}\,(R)$, where $\mathrm{Nil}\,(R)$ denotes the nilradical of R. The ring R is called a nonnil-Noetherian ring if every nonnil ideal of R is finitely generated [**4, 5**].

In [**4, 5**], the authors have investigated nonnil-Noetherian rings with a prime, divided nilradical. They prove that many of the properties of Noetherian rings are true for nonnil-Noetherian rings.

In the first part of this paper, we generalize some of these properties to a nonnil-Noetherian ring without any assumption on the nilradical.

In the second part of this paper, we study the ring of formal power series over a nonnil-Noetherian ring.

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Recall [**1**], that an ideal I of a commutative ring R is an ideal of strong finite type (or an SFT ideal) provided there is a finitely generated ideal $F \subseteq I$ and an integer k such that $x^k \in F$, for any $x \in I$. If each ideal of R is an SFT ideal, then we say that R satisfies the SFT property. In [**1**], Arnold studied the Krull dimension of a power series ring $R[[X]]$ over a commutative ring R and showed that the dimension of $R[[X]]$ is infinite unless R is an SFT ring.

In this section, we begin first by providing an answer to an open question concerning the relationship between the nilradical of R and the nilradical of $R[[X]]$ [**6**, page 284]. We prove that a necessary and sufficient condition on the nilradical of a commutative ring R so that a power series is nilpotent if and only if each of its coefficients is nilpotent is that $\mathrm{Nil}\,(R)$ is an SFT ideal which is equivalent to the existence of an integer $k \in \mathbf{N}$, such that for any $x \in \mathrm{Nil}\,(R)$, $x^k = 0$. The result is also true for a finite number of variables. We prove that for a commutative ring R, $\mathrm{Nil}\,(R)[[X_1, \dots , X_n]] = \mathrm{Nil}\,(R[[X_1, \dots , X_n]])$ if and only if $\mathrm{Nil}\,(R)$ is an SFT ideal of R, and in that case $\mathrm{Nil}\,(R[[X_1, \dots , X_n]])$ is also an SFT ideal of $R[[X_1, \dots , X_n]]$.

Next, we study the stability of the SFT property over the ring of power series over a nonnil-Noetherian ring. In fact, in [**7**], Coykendall showed that for an SFT ring R, the ring of formal power series $R[[X]]$ is not necessarily SFT. Here we prove that if R is an SFT nonnil-Noetherian ring, then for any $n \in \mathbf{N}^*$, $R[[X_1, \dots , X_n]]$ is an SFT ring.

Finally, we study the dimension of $R[[X]]$ over a nonnil-Noetherian ring R. We prove, in contrast to the case of Noetherian rings, that there exist nonnil-Noetherian rings which do not satisfy the SFT property. This fact forces us to consider only SFT rings to study the Krull dimension of the ring of formal power series over a nonnil-Noetherian ring. We prove the following result:

Theorem. *Let R be a nonnil-Noetherian ring. Then*:

1. *If R is an* SFT *ring, then* $\dim R[[X_1, \dots , X_n]] = \dim R + n$.
2. *If R is not* SFT*, then* $\dim R[[X_1, \dots , X_n]] = +\infty$.

In the third part of this paper, we study the possible transfer of the nonnil-Noetherian property from a commutative ring R to the ring of the power series $R[[X]]$. In fact, it is known that a commutative ring R is Noetherian if and only if $R[X]$ is Noetherian if and only if $R[[X]]$ is Noetherian. Here we study the stability of the nonnil-Noetherian property over the ring of formal power series. We prove (in contrast to the case of Noetherian rings) that $R[[X]]$ is nonnil-Noetherian if and only if $R[X]$ is nonnil-Noetherian if and only if R is Noetherian.

1. Nonnil-Noetherian rings.

Proposition 1.1. *Let R be a commutative ring. Then the following are equivalent*:

1. *R is a nonnil-Noetherian ring.*
2. *R satisfies the ascending chain condition on nonnil ideals.*
3. *Every non-empty set of nonnil ideals in R has a maximal element.*

Proof. 1) $\Rightarrow$ 2). Let $I_1 \subset I_2 \subset \cdots \subset I_n \subset \cdots$ be an increasing sequence of nonnil ideals in R. Let $I = \cup_n I_n$; then I is a nonnil ideal. So there exist $x_1, \ldots, x_s \in I$ such that $I = (x_1, \ldots, x_s)$. For any $1 \leq i \leq s$, let $\alpha_i \in \mathbf{N}$, such that $x_i \in I_{\alpha_i}$ and $\alpha = \sup\{\alpha_i, 1 \leq i \leq s\}$; then $x_i \in I_\alpha$, for any $1 \leq i \leq s$. So $I = I_\alpha$. Hence, for any $n \geq \alpha$, $I_n = I_\alpha$.

2) $\Rightarrow$ 3). Let E be a non-empty set of nonnil ideals in R. If E does not have a maximal element (for inclusion), then let $I_1 \in E$. As I_1 is not maximal, then there exists $I_2 \in E$, such that $I_1 \subset I_2$. So we construct inductively a non-terminating strictly increasing sequence $I_1 \subset I_2 \subset \cdots \subset I_n \subset \cdots$ of nonnil ideals in E.

3) $\Rightarrow$ 1). Let I be a nonnil ideal of R and E the set of nonnil finitely generated ideals of R contained in I. Then $E \neq \varnothing$. So E has a maximal element for inclusion. Let J be such an element. If $J \subset I$, then there exists an $x \in I \backslash J$. We consider $J_1 = J + xR$; then J_1 is finitely generated, $J \subset J_1 \subseteq I$ and J_1 is a nonnil ideal. This is impossible as J is maximal. So, $J = I$ and then I is finitely generated. $\square$

The next three propositions are known [**4**, Corollary 2.3], [**5**, Corollary 2.9] and [**4**, Theorem 2.7] under the assumption that Nil (R) is a divided prime ideal, and can be easily generalized to the case of any nonnil-Noetherian ring.

Proposition 1.2. *Let R be a commutative ring. Then R is a nonnil-Noetherian ring if and only if for any P in* $\operatorname{spec}(R)$ *such that $P \not\subseteq \operatorname{Nil}(R)$. Then, P is finitely generated.*

Proposition 1.3. *The homomorphic image of a nonnil-Noetherian ring is also nonnil-Noetherian.*

Example 1.4. If R is a nonnil-Noetherian ring and I is an ideal of R, then R/I is nonnil-Noetherian.

Proposition 1.5 [**4**, Theorem 2.7]. *Any localization R_S of a nonnil-Noetherian ring is nonnil-Noetherian.*

Proposition 1.6. *Let R be a nonnil-Noetherian ring. Then any flat overring of R is also a nonnil-Noetherian ring.*

Proof. Let R_1 be a flat overring of R. Then there exists a multiplicatively closed set S of ideals of R such that

$$R_1 = R_S = \{x \in T; xA \subseteq R \text{ for some } A \in S\},$$

where T is the total quotient ring of R. Moreover, S may be chosen such that $AR_1 = R_1$, for each $A \in S$ and if $Q \in \operatorname{spec}(R_1)$ and $P = Q \cap R$; then $Q = P_S$ [**3**, Theorem 1.3].

Let $Q \in \operatorname{spec}(R_1)$ such that $Q \not\subseteq Nil(R_1)$; then $P = Q \cap R \not\subseteq \operatorname{Nil}(R)$. In fact, if $P \subseteq Nil(R)$ then for any $x \in Q$ there exists an $A \in S$ such that $xA \subseteq P$. So $xAR_1 \subseteq PR_1$. Hence $x \in PR_1$. So $x \in \operatorname{Nil}(R_1)$, for any $x \in Q$, which is impossible. As R is a nonnil-Noetherian ring and $P \not\subseteq \operatorname{Nil}(R)$, then P is finitely generated. Let $x_1, \dots, x_n$ be in P such that $P = x_1R + \cdots + x_nR$. We claim that $Q = x_1R_1 + \cdots + x_nR_1$. In fact, for any $1 \leq i \leq n$, $x_i \in P \subseteq Q$. So $x_1R_1 + \cdots + x_nR_1 \subseteq Q$. Conversely, let $x \in Q$, so there exists an $A \in S$ such that $xA \subseteq P$. Hence, $xAR_1 \subseteq PR_1 = x_1R_1 + \cdots + x_nR_1$. So Q is finitely generated. $\square$

Proposition 1.7. *Let A and B be two commutative rings. Then $A \times B$ is a nonnil-Noetherian ring if and only if $A \times B$ is Noetherian if and only if A and B are Noetherian.*

Proof. Let I be an ideal of $A \times B$. Let $I_1 = \{x \in A; \exists y \in B, (x, y) \in I\}$ and $I_2 = \{y \in B; \exists x \in B, (x, y) \in I\}$. Then $I \subseteq I_1 \times I_2$. Conversely, let $(x, y) \in I_1 \times I_2$, so there exists $(x_1, x_2) \in A \times B$ such that $(x, x_2) \in I$ and $(x_1, y) \in I$. As I is an ideal of $A \times B$, then $(x, x_2)(1, 0) = (x, 0) \in I$ and $(x_1, y)(0, 1) = (0, y) \in I$. So, $(x, y) \in I$ and $I = I_1 \times I_2$. Moreover, clearly I is finitely generated if and only if I_1 and I_2 are.

So $A \times B$ is Noetherian if and only if A and B are Noetherian. In that case $A \times B$ is nonnil-Noetherian. We prove that if $A \times B$ is nonnil-Noetherian then A and B are Noetherian. Indeed, $A \times B$ is nonnil-Noetherian if and only if for any ideal $I \times J$ of $A \times B$, if $I \times J \nsubseteq Nil(A \times B)$, then $I \times J$ is finitely generated. Let I be an ideal in A. So $I \times B$ is a nonnil ideal in $A \times B$, so $I \times B$ is finitely generated, which implies that I is finitely generated. So, A is Noetherian. In the same way, we prove that B is Noetherian. □

Recall that a ring R is said to have a Noetherian prime spectrum if R satisfies the ascending chain condition for radical ideals and that is equivalent to: Any prime ideal of R is the radical of a finitely generated ideal of R [**11**, Corollary 2.4].

Proposition 1.8. *If R is a nonnil-Noetherian ring, then* $\operatorname{spec}(R)$ *is Noetherian. Hence, for any ideal I of R, there are only a finite number of prime ideals minimal over I.*

Proof. We prove that any prime ideal of R is the radical of a finitely generated ideal of R. Indeed, let $p \in \operatorname{Spec}(R)$. If $p \nsubseteq \operatorname{Nil}(R)$, then p is finitely generated, so $p = \sqrt{p}$ is the radical of a finitely generated ideal. If $p \subseteq \operatorname{Nil}(R)$, then $p = \operatorname{Nil}(R) = \sqrt{(0)}$. □

Remark 1.9. The converse of Proposition 1.8 is false. Indeed, if V is a discrete valuation domain of rank ≥ 2, then V is an SFT ring so V has a Noetherian prime spectrum [**2**], but V is not Noetherian. As V is integral, then V is not nonnil-Noetherian.

Proposition 1.10 [**4**, Theorem 2.2]. *Let R be a nonnil-Noetherian ring. Then $R/\mathrm{Nil}\,(R)$ is Noetherian.*

We show later that the converse of this proposition is false in general.

The next three propositions are known in a Noetherian ring [**12**, Théorème 4, Théorème 5 and Théorème 6, page 82]. We have the following similar results for a nonnil ideal of a nonnil-Noetherian ring.

Proposition 1.11. *Any nonnil ideal of a nonnil-Noetherian ring contains a product of prime ideals.*

Recall that a proper ideal I of a commutative ring R is said to be irreducible if I does not have a decomposition of the form $I = I_1 \cap I_2$ with I_1 and I_2 two ideals of R such that $I \subset I_1$ and $I \subset I_2$.

Proposition 1.12. *Let R be a nonnil-Noetherian ring. Then any nonnil ideal is a finite intersection of irreducible ideals.*

Proposition 1.13. *Let R be a nonnil-Noetherian ring. Then any nonnil irreducible ideal is primary.*

Recall that an ideal I of a commutative ring is said to be decomposable if it is a finite intersection of primary ideals.

Corollary 1.14. *In a nonnil-Noetherian ring, each nonnil ideal is decomposable.*

Proposition 1.15. *Let R be a nonnil-Noetherian ring. Then each nonnil ideal contains a power of its radical.*

Proof. Let I be a nonnil ideal of the nonnil-Noetherian ring R. Then $I \subseteq \sqrt{I}$. So $\sqrt{I}$ is a finitely generated ideal. Let $\sqrt{I} = x_1R + \cdots + x_sR$. There exists an $n \in \mathbf{N}^*$, such that for any $1 \leq i \leq s$, $x_i^n \in I$. Let $m = sn$. Then $(\sqrt{I})^m$ is generated by the products of the form

$x_1^{r_1}\cdots x_s^{r_s}$, with $r_1+\cdots+r_s=m$. So, there exists $1\le i\le s$, such that $r_i\ge n$. Then $x_1^{r_1}\cdots x_s^{r_s}\in I$ and $(\sqrt{I})^m\subseteq I$. $\square$

2. Power series ring over an SFT nonnil-Noetherian ring.

First, we recall some definitions. An ideal I of a ring R is an SFT (strong finite type) ideal if there exists an ideal F of finite type with $F\subseteq I$ and an integer n such that for any $a\in I$, $a^n\in F$. The ring R is an SFT ring if every ideal of R is SFT, which is equivalent to each prime ideal of R is SFT.

The aim of the first part of this section is to give a necessary and sufficient condition on the nilradical of a commutative ring R so that a power series in a finite number of variables over R is nilpotent if and only if each of its coefficients is nilpotent. We begin by a characterization of when the nilradical of a commuative ring R is an SFT ideal.

Proposition 2.1. *Let R be a commutative ring and I an ideal of R such that $I\subseteq \mathrm{Nil}\,(R)$. Then I is an SFT ideal of R if and only if there exists $k\in\mathbf{N}$, such that for any $x\in I$, $x^k=0$.*

Proof. Let R be a commutative ring and I an ideal of R such that I is an SFT ideal of R, so there exists an F, a finitely generated ideal contained in I, and $s\in\mathbf{N}$, such that for any $x\in I$, $x^s\in F$. As F is finitely generated and contained in $\mathrm{Nil}\,(R)$, there exists an $r\in\mathbf{N}$, such that for any $x\in F$, $x^r=0$. Then take $k=rs$. Conversely, if there exists a $k\in\mathbf{N}$, such that for any $x\in I$, $x^k=0$, then, it suffices to take $F=(0)$. So I is an SFT ideal. $\square$

Lemma 2.2. *Let I be an ideal of a commutative ring R. Then $I[[X]]$ is an* SFT *ideal of $R[[X]]$ if and only if I is an* SFT *ideal of R.*

Proof. Suppose that $I[[X]]$ is an SFT ideal of $R[[X]]$, there exists an $n\in\mathbf{N}$, $f_1,\dots,f_s\in I[[X]]$ such that for any $f\in I[[X]]$, $f^n\in(f_1,\dots,f_s)$. Let $a_i=f_i(0)\in I$, for any $1\le i\le n$. Then for any $x\in I$, $x^n\in(a_1,\dots,a_s)$. Conversely, if I is an SFT ideal of R, then there exists an $m\in\mathbf{N}^*$ and a finitely generated ideal F of R such that $F\subseteq I$ and for any $x\in I$, $x^m\in F$. For any $\overline{x}\in I/F$ in the ring R/F, we have $\overline{x}^m = 0$. Then by [**1**, Lemma 4], any element of $(I/F)[[X]]$ is nilpotent and there exists $n \in \mathbf{N}^*$ such that for any $f \in I[[X]]$, $\overline{f}^n = 0$ in $(R/F)[[X]]$. So $f^n \in F[[X]] = FR[[X]]$ as F is a finitely generated ideal of R. Moreover, $FR[[X]]$ is a finitely generated ideal of $R[[X]]$ such that $FR[[X]] = F[[X]] \subseteq I[[X]]$. So $I[[X]]$ is an SFT ideal of $R[[X]]$. □

Proposition 2.3. *Let I be an ideal of a commutative ring R such that $I \subseteq \mathrm{Nil}\,(R)$. The following are equivalent*:

1. $I[[X]] \subseteq \mathrm{Nil}\,(R[[X]])$.
2. *I is an* SFT *ideal of R.*
3. *There exists a $k \in \mathbf{N}^*$ such that for any $x \in I$, $x^k = 0$.*
4. *$I[[X]]$ is an* SFT *ideal of $R[[X]]$.*

Proof. Using Lemma 2.2, we have $2 \Leftrightarrow 4$.

Using Proposition 2.1, we have $2 \Leftrightarrow 3$.

For $2 \Rightarrow 1$. As I is an SFT ideal of R, then there exists an $n \in \mathbf{N}$, $a_1, \dots, a_s$ elements of I such that for any $x \in I$, $x^n \in (a_1, \dots, a_s)$. As $I \subseteq \mathrm{Nil}\,(R)$ then there exists an $m \in \mathbf{N}$ such that $a_1^m = \cdots = a_s^m = 0$. So, for any $y \in (a_1, \dots, a_s), y^{ms} = 0$ and for any $x \in I$, $x^{nms} = 0$. By [**1**, Lemma 4], $I[[X]] \subseteq \mathrm{Nil}\,(R[[X]])$.

For $1 \Rightarrow 3$. Suppose that there is no $k \in \mathbf{N}^*$ such that $x^k = 0$ for any $x \in I$. So we can find a sequence $(a_i)_{i \in \mathbf{N}^*}$ in I such that for any $p \in \mathbf{N}^*$ there exists an $i \in \mathbf{N}^*$ such that $a_i^p \neq 0$. Let $f = \sum_{i=1}^{\infty} a_i X^{i!} \in I[[X]] \subseteq \mathrm{Nil}\,(R[[X]])$. There exists a $k \geq 2$ such that $f^k = 0$. We prove that for any $n \geq k$, the coefficient of $X^{k(n!)}$ in f^k is a_n^k. In fact, let $f = g + X^{(n+1)!}h$ with $g = \sum_{i=1}^{n} a_i X^{i!}$ and $h = \sum_{i=n+1}^{\infty} a_i X^{i!-(n+1)!}$. The coefficient of $X^{k(n!)}$ in f^k is equal to that of $X^{k(n!)}$ in g^k which is equal to a_n^k. As $f^k = 0$, we deduce that for any $n \geq k$, $a_n^k = 0$, which is impossible. □

In the next corollary, we provide an answer to an open question concerning the relationship between the nilradical of R and the nilradical of $R[[X]]$ (relation which "is not good in general [**6**, page 284]"). In fact, we give a necessary and sufficient condition on the nilradical of a commutative ring R so that a power series is nilpotent if and only if each of its coefficients is nilpotent.

Corollary 2.4. *Let R be a commutative ring. Then the following are equivalent*:

1. $\mathrm{Nil}\,(R[[X]]) = \mathrm{Nil}\,(R)[[X]]$.
2. $\mathrm{Nil}\,(R)$ *is an* SFT *ideal.*
3. *There exists a $k \in \mathbf{N}$, such that for any $x \in \mathrm{Nil}\,(R)$, $x^k = 0$.*
4. $\mathrm{Nil}\,(R[[X]])$ *is an* SFT *ideal of* $R[[X]]$.

Proof. Take $I = \mathrm{Nil}\,(R)$ in Proposition 2.3 and use the fact that in a commutative ring R, we have $\mathrm{Nil}\,(R[[X]]) \subseteq \mathrm{Nil}\,(R)[[X]]$. □

Corollary 2.5. *Let R be a commutative ring and $n \in \mathbf{N}^*$. Then the following are equivalent*:

1. $\mathrm{Nil}\,(R[[X_1, \dots, X_n]]) = \mathrm{Nil}\,(R)[[X_1, \dots, X_n]]$.
2. $\mathrm{Nil}\,(R)$ *is an* SFT *ideal of R.*
3. $\mathrm{Nil}\,(R)[[X_1, \dots, X_n]]$ *is an* SFT *ideal of* $R[[X_1, \dots, X_n]]$.

Proof. By induction on n. For $n = 1$, use Corollary 2.4.

Suppose that $1 \Leftrightarrow 2 \Leftrightarrow 3$ for an integer $n \geq 1$. We prove the result for $n + 1$.

For $1 \Rightarrow 2$, suppose that $\mathrm{Nil}\,(R[[X_1, \dots, X_n, X_{n+1}]]) = \mathrm{Nil}\,(R)[[X_1, \dots, X_n, X_{n+1}]]$. So $\mathrm{Nil}\,(R[[X_1, \dots, X_n, X_{n+1}]]) \cap R[[X_1]] = \mathrm{Nil}\,(R)[[X_1, \dots, X_n, X_{n+1}]] \cap R[[X_1]]$. Then, $\mathrm{Nil}\,(R[[X_1]]) = \mathrm{Nil}\,(R)[[X_1]]$. So, by Corollary 2.4, $\mathrm{Nil}\,(R)$ is an SFT ideal of R.

For $2 \Rightarrow 3$, as $\mathrm{Nil}\,(R)$ is an SFT ideal of R, then $\mathrm{Nil}\,(R[[X_1, \dots, X_n]]) = \mathrm{Nil}\,(R)[[X_1, \dots, X_n]]$ is an SFT ideal of $R[[X_1, \dots, X_n]]$. Let $B = R[[X_1, \dots, X_n]]$. Then $\mathrm{Nil}\,(B) = \mathrm{Nil}\,(R)[[X_1, \dots, X_n]]$, moreover $Nil(B)$ is an SFT ideal. So, by Corollary 2.4, $\mathrm{Nil}\,(B[[X_{n+1}]]) = \mathrm{Nil}\,(B)[[X_{n+1}]]$ and $\mathrm{Nil}\,(B[[X_{n+1}]])$ is an SFT ideal. So $\mathrm{Nil}\,(R)[[X_1, \dots, X_n, X_{n+1}]]$ is SFT.

For $3 \Rightarrow 1$, suppose that $\mathrm{Nil}\,(R)[[X_1, \dots, X_n, X_{n+1}]]$ is SFT. By Lemma 2.2, $\mathrm{Nil}\,(R)[[X_1, \dots, X_n]]$ is SFT. So $\mathrm{Nil}\,(R[[X_1, \dots, X_n]]) = \mathrm{Nil}\,(R)[[X_1, \dots, X_n]]$ is SFT. Let $B = R[[X_1, \dots, X_n]]$; then $\mathrm{Nil}\,(B)$ is SFT. By Corollary 2.4 $\mathrm{Nil}\,(B)[[X_{n+1}]] = \mathrm{Nil}\,(B[[X_{n+1}]])$. But $\mathrm{Nil}\,(B)[[X_{n+1}]] = \mathrm{Nil}\,(R[[X_1, \dots, X_n]])[[X_{n+1}]]$, which is equal to $\mathrm{Nil}\,(R)[[X_1, \dots, X_n]][[X_{n+1}]] = \mathrm{Nil}\,(R)[[X_1, \dots, X_n, X_{n+1}]]$, and $\mathrm{Nil}\,(B[[X_{n+1}]]) = \mathrm{Nil}\,(R[[X_1, \dots, X_n]][[X_{n+1}]]) = \mathrm{Nil}\,(R[[X_1, \dots, X_n, X_{n+1}]])$. Then $\mathrm{Nil}\,(R[[X_1, \dots, X_n, X_{n+1}]]) = \mathrm{Nil}\,(R)[[X_1, \dots, X_n, X_{n+1}]]$. $\square$

Example 2.6 [**1**, Example 3]. It is possible to have a nonnil-Noetherian ring which is not SFT. In fact, let $A = \mathbf{Q}[Y_i; i \in \mathbf{N}^*]$ and $I = \langle Y_i^n; i \in \mathbf{N}^* \rangle$ the ideal of A generated by Y_i^n, for $i \in \mathbf{N}^*$, with n an integer ≥ 2 and $R = A/I$. Then, R is not an SFT ring. We show that R is a nonnil-Noetherian ring. Let J be an ideal of R such that $J \nsubseteq \mathrm{Nil}\,(R)$; then there exists an $f \in I$, with a non zero constant term (otherwise as $\overline{Y_i}$ is nilpotent for any $i \geq 1$ and f is a polynomial then f would be nilpotent). So, $f = a + g$, with $g \in \langle \overline{Y_i}, i \geq 1 \rangle$ and $a \in \mathbf{Q}^*$, so g is nilpotent and f is invertible. Hence $J = R$, which is finitely generated.

Theorem 2.7. *Let R be a nonnil-Noetherian ring and $n \in \mathbf{N}^*$. The following are equivalent*:

1. *For any $n \in \mathbf{N}$, $R[[X_1, \dots, X_n]]$ is an SFT ring.*
2. *There exists an $n \in \mathbf{N}$ such that $R[[X_1, \dots, X_n]]$ is an* SFT *ring.*
3. *R is* SFT.
4. $\mathrm{Nil}\,(R)$ *is* SFT.

Proof. $1 \Rightarrow 2$ is clear.

For $2 \Rightarrow 3$, if $R[[X_1, \dots, X_n]]$ is an SFT ring, then so is the ring $R \simeq R[[X_1, \dots, X_n]]/(X_1, \dots, X_n)$.

$3 \Rightarrow 4$ is clear.

For $4 \Rightarrow 1$, as $\mathrm{Nil}\,(R)$ is SFT, then by Corollary 2.5, $\mathrm{Nil}\,(R[[X_1, \dots, X_n]]) = \mathrm{Nil}\,(R)[[X_1, \dots, X_n]]$ is SFT. So $R[[X_1, \dots, X_n]]/\mathrm{Nil}\,(R[[X_1, \dots, X_n]]) = R[[X_1, \dots, X_n]]/\mathrm{Nil}\,(R)[[X_1, \dots, X_n]] \simeq (R/\mathrm{Nil}\,(R))[[X_1, \dots, X_n]]$.

As $R/\mathrm{Nil}(R)$ is Noetherian, then so is the ring $(R/\mathrm{Nil}(R))[[X_1,\dots,X_n]]$. Let $P \in \mathrm{spec}(R[[X_1,\dots,X_n]])$; then $P/\mathrm{Nil}(R[[X_1,\dots,X_n]])$ is finitely generated. Let $f_1,\dots,f_s \in P$ such that $P/\mathrm{Nil}(R[[X_1,\dots,X_n]]) = (\overline{f_1},\dots,\overline{f_s})$. So $P = \mathrm{Nil}(R[[X_1,\dots,X_n]]) + (f_1,\dots,f_s)$ is SFT. $\square$

Remark 2.8. Let R be an SFT nonnil-Noetherian ring and $P \in \mathrm{spec}(R[[X]])$.

First case: If $X \in P$, then $P = p + XR[[X]]$. If $p \not\subseteq \mathrm{Nil}(R)$, then p is finitely generated, and so is P. Otherwise, $p \subseteq \mathrm{Nil}(R)$, so $p = \mathrm{Nil}(R)$. As R is an SFT ring, then there exists a $k \in \mathbf{N}$, such that for any $x \in \mathrm{Nil}(R)$, $x^k = 0$. Hence, for any $f \in P$, $f^k \in XR[[X]]$. So, P is an SFT ideal.

Second case: If $X \notin P$.

- If $P \not\subseteq \mathrm{Nil}(R[[X]])$, then there exists an $f \in P$, $f = \sum_{i\in\mathbf{N}} a_iX^i \notin \mathrm{Nil}(R[[X]]) = \mathrm{Nil}(R)[[X]]$ (as $\mathrm{Nil}(R)$ is an SFT ideal of R). So, there exists an $i \in \mathbf{N}$ such that $a_i \notin \mathrm{Nil}(R)$. Let $i_0 = \min\{k \in \mathbf{N}$ such that $a_k \notin \mathrm{Nil}(R)\}$, so for any $j < i_0$, $a_j \in \mathrm{Nil}(R) \subseteq \mathrm{Nil}(R[[X]]) \subseteq P$. Hence, $\sum_{i=i_0}^{\infty} a_iX^i \in P$. But $X \notin P \Rightarrow \sum_{i=i_0}^{\infty} a_iX^{i-i_0} \in P$. So a_{i_0} is a constant term of an element in P. Let $I = \{g(0); g \in P\}$; then I is an ideal of R which is not contained in $\mathrm{Nil}(R)$. So I is finitely generated. Let $I = (\alpha_1,\dots,\alpha_n)$ and for any $1 \le i \le n$, let $f_i \in P$ be such that $f_i(0) = \alpha_i$. We prove that $P = (f_1,\dots,f_n)$. Clearly, we have $(f_1,\dots,f_n) \subseteq P$. Conversely, let $g \in P$; then $g = a + Xg_1$, with $g_1 \in R[[X]]$ and $a \in I$, so $a = \sum_{i=1}^{n} b_i\alpha_i$, with $b_1,\dots,b_n \in R$. Hence, $f - \sum_{i=1}^{n} b_if_i = Xg_1 \in P$; then $g_1 \in P$. We prove the same thing for $g_1 \dots$. Then we prove that $f \in (f_1,\dots,f_n)$. So P is finitely generated.
- If $P \subseteq \mathrm{Nil}(R[[X]])$, then $P = \mathrm{Nil}(R[[X]])$. But $\mathrm{Nil}(R)$ is an SFT ideal, so there exists a $k \in \mathbf{N}^*$, such that for any $x \in \mathrm{Nil}(R)$, $x^k = 0$. Hence there exists an $m \in \mathbf{N}^*$, such that for any $f \in \mathrm{Nil}(R[[X]]) = \mathrm{Nil}(R)[[X]]$, $f^m = 0$ [**1**, Lemma 4]. So P is an SFT ideal of $R[[X]]$ (it suffices to take $F = (0)$).

Theorem 2.9. *Let R be a nonnil-Noetherian ring and $n \in \mathbf{N}^*$. Then*:

1. *If R is an* SFT *ring, then* $\dim R[[X_1, \dots , X_n]] = \dim R + n$.
2. *If R is not* SFT*, then* $\dim R[[X_1, \dots , X_n]] = +\infty$.

Proof. Using [**1**, Theorem 1], 2) is clear.

We prove 1). As $\mathrm{Nil}\,(R) = \cap_{P\in \mathrm{spec}\,(R)} P$, then $\dim R = \dim R/\mathrm{Nil}\,(R)$, for any commutative ring R.

Since R is an SFT, nonnil-Noetherian ring, then

$$\mathrm{Nil}\,(R[[X_1, \dots , X_n]]) = \mathrm{Nil}\,(R)[[X_1, \dots , X_n]].$$

So,

$$\begin{aligned}\dim R[[X_1, \dots , X_n]] &= \dim\,(R[[X_1, \dots , X_n]]/\mathrm{Nil}\,(R[[X_1, \dots , X_n]]))\\ &= \dim\,(R[[X_1, \dots , X_n]]/\mathrm{Nil}\,(R)[[X_1, \dots , X_n]])\\ &= \dim\,(R/\mathrm{Nil}\,(R))[[X_1, \dots , X_n]].\end{aligned}$$

Hence,

$$\dim\,(R/\mathrm{Nil}\,(R))[[X_1, \dots , X_n]] = \dim\,(R/\mathrm{Nil}\,(R)) + n = \dim\,(R) + n.$$

So, $\dim R[[X_1, \dots , X_n]] = \dim R + n$. □

Remark 2.10. If R is a nonnil-Noetherian ring, then $\dim R[X] = \dim R + 1$. (In fact, we have always $\mathrm{Nil}\,(R[X]) = \mathrm{Nil}\,(R)[X]$).

Example 2.11. Let $A = \mathbf{Q}[Y_i; i \in \mathbf{N}^*]$, $I = \langle Y_iY_j; (i,j) \in (\mathbf{N}^*)^2\rangle$ and $R = A/I$. Then, for any $i \in \mathbf{N}^*$, $\overline{Y_i}$ is nilpotent. As $\mathrm{spec}\,(R) = \{\mathrm{Nil}\,(R)\}$, then R is nonnil-Noetherian. Moreover, for any $f \in \mathrm{Nil}\,(R)$, $f^2 = 0$. So $\mathrm{Nil}\,(R)$ is an SFT ideal. By Theorem 2.7, R is an SFT ring. But, as $\mathrm{Nil}\,(R)$ is not finitely generated as it contains $\overline{Y_i}$ for any $i \geq 1$, then $R[[X]]$ is not Noetherian (in fact, it is not a nonnil-Noetherian ring: take the ideal $\mathrm{Nil}\,(R) + XR[[X]] \not\subseteq \mathrm{Nil}\,(R[[X]])$ but is not finitely generated). So R is an SFT, nonnil-Noetherian ring which is not Noetherian, and $\dim R = 0$. Hence, for any $n \in \mathbf{N}$, $\dim R[[X_1, \dots , X_n]] = n$.

3. Nonnil-Noetherian stability via power series. Now we study the stability of the nonnil-Noetherian property when we pass to the formal power series ring.

Example 3.1. We construct an example of a nonnil-Noetherian ring R (which is not an SFT ring) and such that $R[[X]]$ is not nonnil-Noetherian. Let $R = A/I$ with $A = \mathbf{Q}[Y_i; i \in \mathbf{N}^*]$ and $I = \langle Y_i^i; i \in \mathbf{N}^* \rangle$; then R is a nonnil-Noetherian ring which does not satisfy the SFT property. In fact, for any $i \in \mathbf{N}^*$, $\overline{Y_i}$ is nilpotent and $\overline{Y_i}^i = 0$ but $\overline{Y_i}^j \neq 0$, for any $j < i$. As $\operatorname{spec}(R) = \{\operatorname{Nil}(R)\}$, so R is nonnil-Noetherian. Moreover, R is not an SFT ring, otherwise there exists a $k \in \mathbf{N}$, such that for any $x \in \operatorname{Nil}(R)$, $x^k = 0$, [Proposition 2.1]. We show that $R[[X]]$ is not a nonnil-Noetherian ring. Let $f = X \notin Nil(R[[X]])$ and for any $n \in \mathbf{N}^*$, $f_n = \overline{Y_n}$. Let $J = (f, f_n$, for any $n \in \mathbf{N}^*)$. Then $J \nsubseteq \operatorname{Nil}(R[[X]])$. We show that J is not finitely generated. Otherwise, there exists an $n_0 \in \mathbf{N}^*$ such that $J = (f, f_1, \dots, f_{n_0})$. So, for any $n > n_0$, $\overline{Y_n} = \sum_{i=1}^{n_0} f_i g_i + fg$ with $g \in R[[X]]$. Take in this equality $X = 0$. Then we obtain $Y_n - \sum_{i=1}^{n_0} Y_i g_i(0) \in I$, which implies, when we take $Y_i = 0$, for any $i \neq n$, that for any $n > n_0$, $Y_n \in Y_n^n \mathbf{Q}[Y_n]$, which is impossible $(n \geq 2)$. (In the same way, we prove that $R[X]$ is not a nonnil-Noetherian ring).

We remark that for a ring B which does not satisfy the SFT property, $B[[X]]$ is never nonnil-Noetherian. In fact, as B is not SFT, then by [**1**, Theorem 1], there exists an infinite chain of prime ideals of $B[[X]] : P_1 \subset P_2 \subset \cdots$. As $\operatorname{Nil}(B[[X]]) \subseteq P_1$, so there exists an infinite chain of nonnil ideals in $B[[X]]$. So, using Proposition 1.1, we deduce that $B[[X]]$ is not a nonnil-Noetherian ring.

Example 3.2. Here we give an example of an SFT, nonnil-Noetherian ring such that $R[[X]]$ is not a nonnil-Noetherian ring. Let R be the ring of Example 2.11. Then R is an SFT nonnil-Noetherian ring such that $R[[X]]$ is not nonnil-Noetherian (take the ideal $Nil(R) + XR[[X]] \nsubseteq Nil(R[[X]])$ but is not finitely generated).

This example shows also that the converse of Proposition 1.10 is false, in general. In fact, if we put $R_1 = R[[X]]$, then R_1 is not a nonnil-Noetherian ring. On the other hand, $R_1/\operatorname{Nil} R_1 = R[[X]]/\operatorname{Nil} R[[X]]$. As R is an SFT ring, then $\operatorname{Nil}(R)[[X]] = \operatorname{Nil}(R[[X]])$. So, $R[[X]]/\operatorname{Nil}(R)[[X]] = R/\operatorname{Nil}(R)[[X]]$ which is a Noetherian ring.

Theorem 3.3. *Let R be a commutative ring. Then the following are equivalent*:

1. *R is nonnil-Noetherian and* $\mathrm{Nil}\,(R)$ *is finitely generated.*
2. *R is Noetherian.*
3. *$R[X]$ is Noetherian.*
4. *$R[X]$ is nonnil-Noetherian.*
5. *$R[[X]]$ is Noetherian.*
6. *$R[[X]]$ is nonnil-Noetherian.*

Proof. Clearly, we have $2 \Leftrightarrow 3 \Leftrightarrow 5$, $5) \Rightarrow 6)$ and $3) \Rightarrow 4)$. To prove $1) \Rightarrow 2)$, let $P \in \mathrm{spec}\,(R)$. If $P \nsubseteq \mathrm{Nil}\,(R)$, then P is finitely generated. If $P \subseteq \mathrm{Nil}\,(R)$, then $P = \mathrm{Nil}\,(R)$, so P is finitely generated.

To prove $6) \Rightarrow 1)$: as $R \simeq R[[X]]/XR[[X]]$, then R is a nonnil-Noetherian ring (as a homomorphic image of a nonnil-Noetherian ring). Moreover, the ideal $\mathrm{Nil}\,(R) + XR[[X]] \nsubseteq \mathrm{Nil}\,(R[[X]])$ (as it contains X which is not nilpotent) so, $\mathrm{Nil}\,(R) + XR[[X]]$ and so $\mathrm{Nil}\,(R)$ are finitely generated. In the same way we prove $4) \Rightarrow 1)$. $\square$

Remark 3.4. The preceding theorem and example show that for a nonnil-Noetherian ring, we don't have in general the equivalence: R is an SFT ring if and only if $\mathrm{Nil}\,(R)$ is finitely generated. Moreover a nonnil-Noetherian, SFT ring needs not to be Noetherian in general.

Proposition 3.5. *Let R be a nonnil-Noetherian ring and $P \in \mathrm{spec}\,(R[[X]])$. Let $p = \{f(0); f \in P\}$. If $p \nsubseteq \mathrm{Nil}\,(R)$, then P is finitely generated.*

Proof. As p is an ideal of R which is not contained in $\mathrm{Nil}\,(R)$, so p is finitely generated. Let $a_1, \dots, a_n \in p$ be such that $p = \langle a_1, \dots, a_n \rangle$ and for any $1 \leq i \leq n$, let $f_i \in P$ such that $f_i(0) = a_i$.

First case: $X \in P$. We show that $P = \langle f_1, \dots, f_n, X\rangle$. In fact, $\langle f_1, \dots, f_n, X\rangle \subseteq P$ is clear. Conversely, let $f \in P$. So $f(0) \in p$; then there exist $\alpha_1, \dots, \alpha_n$ in R such that $f(0) = \sum\limits_{i=1}^{n} \alpha_i a_i$. So $f - \sum\limits_{i=1}^{n} \alpha_i f_i \in XR[[X]]$. Hence $f \in \langle f_1, \dots, f_n, X\rangle$.

Second case: $X \notin P$. We show that $P = \langle f_1, \dots, f_n \rangle$. In fact, $\langle f_1, \dots, f_n \rangle \subseteq P$ is clear. Conversely, let $f \in P$. So $f(0) \in p$; then there exist $\alpha_1, \dots, \alpha_n$ in R such that $f(0) = \sum_{i=1}^{n} \alpha_i a_i$. So $f - \sum_{i=1}^{n} \alpha_i f_i = X g_1(X)$, where $g_1 \in R[[X]]$. As $X \notin P$ and $P \in \operatorname{spec}(R[[X]])$, so $g_1 \in P$. In the same way, we write $g_1 = \sum_{i=1}^{n} \beta_i f_i + X g_2(X)$. Continuation of the process leads as to $h_1, \dots, h_n \in R[[X]]$ such that $f = \sum_{i=1}^{n} h_i f_i$. So $f \in \langle f_1, \dots, f_n \rangle$. □

Corollary 3.6. *Let R be an* SFT *nonnil-Noetherian ring and $P \in \operatorname{spec}(R[[X]])$ such that $\operatorname{ht}_{R[[X]]} P > 1$. Then P is finitely generated.*

Proof. If P is not finitely generated, then $p = \{f(0), f \in P\}$ is contained in $\operatorname{Nil}(R)$ [Proposition 3.5]. As $P \cap R \subseteq p \subseteq \operatorname{Nil}(R) = \cap_{Q \in \operatorname{spec}(R)} Q$ and $P \cap R \in \operatorname{spec}(R)$, then $P \cap R = p = \operatorname{Nil}(R) \in \operatorname{spec}(R)$. So, $P \subseteq p + XR[[X]] = \operatorname{Nil}(R) + XR[[X]]$. We show then that $\operatorname{ht}(\operatorname{Nil}(R) + XR[[X]]) \leq 1$.

Let $Q \in \operatorname{spec}(R[[X]])$ such that $Q \subset \operatorname{Nil}(R) + XR[[X]]$; we prove that $Q = \operatorname{Nil}(R)[[X]]$. As $\operatorname{Nil}(R) \in \operatorname{spec}(R)$, so $\operatorname{Nil}(R) \subseteq Q \cap R \subseteq Q$. But R is an SFT ring, then $\operatorname{Nil}(R)[[X]] \subseteq Q$ [**1**, Theorem 1]. Conversely, let $f = \sum_{i=0}^{\infty} a_i X^i \in Q \subset \operatorname{Nil}(R) + XR[[X]]$. So $a_0 \in \operatorname{Nil}(R) \subseteq \operatorname{Nil}(R[[X]]) \subseteq Q$. So $\sum_{i=1}^{\infty} a_i X^i \in Q$. But $X \notin Q$, otherwise $Q = \operatorname{Nil}(R) + XR[[X]]$. So $\sum_{i=1}^{\infty} a_i X^{i-1} \in Q$, and then $a_1 \in \operatorname{Nil}(R)$. We show by induction that for any $i \in \mathbf{N}$, $a_i \in \operatorname{Nil}(R)$. Hence $Q = \operatorname{Nil}(R)[[X]]$. So $\operatorname{ht}(P) \leq 1$, which is impossible and then P is finitely generated. □

Proposition 3.7. *Let $A \subset B$ be an extension of commutative rings. Then $A + XB[[X]]$ is a nonnil-Noetherian ring if and only if $A + XB[[X]]$ is Noetherian if and only if A is Noetherian and B is a finitely generated A-module.*

Proof. The second equivalence is proved in [**9**]. Clearly, if $A + XB[[X]]$ is Noetherian, then it is a nonnil-Noetherian ring. Conversely, we suppose that $A + XB[[X]]$ is nonnil-Noetherian.We show that A is Noetherian. Let I be an ideal of A. Then the ideal $I + XB[[X]]$ of $A + XB[[X]]$ is not contained in $\mathrm{Nil}\,(A + XB[[X]])$ (as it contains X). So, $I+XB[[X]]$ is finitely generated. Then, I is finitely generated ($A \simeq (A + XB[[X]])/XB[[X]]$). We now show that B is a finitely generated A-module. The ideal $XB[[X]]$ of $A + XB[[X]]$ is not contained in $\mathrm{Nil}\,(A+XB[[X]])$, so it is finitely generated. So there exist $f_1, \dots, f_n \in B[[X]]$ such that $XB[[X]] = Xf_1(A + XB[[X]]) + \cdots + Xf_n(A + XB[[X]])$. So $B[[X]] = f_1(A + XB[[X]]) + \cdots + f_n(A + XB[[X]])$. We obtain then $B = f_1(0)A+\cdots+f_n(0)A$. So B is a finitely generated A-module. $\square$

Now, we construct a ring R of power series for which we don't have the equivalence R Noetherian if and only if R is nonnil-Noetherian.

Lemma 3.8. *Let A be a commutative ring with unity and I a finitely generated ideal of A such that $I^2 = I$. Then there exists an $a \in I$ with $a^2 = a$ and $I = aA$.*

Proof. Since I is finitely generated then $I = (x_1, \dots, x_n)$. We define the sequence of ideals of A: $I_1 = (x_1, \dots, x_n)$, $I_2 = (x_2, \dots, x_n), \dots, I_n = (x_n)$ and $I_{n+1} = (0)$. Then for $1 \le i \le n + 1$, there exists a $z_i \in I$ such that $(1 - z_i)I \subseteq I_i$. Indeed, we prove this by induction on i. Take $z_1 = 0$, and suppose that there exists a $z_i \in I$ such that $(1 - z_i)I \subseteq I_i$, for $1 \le i \le n$; then $(1 - z_i)I = (1 - z_i)I^2 \subseteq I_iI$. In particular, $(1 - z_i)x_i = \sum_{j=i}^{n} x_jz_{ij}$, with $z_{ij} \in I$; then $(1 - z_i - z_{ii})x_i = \sum_{j=i+1}^{n} x_jz_{ij} \in I_{i+1}$. We can take $1 - z_{i+1} = (1 - z_i)(1 - z_i - z_{ii})$. Then $(1-z_{i+1})I = (1-z_i)(1-z_i-z_{ii})I \subseteq (1-z_i-z_{ii})I_i = (1-z_i-z_{ii})(I_{i+1}+ x_iA) = (1 - z_i - z_{ii})I_{i+1} + (1 - z_i - z_{ii})x_iA \subseteq I_{i+1}$. We deduce that $(1 - z_{n+1})I = (0)$. So for each $x \in I$, $x = xz_{n+1} \Rightarrow I = z_{n+1}A$, with $z_{n+1}^2 = z_{n+1}$. $\square$

Proposition 3.9. *Let A be a commutative ring and I an ideal of A. Then $A+XI[[X]]$ is Noetherian if and only if A is Noetherian and $I^2 = I$.*

Proof. Since $A + XI[[X]]$ is Noetherian then so is the ring $A + XI[[X]]/XI[[X]] \simeq A$. On the other hand $I[[X]]$ is an ideal of

$A + XI[[X]]$; then it is finitely generated. Let $f_1, \dots, f_s \in I[[X]]$ such that, $I[[X]] = f_1 \cdot (A + XI[[X]]) + \cdots + f_n \cdot (A + XI[[X]])$. Reducing modulo I^2, we then get $I/I^2[[X]] = \overline{f_1}A/I^2 + \cdots + \overline{f_s}A/I^2$. Let for $1 \leq i \leq s$, $f_i = \sum_{k=0}^{\infty} a_k^i X^k$. Let $a \in I$; for each $n \in \mathbf{N}$, there exists $\overline{\alpha_1^n}, \dots, \overline{\alpha_s^n} \in A/I^2$ such that $\overline{a}X^n = \overline{\alpha_1^n}\overline{f_1} + \cdots + \overline{\alpha_s^n}\overline{f_s}$. So, for each $n \in \mathbf{N}$, $\overline{a} = \overline{\alpha_1^n}\overline{a_n^1} + \cdots + \overline{\alpha_s^n a_n^s}$ and, for any $k \neq n$, $0 = \overline{\alpha_1^n}\overline{a_k^1} + \cdots + \overline{\alpha_s^n}\overline{a_k^s}$. On the other hand, $(A/I^2)^s$ is a finitely generated module on the Noetherian ring A/I^2; then it is a Noetherian module. For $n \in \mathbf{N}$, let $M_n = \{(\overline{\alpha_1}, \dots, \overline{\alpha_s}) \in (A/I^2)^s$; for any $k \geq n$, $\overline{\alpha_1}\overline{a_k^1} + \cdots + \overline{\alpha_s}\overline{a_k^s} = 0\}$, then $(M_n)_n$ is an increasing sequence of submodules of $(A/I^2)^s$. Then there exists an $n \in \mathbf{N}$ such that $M_n = M_{n+1}$. But $(\overline{\alpha_1^n}, \dots, \overline{\alpha_s^n}) \in M_{n+1}$, so $(\overline{\alpha_1^n}, \dots, \overline{\alpha_s^n}) \in M_n$, which implies that $\overline{\alpha_1^n}\overline{a_n^1} + \cdots + \overline{\alpha_s^n a_n^s} = 0$, so $\overline{a} = 0$ and $I = I^2$.

Conversely, we suppose that A is Noetherian and $I^2 = I$. In particular $I = aA$, with $a \in A$ such that $a^2 = a$ (Lemma 3.8). Consider the unique A-homomorphism $\Psi : A[[T]] \to A[[X]]$ such that $\Psi(T) = aX$. Then $\operatorname{Im}(\Psi) = A + XI[[X]]$. As A is Noetherian, then $A[[T]]$ is also Noetherian and then $A + XI[[X]]$ is Noetherian. $\square$

Proposition 3.10. *Let A be a commutative ring and I an ideal of A such that $I[[X]] \nsubseteq \operatorname{Nil}(A[[X]])$. Then $A+XI[[X]]$ is nonnil-Noetherian if and only if $A + XI[[X]]$ is Noetherian.*

Proof. Clearly, if $A + XI[[X]]$ is Noetherian then it is nonnil-Noetherian. Conversely, we suppose that $A + XI[[X]]$ is nonnil-Noetherian. We show that A is Noetherian and $I^2 = I$. Let $p \in \operatorname{spec}(A)$. Then $p + XI[[X]] \in \operatorname{spec}(A + XI[[X]])$ and $p + XI[[X]] \nsubseteq \operatorname{Nil}(A + XI[[X]])$, so it is a finitely generated ideal which implies that $p \simeq p + XI[[X]]/XI[[X]]$ is also finitely generated. Then A is Noetherian. On the other hand, $I[[X]] \nsubseteq \operatorname{Nil}(A + XI[[X]])$, so $I[[X]]$ is a finitely generated ideal of $A + XI[[X]]$. Then, using the proof of Proposition 3.10, we show that $I^2 = I$. $\square$

Remark 3.11. If A is a commutative ring and I a proper ideal of A such that $I[[X]] \subseteq \operatorname{Nil}(A[[X]])$, then $A + XI[[X]]$ is never Noetherian. In fact, if $A + XI[[X]]$ is Noetherian, then $I = aA$, with $a^2 = a$. Moreover $I \subseteq \operatorname{Nil}(R)$, so $a \in \operatorname{Nil}(A)$, which implies that $a = 0$, and then $I = (0)$.

In that case we have $\mathrm{Nil}\,(A + XI[[X]]) = \mathrm{Nil}\,(A) + XI[[X]]$ and $\mathrm{spec}\,(A + XI[[X]]) = \{p + XI[[X]]; p \in \mathrm{spec}\,(A)\}$.

However, $A+XI[[X]]$ may be a nonnil-Noetherian ring. To illustrate that case, take in Example 3.2, $J = \mathrm{Nil}\,(R)$, then $R + XJ[[X]]$ is a nonnil-Noetherian ring ($\mathrm{spec}\,(R + XJ[[X]]) = \{\mathrm{Nil}\,(R + XJ[[X]])\} = \{\mathrm{Nil}\,(R) + XJ[[X]]\}$).

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