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ISOMORPHISM BETWEEN MORSE AND LAGRANGIAN FLOER COHOMOLOGY RINGS

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ABSTRACT. We investigate cup products in Morse and Floer theory for Lagrangian intersections in cotangent bundle. We prove that these products are natural with respect to canonical isomorphisms for different Morse functions (or Hamiltonians, in Floer case) and with respect to Piunikhin-Salamon-Schwarz isomorphisms between Morse and Floer cohomologies.

1. Introduction. Let M be a compact manifold and T^*M its cotangent bundle with the standard symplectic form. For a given Morse function $f : M \to \mathbf{R}$, denote by $\mathrm{Crit}_k(f)$ the set of all critical points p of f of Morse index $m_f(p)$ equal to k. Denote by $CM_k(f)$ the $\mathbf{Z}_2-$vector spaces generated by $\mathrm{Crit}_k(f)$. Let $HM_k(f)$ be the homology groups of $CM_k(f)$ with respect to the boundary operator

$$\partial_M : CM_k(f) \longrightarrow CM_{k-1}(f), \quad \partial_M(p) := \sum_{m_f(q)=m_f(p)-1} n(p,q)q$$

where $n(p,q)$ is the number (mod 2) of solutions of

$$\begin{cases} \frac{d\gamma}{ds} + \nabla f(\gamma) = 0 \\ \gamma(-\infty) = p, \ \gamma(+\infty) = q. \end{cases} \tag{1}$$

Morse cohomology $HM^*(f)$ groups are defined a standard way, by taking a cochain complex $CM^*(f)$ to be $\mathrm{Hom}\,(CM_*(f), \mathbf{Z}_2)$ and the coboundary operator δ:

$$\langle \delta a, \alpha \rangle := \langle a, \partial \alpha \rangle. \tag{2}$$

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By using an isomorphism

$$\Gamma : CM_*(f) \xrightarrow{\cong} CM_{n-*}(-f)$$

which is compatible with the boundary operators in homology we obtain the following isomorphism

$$\sigma_f : HM_{n-*}(-f) \xrightarrow{\cong} HM^*(f). \tag{3}$$

There are several equivalent definitions of a cup product in Morse case. We will consider the one that uses the mappings from a tree T to M. Let T be a tree with three edges. We identify two edges e_1, e_2 with $(-\infty, 0]$ (incoming edges) and one e_3 with $[0, +\infty)$ (outgoing edge). Let f_i, for $i = 1, 2, 3$ be three Morse functions on M, and let $g_i := -f_i$. Consider the mapping $I : T \to M$ such that, for $\gamma_i := I|_{e_i}$, $i = 1, 2, 3$, it holds:

$$\begin{cases} \frac{d\gamma_i}{ds} = -\nabla(g_i)(\gamma_i) \\ \gamma_i(-\infty) = p_i, \ i = 1, 2 \\ \gamma_3(+\infty) = p_3 \end{cases} \tag{4}$$

where p_i are critical points of g_i such that $m_{g_1}(p_1) = n - k$, $m_{g_2}(p_2) = n - l$, $m_{g_3}(p_3) = n - (k + l)$ (see Figure 1). If we denote by $\vec{g} = (g_1, g_2, g_3)$, $\vec{p} = (p_1, p_2, p_2)$ and by $\mathcal{M}(\vec{p}, \vec{g})$ the set of I's that satisfy (4), then

$$\mathcal{M}(\vec{p}, \vec{g}) = W^u(g_1, p_1) \cap W^u(g_2, p_2) \cap W^s(g_3, p_3)$$

where $W^u(g, p)$ (respectively $W^s(g, p)$) denotes the unstable (respectively stable) manifold of a critical point p of a Morse function g. For generic choices $\mathcal{M}(\vec{p}, \vec{g})$ is a smooth finite-dimensional manifold and for the above choice of Morse indices it is zero dimensional. Denote by $n(\vec{p}, \vec{g})$ the cardinality of $\mathcal{M}(\vec{p}, \vec{g})$ mod 2. If we set

$$\begin{aligned} \Psi_M(\vec{g}) := \sum_{\vec{p}} n(\vec{p}, \vec{g}) p_1 \otimes p_2 \otimes p_3 \in CM_{n-k}(g_1) \\ \otimes CM_{n-l}(g_2) \otimes CM_{n-(k+l)}(g_3), \end{aligned}$$

then Ψ_M is well defined and it actually holds

$$\Psi_M(\vec{g}) \in HM_{n-k}(g_1) \otimes HM_{n-l}(g_2) \otimes HM_{n-(k+l)}(g_3).$$

For $[a_1] \in HM^k(f_1) \cong HM^{n-k}(g_1)$ and $[a_2] \in HM^l(f_2) \cong HM^{n-l}(g_2)$, where $[\cdot]$ stands for a cohomological class, i.e., $a_1 \in CM^k(f_1)$, $a_2 \in CM^l(f_2)$, we have the contraction

$$\langle a_1 \otimes a_2, \Psi_M(\vec{g})\rangle \in CM_{n-(k+l)}(g_3).$$

Finally, define

$$[a_1] \cup_M [a_2] := [\sigma_{f_3}(\langle a_1 \otimes a_2, \Psi_M(\vec{g})\rangle)] \in HM^{k+l}(f_3).$$

One can easily show that the cup product does not depend on the choice of cycles a_1 and a_2 within the same cohomological classes (see [**11**] for the details).

We briefly recall the construction of Floer (co)homology and the cup product in Floer cohomology. For a smooth compactly supported Hamiltonian function $H_t : T^*M \to \mathbf{R}$, let $CF_k(H)$ denote Floer chain groups, i.e., $\mathbf{Z}_2$- vector space generated by the set of Hamiltonian paths:

$$\begin{cases} x : [0,1] \to T^*M \\ \dot{x}(t) = X_H(x(t)) \\ x(0), x(1) \in O_M, \end{cases}$$

where O_M is a zero section of T^*M. The grading here is determined by Maslov index $\mu_H(x)$ of Hamiltonian path $x(t)$, such that

$$x \in CF_k(H) \Longleftrightarrow k = \mu_H(x) + \frac{n}{2}$$

where $n = \dim M$ (for a definition of Maslov index see [**13, 14**] and [**10**] for the application in grading in Floer homology). Denote by $HF_k(H)$ the homology groups of $CF_k(H)$ with respect to the boundary operator

$$\partial_F : CF_k(H) \longrightarrow CF_k(H), \quad \partial_F(x) := \sum_{\mu_H(y)=\mu_H(x)-1} n(x,y)y$$

where $n(x, y)$ are the numbers of the solutions of an elliptic system

$$\begin{cases} \frac{\partial u}{\partial s} + J(\frac{\partial u}{\partial t} - X_H(u)) = 0 \\ u(s,i) \in O_M, \ i \in \{0,1\} \\ u(-\infty, t) = x(t), \ u(+\infty, t) = y(t) \end{cases} \tag{5}$$

mod 2. Floer cohomology groups are constructed analogously as in Morse case, by defining a cochain by $CF^*(H) := \mathrm{Hom}\,(CF_*(H), \mathbf{Z}_2)$ and coboundary operator as in (2). By using the following transformations:

$$\begin{aligned} x &\longmapsto \widetilde{x}, & \widetilde{x}(t) &:= x(1-t) \\ H &\longmapsto \widetilde{H}, & \widetilde{H}(p,t) &:= -H(p,1-t) \\ J &\longmapsto \widetilde{J}, & \widetilde{J}(p,t) &:= J(p,1-t) \end{aligned}$$

one can easily obtain a Poincaré duality map:

$$\sigma_H : HF_{n-*}(\widetilde{H}) \xrightarrow{\cong} HF^*(H) \tag{6}$$

(see [**11**] for details). The cup product in Floer cohomology is constructed by means of "pair of pants" mappings (see Figure 1). More precisely, let H_i, for $i = 1, 2, 3$, be three compactly supported Hamiltonians and x_i three corresponding Hamiltonian paths with ends in O_M. Let Σ be a Riemannian surface with boundary of genus zero with three semi-strips-ends (surface $\overline{\Sigma}$ is conformally equivalent to a disc with three marked boundary points). Denote by

$$\begin{aligned} &\Sigma_i := \phi_i((-\infty, 0] \times [0,1]) \subset \Sigma, \quad i = 1, 2 \\ &\Sigma_3 := \phi_3([0, +\infty) \times [0,1]) \subset \Sigma \\ &\Sigma_0 := \Sigma - (\Sigma_1 \cup \Sigma_2 \cup \Sigma_3), \end{aligned}$$

where ϕ_i are conformal parameterizations of the corresponding semi-strips. Denote again by $G_i := \widetilde{H_i}$, $y_i := \widetilde{x}_i$, $\overrightarrow{G} := (G_1, G_2, G_3)$ and $\vec{y} := (y_1, y_2, y_3)$. Denote by $\mathcal{M}(\vec{y}, \overrightarrow{G})$ the set of maps $u : \Sigma \to T^*M$, such that ($u_i := u \circ \phi_i$):

$$\begin{cases} \frac{\partial u_i}{\partial s} + J\left(\frac{\partial u_i}{\partial t} - X_{\tilde{\rho}(s)G_i}(u_i)\right) = 0, \ i = 1, 2 \\ \frac{\partial u_3}{\partial s} + J\left(\frac{\partial u_3}{\partial t} - X_{\rho(s)G_3}(u_3)\right) = 0 \\ u_i(-\infty, t) = y_i(t), \ i = 1, 2, \quad u_3(+\infty, t) = y_3(t) \\ u(\partial\Sigma) \subset O_M \\ \overline{\partial}(u|_{\Sigma_0}) = 0 \end{cases} \tag{7}$$

where $\rho : [0, +\infty) \to \mathbf{R}$ is a smooth function such that

$$\rho(s) = \begin{cases} 1 & s \geq 2 \\ 0 & s \leq 1 \end{cases}$$

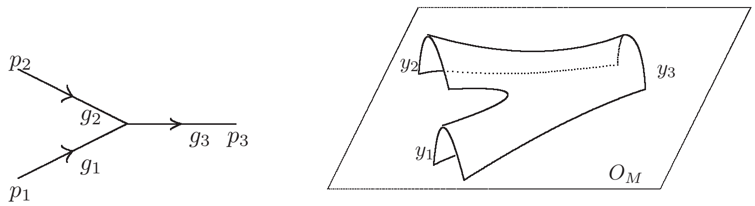

FIGURE 1. Cup products: tree in Morse and "pair-of-pants" in Floer theory.

and $\widetilde{\rho}(s) := \rho(-s)$. For generic choices, the set $\mathcal{M}(\vec{y}, \overrightarrow{G})$ is manifold of a dimension

$$d = \Big(-\mu_{G_3}(y_3) + \frac{n}{2}\Big) + \Big(\mu_{G_1}(y_1) - \frac{n}{2}\Big) + \Big(\mu_{G_2}(y_2) - \frac{n}{2}\Big).$$

For $\mu_{G_1}(y_1) + n/2 = n-k$, $\mu_{G_2}(y_2) + n/2 = n-l$ and $\mu_{G_3}(y_3) + n/2 = n-(k+l)$, i.e., $d = 0$, denote by $n(\vec{y}, \overrightarrow{G})$ the cardinal number of a set $\mathcal{M}(\vec{y}, \overrightarrow{G})$ mod 2 and define

$$\Psi_F(G) := \sum_{\vec{y}} n(\vec{y}, \overrightarrow{G}) y_1 \otimes y_2 \otimes y_3 \in HF_{n-k}(G_1) \otimes HF_{n-l}(G_2) \otimes HF_{n-(k+l)}(G_3).$$

Again by using the contraction map and the isomorphism σ_{H_3}, we set, for given $a_1 \in HF^k(H_1) \cong HF^{n-k}(G_1)$ and $a_2 \in HF^l(H_2) \cong HF^{n-l}(G_2)$:

$$a_1 \cup_F a_2 := \sigma_{H_3}\Big(\Big\langle a_1 \otimes a_2, \Psi_F(\overrightarrow{G})\Big\rangle\Big) \in HM^{k+l}(H_3)$$

(see [**11**] for more details).

The Piunikhin-Salamon-Schwarz type isomorphism between $HM_*(f)$ and $HF_*(H)$ was constructed in [**8**], following [**12**] (where it was originally given, for the case of periodic orbits). It is based on counting mixed type objects. More precisely, for a given critical point p of a Morse function f and Hamiltonian path x with ends in O_M assigned to Hamiltonian function H, we consider the space of pairs of maps

$$\gamma : (-\infty, 0] \longrightarrow M, \quad u : \mathbf{R} \times [0,1] \longrightarrow T^*M$$

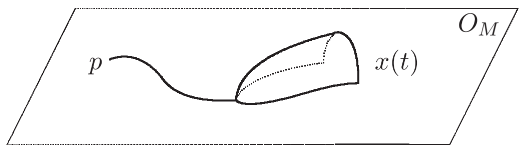

FIGURE 2. Mixed object from $\mathcal{M}(p, f; x, H)$.

that satisfy

$$\begin{cases} \frac{d\gamma}{ds} = -\nabla f(\gamma(s)) \\ \frac{\partial u}{\partial s} + J(\frac{\partial u}{\partial t} - X_{\rho_R H}(u)) = 0 \\ u(\partial(\mathbf{R} \times [0,1])) \in O_M \\ \gamma(-\infty) = p, \; u(+\infty, t) = x(t) \\ \gamma(0) = u(-\infty, t) \\ E(u) := \int_{-\infty}^{+\infty} \int_0^1 \left\| \frac{\partial u}{\partial s} \right\|^2 dt\, ds < +\infty, \end{cases} \tag{8}$$

where $\rho_R : \mathbf{R} \to \mathbf{R}$ is a smooth function such that

$$\rho_R(s) = \begin{cases} 1 & s \geq R+1 \\ 0 & s \leq R. \end{cases} \tag{9}$$

Let $\mathcal{M}(p, f; x, H)$ denote the set of solutions of (8) (see Figure 2).

For generic choices $\mathcal{M}(p, f; x, H)$ is a smooth manifold of dimension $m_f(p) - (\mu_H(x) + n/2)$, compact in dimension zero. So denote by $n(p, f; x, H)$ the cardinality mod 2 of the set $\mathcal{M}(p, f; x, H)$. The map

$$\psi : CM_k(f) \longrightarrow CF_k(H), \qquad p \longmapsto \sum_{\mu_H(x)+n/2=k} n(p, f; x, H)x \tag{10}$$

is well defined, it is also defined on $HM_k(f)$ and it is an isomorphism. Its inverse ϕ is defined similarly. More precisely, let $\mathcal{M}(x, H; p, f)$ be the space of pairs of maps

$$u : \mathbf{R} \times [0,1] \longrightarrow T^*M, \quad \gamma : [0, +\infty) \longrightarrow M$$

that satisfy

$$
\text{(11)} \qquad \begin{cases} \frac{d\gamma}{ds} = -\nabla f(\gamma(s)) \\ \frac{\partial u}{\partial s} + J(\frac{\partial u}{\partial t} - X_{\tilde{\rho}_R H}(u)) = 0 \\ u(\partial(\mathbf{R}\times[0,1])) \subset O_M \\ u(-\infty,t) = x(t), \gamma(+\infty) = p \\ \gamma(0) = u(+\infty,t) \\ \int_{-\infty}^{+\infty}\int_0^1 \left\| \frac{\partial u}{\partial s}\right\|^2 dt\, ds < +\infty. \end{cases}
$$

Again, $\widetilde{\rho}_R(s) := \rho_R(-s)$ for ρ_R defined in (9). For generic choices, $\mathcal{M}(x,H;p,f)$ is a smooth manifold of dimension $\mu_H(x)+n/2-m_f(p)$, compact in dimension zero. If $n(x,H;p,f)$ is the cardinal number mod 2 of the set $\mathcal{M}(x,H;p,f)$, then the map ϕ is defined as

$$
\text{(12)} \qquad \phi : CF_k(H) \longrightarrow CM_k(f), \qquad x \longmapsto \sum_{m_f(p)=k} n(x,H;p,f)p.
$$

For the proofs of the above results see [**8, 12**]. The generalization of the previous construction to more general symplectic manifolds does not give isomorphisms, but only homomorphisms. It was done by Albers in [**1**]. Biran and Cornea gave some extensions of mentioned construction in [**3**]. Leclercq used the mixed type objects described above to construct a product that turns Floer homology into module over Morse homology ring. This construction and natural isomorphism between Floer homologies as morphisms of modules are considered in [**9**]. Cornea and Lalonde also considered the Lagrangian PSS homomorphism and its algebraic properties (see [**4**]).

In this paper we prove that the above PSS isomorphism naturally intertwines with cup products. We establish PSS isomorphism between Morse and Floer homologies of M and T^*M (defined independently on Morse function or Hamiltonian) and prove that such an isomorphism also preserves ring structure induced by cup products.

2. Cup products and PSS isomorphisms. In this section we prove the naturality of cup products with respect to PSS isomorphism. This is the content of the third author's Master thesis [**16**]. It is stated in the following

Theorem 1. *Let f_i, $i = 1, 2, 3$ be three Morse functions on M and H_i three compactly supported Hamiltonian functions on T^*M. There exist isomorphisms*

$$\tau_i : HM^*(f_i) \xrightarrow{\cong} HF^*(H_i), \quad i = 1, 2, 3,$$

induced by Piunikhin-Salamon-Schwarz isomorphisms (10)*, such that the diagram*

$$\begin{tikzcd} HF^k(H_1) \otimes HF^l(H_2) \arrow[r, "\cup_F"] & HF^{k+l}(H_3) \\ HM^k(f_1) \otimes HM^l(f_2) \arrow[r, "\cup_M"] \arrow[u, "\tau_1 \otimes \tau_2"] & HM^{k+l}(f_3) \arrow[u, "\tau_3"'] \end{tikzcd} \tag{13}$$

commutes.

Proof. Using the Poincaré duality isomorphisms between homology and cohomology groups (3) and (6) and Piunikhin-Salamon-Schwarz isomorphisms (10) between $HM_*(f_i)$ and $HF_*(H_i)$, we define τ_i in the cochain level such that the following diagram commutes

$$\begin{tikzcd} CF_{n-*}(G_i) \arrow[r, "\sigma_{H_i}"] & CF^*(H_i) \\ CM_{n-*}(g_i) \arrow[r, "\sigma_{f_i}"] \arrow[u, "\psi_i"] & CM^*(f_i), \arrow[u, "\tau_i"'] \end{tikzcd}$$

i.e.,

$$\tau_i := \sigma_{H_i} \circ \psi_i \circ \sigma_{f_i}^{-1}. \tag{14}$$

Here g_i and G_i are as above. Since ψ_i, σ_{f_i} and σ_{H_i} induce the isomorphisms on the (co)homology levels, the same is true for τ_i. Consider the following diagram:

(15)

$$\begin{tikzcd} H_*(G_1)\otimes H_*(G_2) \arrow[r, "\sigma_{H_1}\otimes\sigma_{H_2}"] & H^*(H_1)\otimes H^*(H_2) \arrow[r, "\cup_F"] & H^*(H_3) \arrow[r, "\sigma_{H_3}^{-1}"] & H_*(G_3) \\ H_*(g_1)\otimes H_*(g_2) \arrow[r, "\sigma_{f_1}\otimes\sigma_{f_2}"] \arrow[u, "\psi_1\otimes\psi_2"] & H^*(f_1)\otimes H^*(f_2) \arrow[r, "\cup_M"] \arrow[u, "\tau_1\otimes\tau_2"] & H^*(f_3) \arrow[r, "\sigma_{f_3}^{-1}"] \arrow[u, "\tau_3"] & H_*(g_3). \arrow[u, "\psi_3"] \end{tikzcd}$$

We used the abbreviations $H_*(H_i)$ for $HF_*(H_i)$ and $H_*(f_i)$ for $HM_*(f_i)$; we also assumed that $*$ stands for the corresponding index. Since τ is defined such that the left and the right diagrams in (15) commute, and since the inner diagram in (15) is actually the diagram (13), the proof of the theorem will be completed if we prove that the outer diagram, that is,

$$\begin{tikzcd} HF_{n-k}(G_1)\otimes HF_{n-l}(G_2) \arrow[r, "\tilde{l}_H"] & HF_{n-(k+l)}(G_3) \\ HM_{n-k}(g_1)\otimes HM_{n-l}(g_2) \arrow[r, "\tilde{l}_f"] \arrow[u, "\psi_1\otimes\psi_2"] & HM_{n-(k+l)}(g_3) \arrow[u, "\psi_3"'] \end{tikzcd} \tag{16}$$

commutes. Here

$$\tilde{l}_f := {\sigma_{f_3}}^{-1}\circ\cup\circ(\sigma_{f_1}\otimes\sigma_{f_2}), \quad \tilde{l}_H := {\sigma_{H_3}}^{-1}\circ\cup\circ(\sigma_{H_1}\otimes\sigma_{H_2}).$$

Set

$$\begin{aligned} & l_f : \bigoplus_{*_1+*_2=n+*_3} CM_{*_1}(g_1)\otimes CM_{*_2}(g_2) \longrightarrow CM_{*_3}(g_3) \\ & l_f(p_1\otimes p_2) := \sum_{m_{g_3}(p_3)=m_{g_1}(p_1)+m_{g_2}(p_2)-n} n(\vec{p},\vec{g})p_3 \end{aligned} \tag{17}$$

and

$$\begin{aligned} & l_H : \bigoplus_{*_1+*_2=n+*_3} CF_{*_1}(G_1)\otimes CF_{*_2}(G_2) \longrightarrow CF_{*_3}(G_3) \\ & l_H(y_1\otimes y_2) := \sum_{\mu_{G_3}(y_3)+n/2=*_1+*_2-n} n(\vec{y},\overrightarrow{G})y_3, \end{aligned}$$

where, as before, $n(\vec{p},\vec{g})$ and $n(\vec{y},\overrightarrow{G})$ are, respectively, the numbers of solutions of (4) and (7) $\bmod 2$. One easily checks that l_f and l_H induce mappings in the homology level, i.e., that it holds $l_f\circ(\partial_M\otimes\mathrm{Id}+\mathrm{Id}\otimes\partial_M)=\partial_M\circ l_f$ and $l_H\circ(\partial_F\otimes\mathrm{Id}+\mathrm{Id}\otimes\partial_F)=\partial_F\circ l_H$ where ∂_M and ∂_F are boundary operators with respect to the corresponding Morse functions and Hamiltonians. We will use the same notations for the mappings obtained from l_f and l_H in the homology level. From the definitions of cup products $\cup_M$ and $\cup_F$ we see that the equalities

$$l_f=\tilde{l}_f, \quad l_H=\tilde{l}_H$$

hold in the homology level. So, to prove that the diagram (16) (hence (13)) commutes we need to prove that, in the homology level, it holds

$$l_f = \psi_3 \circ l_H \circ (\psi_1 \otimes \psi_2). \tag{18}$$

Since the mappings l_f and l_H are defined on the chain groups CM and CF, the equality (18) is equivalent to equality

$$\psi_3 \circ l_H \circ (\psi_1 \otimes \psi_2) - l_f = K \circ (\partial_1 \otimes \mathrm{Id} + \mathrm{Id} \otimes \partial_2) + \partial_3 \circ K, \tag{19}$$

for some mapping

$$K : \bigoplus_{*_1 + *_2 = n + *_3} CM_{*_1}(g_1) \otimes CM_{*_2}(g_2) \to CM_{*_3+1}(g_3).$$

The symbols ∂_1, ∂_2 and ∂_3 in (19) denote the Morse boundary operators (1) for Morse functions f_1, f_2 and f_3. We will deduce the equality (19) from cobordism arguments by introducing the following auxiliary one-dimensional manifold. For $\vec{p} = (p_1, p_2, p_3)$ as above (such that $m_{g_1}(p_1) = n - k$, $m_{g_2}(p_2) = n - l$, $m_{g_3}(p_3) = n - (k + l)$), Σ, u as before, denote by $\mathcal{M}_R(\vec{p}, \vec{g}; \overrightarrow{G})$ the set of all $(\gamma_1, \gamma_2, \gamma_3, u)$ that satisfy:

$$\begin{cases} \gamma_i : (-\infty, 0] \to M, \quad i = 1, 2, \ \gamma_3 : [0, +\infty) \to M \\ u : \Sigma \to T^*M \\ \frac{\partial u_j}{\partial s} + J(\frac{\partial u_j}{\partial t} - X_{\tilde{\rho}_R H_j}(u_j)) = 0, \ u_j := u \circ \phi_j, j = 1, 2 \\ \frac{\partial u_3}{\partial s} + J(\frac{\partial u_3}{\partial t} - X_{\rho_R H_3}(u_3)) = 0, \ u_3 := u \circ \phi_3 \\ u(\partial\Sigma) \subset O_M \\ \overline{\partial}(u|_{\Sigma_0}) = 0 \\ \gamma_i(0) = u_i(-\infty, t), \ i = 1, 2, \ \gamma_3(0) = u_3(+\infty, t) \\ \gamma_i(-\infty) = p_i, \ i = 1, 2, \gamma_3(+\infty) = p_3. \end{cases} \tag{20}$$

Now $\rho_R : [0, +\infty) \to [0, 1]$ is a smooth function such that

$$\rho_R(s) = \begin{cases} 1, & 2 \le s \le R, \\ 0, & s \le 1, \ s \ge R + 1 \end{cases}$$

and $\widetilde{\rho}_R(s) := \rho_R(-s)$. Denote by

$$\mathcal{M}(R, \vec{p}, \vec{g}; \overrightarrow{G}) := \left\{ (R, \gamma_1, \gamma_2, \gamma_3, u) \mid (\gamma_1, \gamma_2, \gamma_3, u) \in \mathcal{M}_R(\vec{p}, \vec{g}; \overrightarrow{G}) \right\}.$$

For the above choice of indices the set $\mathcal{M}_R(\vec{p}, \vec{g}; \overrightarrow{G})$ is a zero dimensional manifold (for generic choices of $\vec{g}$, $\overrightarrow{G}$) and $\mathcal{M}(R, \vec{p}, \vec{g}; \overrightarrow{G})$ is a manifold of dimension one. Let $\mathcal{M}(p, q, f)$ denote the set of solutions of (1) and $\widehat{\mathcal{M}}(p, q, f)$ denote the same set modulo $\mathbf{R}$-action. The description of the topological boundary of $\widehat{\mathcal{M}}(p, q, f)$:

$$\partial\left(\mathcal{M}(R, \vec{p}, \vec{g}; \overrightarrow{G})\right) = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3 \cup \mathcal{B}_4 \cup \mathcal{B}_5 \tag{21}$$

is obtained by using the standard compactness and gluing arguments. One inclusion (compactness) in (21) follows from Gromov compactness and the Arzela-Ascoli theorem (see [**7**]). The other one (gluing) can be proved using the standard pre-gluing construction and Banach fixed point theorem as in the appendix in [**2**] (see also [**6**] for a similar case). The parts $\mathcal{B}_i$ of the boundary are the following:

$$\begin{aligned}
\mathcal{B}_1 &= \mathcal{M}_{R_0}(\vec{p}, \vec{g}, \overrightarrow{G})\\
\mathcal{B}_2 &= \bigcup_{m_{g_1}(q_1)=m_{g_1}(p_1)-1} \widehat{\mathcal{M}}(p_1, q_1, g_1) \times \mathcal{M}(R, (q_1, p_2, p_3), \vec{g}; \overrightarrow{G})\\
\mathcal{B}_3 &= \bigcup_{m_{g_2}(q_2)=m_{g_2}(p_2)-1} \widehat{\mathcal{M}}(p_2, q_2, g_2) \times \mathcal{M}\left(R, (p_1, q_2, p_3), \vec{g}; \overrightarrow{G}\right)\\
\mathcal{B}_4 &= \bigcup_{m_{g_3}(q_3)=m_{g_3}(p_3)+1} \mathcal{M}\left(R, (p_1, p_2, g_3), \vec{g}; \overrightarrow{G}\right) \times \widehat{\mathcal{M}}(q_3, p_3, f_3)\\
\mathcal{B}_5 &= \bigcup_{y_i, q_i} \mathcal{M}(p_1, g_1; y_1, G_1) \times \mathcal{M}(p_2, g_2; y_2, G_2)\\
&\quad \times \mathcal{M}(\vec{y}, \overrightarrow{G}) \times \mathcal{M}(y_3, G_3; q_3, g_3)
\end{aligned}$$

(see Figure 3). In the last equation all q_i's are critical points of g_i's and all y_i's are Hamiltonian path with respect to G_i's. The last union is taken by all q_i and y_i such that $\mu_{G_i}(y_i) = m_{g_i}(q_i) = m_{g_i}(p_i)$. The first four types of the boundary correspond to the case when R is bounded, the fifth type $\mathcal{B}_5$ appears when R is unbounded. Now define the mentioned mapping K as

$$\begin{aligned}
K : \bigoplus_{*_1+*_2=n+*_3} CM_{*_1}(g_1) \otimes CM_{*_2}(g_2) &\longrightarrow CM_{*_3+1}(g_3)\\
K(p \otimes q) := \sum_{m_{g_3}(r)=*_1+*_2-n+1} n\left(R, (p, q, r), \vec{g}; \overrightarrow{G}\right) r,
\end{aligned}$$

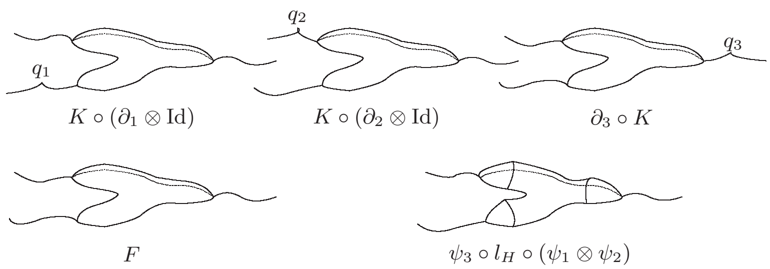

FIGURE 3. Five types of boundary of $\mathcal{M}(R, \vec{p}, \vec{g}; \overrightarrow{G})$ and corresponding mappings.

where $n(R, (p, q, r), \vec{g}; \overrightarrow{G})$ stands for the cardinal number mod 2 of $\mathcal{M}(R, (p, q, r), \vec{g}; \overrightarrow{G})$. Define also the following mapping:

$$\begin{aligned} F &: \bigoplus_{*_1 + *_2 = n + *_3} CM_{*_1}(g_1) \otimes CM_{*_2}(g_2) \longrightarrow CM_{*_3}(g_3) \\ F(p_1 \otimes p_2) &:= \sum_{m_{g_3}(p_3) = *_1 + *_2 - n} n_{R_0}\left((p_1, p_2, p_3), \vec{g}; \overrightarrow{G}\right) p_3, \end{aligned} \tag{22}$$

where $n_{R_0}((p_1, p_2, p_3), \vec{g}; \overrightarrow{G})$ is the cardinal number mod 2 of $\mathcal{M}_{R_0}((p_1, p_2, p_3), \vec{g}; \overrightarrow{G})$. Now we see that counting the elements in $\partial(\mathcal{M}(R, \vec{p}, \vec{g}; \overrightarrow{G}))$ defines the mappings that we already defined, in the following way (see Figure 3):

- counting the elements of $\mathcal{B}_1$ defines the mapping F;
- counting the elements of $\mathcal{B}_2$ defines the mapping $K \circ (\partial_1 \otimes \mathrm{Id})$;
- counting the elements of $\mathcal{B}_3$ defines the mapping $K \circ (\mathrm{Id} \otimes \partial_2)$;
- counting the elements of $\mathcal{B}_4$ defines the mapping $\partial_3 \otimes K$;
- counting the elements of $\mathcal{B}_5$ defines the mapping $\psi_3 \circ l_H \circ (\psi_1 \otimes \psi_2)$.

Since the cardinality of $\partial(\mathcal{M}(R, \vec{p}, \vec{g}; \overrightarrow{G}))$ is an even number, we conclude that it holds

$$\psi_3 \circ l_H \circ (\psi_1 \otimes \psi_2) - F = K \circ (\partial_1 \otimes \mathrm{Id} + \mathrm{Id} \otimes \partial_2) + \partial_3 \circ K,$$

so the mappings F and $\psi_3 \circ l_H \circ (\psi_1 \otimes \psi_2)$ are chain homotopic, hence the same in the homology level.

The homomorphism F defined in (22) is independent of the choice of Hamiltonians $\overrightarrow{G}$. Indeed, let $\overrightarrow{G_0}$, $\overrightarrow{G_1}$ be two triples of Hamiltonians, $\overrightarrow{G}_\lambda$, $0 \leq \lambda \leq 1$, a homotopy between them, and F_0, F_1-chain homomorphisms that correspond to $\overrightarrow{G}_0$, $\overrightarrow{G}_1$. Consider the space

$$\begin{aligned}&\mathcal{M}_{R_0}(\lambda, \vec{p}, \vec{g}; \overrightarrow{G}_\lambda)\\ &\qquad := \{(\gamma_1, \gamma_2, \gamma_3, u, \lambda) \mid (\gamma_1, \gamma_2, \gamma_3, u) \in \mathcal{M}_{R_0}(\vec{p}, \vec{g}; \overrightarrow{G}_\lambda)\}.\end{aligned}$$

Its dimension is equal to 1 and its boundary is

$$\begin{aligned}\partial \left(\mathcal{M}_{R_0}(\lambda, \vec{p}, \vec{g}; \overrightarrow{G}_\lambda)\right) &= \mathcal{M}(\vec{p}, \vec{g}; \overrightarrow{G}_0) - \mathcal{M}(\vec{p}, \vec{g}; \overrightarrow{G}_1)\\ &+ \bigcup_{m_{g_1}(q_1)=m_{g_1}(p_1)-1} \widehat{\mathcal{M}}(p_1, q_1, g_1) \times \mathcal{M}_{R_0}\left(\lambda, (q_1, p_2, p_3), \vec{g}; \overrightarrow{G}_\lambda\right)\\ &+ \bigcup_{m_{g_2}(q_2)=m_{g_2}(p_2)-1} \widehat{\mathcal{M}}(p_2, q_2, g_2) \times \mathcal{M}_{R_0}\left(\lambda, (p_1, q_2, p_3), \vec{g}; \overrightarrow{G}_\lambda\right)\\ &+ \bigcup_{m_{g_3}(q_3)=m_{g_3}(p_3)+1} \mathcal{M}_{R_0}\left(\lambda, (p_1, p_2, q_3), \vec{g}; \overrightarrow{G}_\lambda\right) \times \widehat{\mathcal{M}}(q_3, p_3, g_3).\end{aligned}$$

It follows that $F_1 - F_0 = L \circ (\partial_1 \otimes \mathrm{Id} + \mathrm{Id} \otimes \partial_2) + \partial_3 \circ L$ for

$$\begin{aligned}L &: \bigoplus_{*_1+*_2=n+*_3} CM_{*_1}(g_1) \otimes CM_{*_2}(g_2) \to CM_{*_3+1}(g_3)\\ L(p \otimes q) &:= \sum_{m_{g_3}(r)=*_1+*_2-n+1} n_{R_0}\left(\lambda, (p, q, r), \vec{g}; \overrightarrow{G}_\lambda\right) r\end{aligned}$$

where

$$n_{R_0}\left(\lambda, (p, q, r), \vec{g}; \overrightarrow{G}_\lambda\right) = \sharp\, \mathcal{M}_{R_0}\left(\lambda, (p, q, r), \vec{g}; \overrightarrow{G}_\lambda\right) \pmod 2,$$

i.e., F_0 and F_1 are chain homotopic. Choose now the homotopy between $\overrightarrow{G}$ and $\vec{0} = (0, 0, 0)$. We see that the map (22) is chain homotopic to the map defined by counting the objects of the type (20), such that u is holomorphic on the whole domain, with the boundary on the zero section. By Stokes' formula, this u has to be a constant map, so the mapping F (when $\overrightarrow{G} = \vec{0}$) is defined by counting Morse trees, i.e., it is the same mapping as l_f. So the proof of Theorem 1 follows. □

3. Ring structure isomorphism. In this section we prove that the above PSS isomorphism establishes the isomorphism between Morse and Floer cohomological rings (with respect to cup products).

We construct Morse cohomology of M that is independent of a Morse function by means of the natural isomorphisms between Morse cohomologies for different Morse functions. Recall that for two Morse functions f^α and f^β there exists a canonical isomorphism

$$T^{\alpha\beta} : HM_*(f^\alpha) \longrightarrow HM_*(f^\beta)$$

defined in a following way. Let $f_s^{\alpha\beta}$ be a smooth homotopy between f^α and f^β, i.e., a smooth function defined on $M \times \mathbf{R}$ such that

$$f_s^{\alpha\beta}(\cdot) = \begin{cases} f^\alpha(\cdot) & s \leq -T, \\ f^\beta(\cdot) & s \geq T, \end{cases}$$

for some $T > 0$. For a critical point p^α of f^α and p^β of f^β, $m_{f^\alpha}(p^\alpha) = m_{f^\beta}(p^\beta)$, denote by $n(p^\alpha, p^\beta)$ the number of solutions $\pmod 2$ of

$$\mathcal{M}(p^\alpha, p^\beta, f_s^{\alpha\beta}) \\ := \left\{ \gamma : \mathbf{R} \to M \;\middle|\; \frac{d\gamma}{ds} = -\nabla(f_s^{\alpha\beta}(\gamma)), \gamma(-\infty) = p^\alpha, \gamma(+\infty) = p^\beta \right\}.$$

Now set

$$T^{\alpha\beta}(p^\alpha) := \sum_{m_{f^\alpha}(p^\alpha) = m_{f^\beta}(p^\beta)} n(p^\alpha, p^\beta) p^\beta.$$

The mapping $T^{\alpha\beta}$ is well defined also in the homology level and it is an isomorphism (see [**15**] for a detailed proof of these facts). Now define

$$\widetilde{T}^{\alpha\beta} := \sigma_{f^\beta} \circ T^{\alpha\beta} \circ {\sigma_{f^\alpha}}^{-1} : HM^*(f^\alpha) \xrightarrow{\cong} HM^*(f^\beta).$$

The Morse cohomology $HM^*(M)$ is defined as an inverse limit of Morse cohomology groups $HM^*(f)$ with respect to above isomorphisms $\widetilde{T}^{\alpha\beta}$. More precisely, consider the product of groups

$$\widetilde{HM}^k(M) = \prod_{f^\alpha \text{ Morse}} HM^k(f^\alpha),$$

and define $HM^k(M)$ as

$$HM^k(M) := \left\{(\dots, a_k^\alpha, \dots, b_k^\beta \dots) \in \widetilde{HM}^k(M) \mid \widetilde{T}^{\alpha\beta}(a_k^\alpha) = b_k^\beta\right\}.$$

The above-defined cup product turns the Morse cohomology into ring. To see this, we need to prove that the cup product is independent of the choice of equivalence classes, where the equivalence relation is given by

$$\{a_k^\alpha\} \sim \{b_k^\beta\} \Longleftrightarrow \widetilde{T}^{\alpha\beta}(a_k^\alpha) = b_k^\beta \quad \text{for all } a^\alpha, b^\beta,$$

i.e., we need to check that two different pairs of equivalent elements give rise to two equivalent elements. Obviously, it is enough to show that the diagram

$$\begin{array}{ccc} HM^k(f_1^\beta) \otimes HM^l(f_2^\beta) & \xrightarrow{\cup_\beta} & HM^{k+l}(f_3^\beta) \\ {\scriptstyle \widetilde{T}_1^{\alpha\beta} \otimes \widetilde{T}_2^{\alpha\beta}} \uparrow & & \uparrow {\scriptstyle \widetilde{T}_3^{\alpha\beta}} \\ HM^k(f_1^\alpha) \otimes HM^l(f_2^\alpha) & \xrightarrow{\cup_\alpha} & HM^{k+l}(f_3^\alpha) \end{array} \tag{23}$$

commutes. Here $\widetilde{T}_i^{\alpha\beta}$ are the corresponding isomorphisms between $HM^*(f_i^\alpha)$ and $HM^*(f_i^\beta)$. Since we defined the isomorphisms $\widetilde{T}_i^{\alpha\beta}$ via the maps $T_i^{\alpha\beta}$ and σ_{f_i}, we consider the following diagram

$$\begin{array}{ccccccc} H_*(g_1^\beta)\otimes H_*(g_2^\beta) & \xrightarrow{\sigma_{f_1^\beta}\otimes\sigma_{f_2^\beta}} & H^*(f_1^\beta)\otimes H^*(f_2^\beta) & \xrightarrow{\cup_\beta} & H^*(f_3^\beta) & \xrightarrow{{\sigma_{f_3^\beta}}^{-1}} & H_*(f_3^\beta) \\ {\scriptstyle T_1^{\alpha\beta}\otimes T_2^{\alpha\beta}}\uparrow & & {\scriptstyle \widetilde{T}_1^{\alpha\beta}\otimes\widetilde{T}_2^{\alpha\beta}}\uparrow & & {\scriptstyle \widetilde{T}_3^{\alpha\beta}}\uparrow & & {\scriptstyle T_3^{\alpha\beta}}\uparrow \\ H_*(g_1^\alpha)\otimes H_*(g_2^\alpha) & \xrightarrow{\sigma_{f_1^\alpha}\otimes\sigma_{f_2^\alpha}} & H^*(f_1^\alpha)\otimes H^*(f_2^\alpha) & \xrightarrow{\cup_\alpha} & H^*(f_3^\alpha) & \xrightarrow{{\sigma_{f_3^\alpha}}^{-1}} & H_*(f_3^\alpha). \end{array}$$

We used the same notations and abbreviations as in the previous section, except that here we deal only with Morse homology (so all H's stand for HM). Obviously it is enough to prove the commutativity of

$$\begin{array}{ccc} HM_{n-k}(g_1^\beta) \otimes HM_{n-l}(g_2^\beta) & \xrightarrow{\tilde{\iota}_\beta} & HM_{n-(k+l)}(g_3^\beta) \\ {\scriptstyle T_1^{\alpha\beta} \otimes T_2^{\alpha\beta}} \uparrow & & \uparrow {\scriptstyle T_3^{\alpha\beta}} \\ HM_{n-k}(g_1^\alpha) \otimes HM_{n-l}(g_2^\alpha) & \xrightarrow{\tilde{\iota}_\alpha} & HM_{n-(k+l)}(g_3^\alpha) \end{array} \tag{24}$$

where, recall

$$\tilde{l}_\alpha = {\sigma_{f_3^\alpha}}^{-1} \circ \cup_\alpha \circ \left(\sigma_{f_1^\alpha} \otimes \sigma_{f_2^\alpha}\right),$$
$$\tilde{l}_\beta = {\sigma_{f_3^\beta}}^{-1} \circ \cup_\beta \circ \left(\sigma_{f_1^\beta} \otimes \sigma_{f_2^\beta}\right).$$

It follows from the definitions of mappings involved that:

$$\tilde{l}_\alpha = l_\alpha := l_{f^\alpha}, \qquad \tilde{l}_\beta = l_\beta := l_{f^\beta}$$

where l_{f^α} and l_{f^β} are the mappings defined in the previous section (see (17)). In order to show that it holds

$$l_\beta \circ \left(T_1^{\alpha\beta} \otimes T_2^{\alpha\beta}\right) = T_3^{\alpha\beta} \circ l_\alpha$$

it is enough to show (since all the maps are defined also in the chain level) that, for some $K : \bigoplus_{*_1+*_2=n+*_3} CM_{*_1}(g_1) \otimes CM_{*_2}(g_2) \to CM_{*_3+1}(g_3)$ it holds

$$l_\alpha - {T_3^{\alpha\beta}}^{-1} \circ l_\beta \circ \left(T_1^{\alpha\beta} \otimes T_2^{\alpha\beta}\right) + K \circ (\partial_1 \otimes \mathrm{Id} + \mathrm{Id} \otimes \partial_2) + \partial_3 \circ K = 0. \tag{25}$$

For a fixed $T > 0$, denote by $g_{s,i}^{\alpha\beta} : M \to \mathbf{R}$ smooth functions that satisfy:

$$g_{s,i}^{\alpha\beta}(\cdot) = \begin{cases} g_i^\alpha(\cdot), & s \leq -T-1, \\ g_i^\beta(\cdot) & s \geq -T, \text{ for } i = 1,2, \end{cases}$$
$$g_{s,3}^{\alpha\beta}(\cdot) = \begin{cases} g_3^\beta(\cdot) & s \leq T, \\ g_3^\alpha(\cdot) & s \geq T+1. \end{cases}$$

For p_i^α a critical point of g_i^α, denote by $\vec{p}^\alpha := (p_1^\alpha, p_2^\alpha, p_3^\alpha)$, $\vec{g}_s^{\alpha\beta} := (g_{s,1}^{\alpha\beta} g_{s,2}^{\alpha\beta}, g_{s,3}^{\alpha\beta})$ and consider the following manifolds:

$$\mathcal{M}_T(\vec{p}^\alpha, \vec{g}_s^{\alpha\beta}) := \left\{ (\gamma_1,\gamma_2,\gamma_3) \;\middle|\; \begin{array}{l} \gamma_i : (-\infty, 0] \to M, \text{ for } i=1,2, \ \gamma_3 : [0,+\infty) \to M \\ \dfrac{d\gamma_i}{ds} = -\nabla g_{s,i}^{\alpha\beta}(\gamma_i(s)) \\ \gamma_i(-\infty) = p_i^\alpha, \text{ for } i=1,2, \gamma_3(+\infty) = p_3^\alpha \\ \gamma_1(0) = \gamma_2(0) = \gamma_3(0) \end{array} \right\} \tag{26}$$

and

$$\mathcal{M}(T,\vec{p}^{\alpha},\vec{g}_s^{\alpha\beta}) := \left\{(T,\gamma_1,\gamma_2,\gamma_3) \mid (\gamma_1,\gamma_2,\gamma_3) \in \mathcal{M}_T(\vec{p}^{\alpha},\vec{g}_s^{\alpha\beta})\right\}. \tag{27}$$

For $m_{g_i^\alpha}(p_1^\alpha) = n-k$, $m_{g_2^\alpha}(p_2^\alpha) = n-l$ $m_{g_3^\alpha}(p_3^\alpha) = n-(k+l)$ the manifold $\mathcal{M}_T(\vec{p}^{\alpha},\vec{g}_s^{\alpha\beta})$ is zero-dimensional and $\mathcal{M}(T,\vec{p}^{\alpha},\vec{g}_s^{\alpha\beta})$ is one-dimensional. Moreover, its topological, zero-dimensional boundary can be identified with

$$\partial\left(\mathcal{M}(T,\vec{p}^{\alpha},\vec{g}_s^{\alpha\beta})\right) = \mathcal{B}_1\cup\mathcal{B}_2\cup\mathcal{B}_3\cup\mathcal{B}_4\cup\mathcal{B}_5,$$

where

$$\begin{aligned}
\mathcal{B}_1 &= \mathcal{M}_{T_0}(\vec{p}^{\alpha},\vec{g}_s^{\alpha\beta})\\
\mathcal{B}_2 &= \bigcup_{m_{g_1^\alpha}(q_1^\alpha)=m_{g_1^\alpha}(p_1^\alpha)-1} \widehat{\mathcal{M}}(p_1^\alpha,q_1^\alpha,g_1^\alpha)\times\mathcal{M}\left(T,(q_1^\alpha,p_2^\alpha,p_3^\alpha),\vec{g}_s^{\alpha\beta}\right)\\
\mathcal{B}_3 &= \bigcup_{m_{g_2^\alpha}(q_2^\alpha)=m_{g_2^\alpha}(p_2^\alpha)-1} \widehat{\mathcal{M}}(p_2^\alpha,q_2^\alpha,g_2^\alpha)\times\mathcal{M}\left(T,(p_1^\alpha,q_2^\alpha,p_3^\alpha),\vec{g}_s^{\alpha\beta}\right)\\
\mathcal{B}_4 &= \bigcup_{m_{g_3^\alpha}(q_3^\alpha)=m_{g_3^\alpha}(p_3^\alpha)+1} \mathcal{M}\left(T,(p_1^\alpha,p_2^\alpha,q_3^\alpha),\vec{g}_s^{\alpha\beta}\right)\times\widehat{\mathcal{M}}(q_3^\alpha,p_3^\alpha,g_3^\alpha)\\
\mathcal{B}_5 &= \bigcup\mathcal{M}(p_1^\alpha,p_1^\beta,g_{1,s}^{\alpha\beta})\times\mathcal{M}(p_2^\alpha,p_2^\beta,g_{2,s}^{\alpha\beta})\times\mathcal{M}(\vec{p}^{\beta},\vec{g}^{\beta})\\
&\quad\times\mathcal{M}(p_3^\beta,p_3^\alpha,g_{3,s}^{\alpha\beta}).
\end{aligned}\tag{28}$$

The last union is taken by all p_i^β such that $m_{g_i^\alpha}(p_i^\alpha) = m_{g_i^\beta}(p_i^\beta)$. All q_i^α's are critical points of g_i^α's and all p_i^β's are critical points of g_i^β's. The spaces that figure in (28) are defined in (1), (4), (26) and (27). The first four types of the boundary correspond to the case when T is bounded, the fifth type $\mathcal{B}_5$ happens when T is unbounded. If we define mapping K by counting the objects from $\mathcal{M}(T,\vec{p}^{\alpha},\vec{g}_s^{\alpha\beta})$, when the latter is zero-dimensional, and the mapping F_{T_0} by counting the elements from $\mathcal{M}_{T_0}(\vec{p}^{\alpha},\vec{g}_s^{\alpha\beta})$ (in dimension zero), then we conclude that

- counting the elements of $\mathcal{B}_1$ defines the mapping F_{T_0};
- counting the elements of $\mathcal{B}_2$ defines the mapping $K\circ(\partial_1\otimes\mathrm{Id})$;
- counting the elements of $\mathcal{B}_3$ defines the mapping $K\circ(\mathrm{Id}\otimes\partial_2)$;

• counting the elements of $\mathcal{B}_4$ defines the mapping $\partial_3 \otimes K$;

• counting the elements of $\mathcal{B}_5$ defines the mapping ${T_3^{\alpha\beta}}^{-1} \circ l_\beta \circ (T_1^{\alpha\beta} \otimes T_2^{\alpha\beta})$.

By taking the homotopy $\vec{g}_\lambda$, $\lambda \in [0,1]$ between $\vec{g}_s^{\alpha\beta}$ and $\vec{g}^\alpha$ and by using the similar cobordism arguments as we used by now, one can show that the mappings F_{T_0} and l_α are the same in the homology level. Hence we proved (25) which implies that the cup product is well defined on Morse cohomology $HM^*(M)$. Since $HM^*(f)$ is a ring with respect to cup product, and since all the properties of $\cup$ remain true for the equivalence classes, $HM^*(M)$ also becomes a ring.

The natural homomorphism between two Floer homologies for two different Hamiltonians is defined as following. Fix $T > 0$. Let $H_s^{\alpha\beta}(t,x)$ be a smooth function such that $H_s^{\alpha\beta}(t,x) = H^\alpha(t,x)$ for $s \leq -T$ and $H_s^{\alpha\beta}(t,x) = H^\beta(t,x)$ for $s \geq T$. The isomorphism

$$S^{\alpha\beta} : HF_*(H^\alpha) \longrightarrow HF_*(H^\beta), \qquad S^{\alpha\beta}(x^\alpha) = \sum_{x^\beta} n(x^\alpha, x^\beta) x^\beta$$

is defined using the numbers $n(x^\alpha, x^\beta)$ of the solutions of the system

$$\begin{cases} \frac{\partial u}{\partial s} + J(\frac{\partial u}{\partial t} - X_{H_s^{\alpha\beta}}(u)) = 0 \\ u(s,i) \in O_M, \ i \in \{0,1\} \\ u(-\infty,t) = x^\alpha(t), \ u(+\infty,t) = x^\beta(t), \end{cases}$$

where x^α and x^β are Hamiltonian (with respect to the functions H^α and H^β) paths with ends in O_M. The definition of Floer cohomology $HF^*(T^*M)$ is completely analogous to the definition of $HM^*(M)$. The cup product defined by use of "pair-of-pants" can also be defined for the equivalence classes in $HF^*(T^*M)$, i.e., the diagram

$$\begin{array}{ccc} HF^k(H_1^\beta) \otimes HF^l(H_2^\beta) & \xrightarrow{\cup_\beta} & HF^{k+l}(H_3^\beta) \\ {\scriptstyle \widetilde{S}_1^{\alpha\beta} \otimes \widetilde{S}_2^{\alpha\beta}} \Big\uparrow & & {\scriptstyle \widetilde{S}_3^{\alpha\beta}} \Big\uparrow \\ HF^k(H_1^\alpha) \otimes HF^l(H_2^\alpha) & \xrightarrow{\cup_\alpha} & HF^{k+l}(H_3^\alpha) \end{array}$$

commutes, where $\widetilde{S}_i^{\alpha\beta} : HF^*(H_i^\alpha) \to HF^*(H_i^\alpha)$ are natural isomorphisms between Floer cohomology groups, defined as $\widetilde{S}_i^{\alpha\beta} := \sigma_{H_i^\beta} \circ$

$S_i^{\alpha\beta} \circ \sigma_{H_i^\alpha}{}^{-1}$. The proof is the same as in the Morse case. Now we are able to state the following

Theorem 2. *Piunikin-Salamon-Schwarz isomorphism defined above induces the isomorphism of rings*

$$\mathcal{T} : (HM^*(M), \cup_M) \longrightarrow (HF^*(T^*M), \cup_F)\,.$$

Proof. The commutativity of the diagram

$$\begin{array}{ccc} HF_*(H^\alpha) & \xrightarrow{S^{\alpha\beta}} & HF_*(H^\beta) \\ {\scriptstyle\psi_\alpha}\big\uparrow & & {\scriptstyle\psi_\beta}\big\uparrow \\ HM_*(f^\alpha) & \xrightarrow{T^{\alpha\beta}} & HM_*(f^\beta) \end{array}$$

that was proved in [**8**] immediately implies (due to the definitions of τ and $\widetilde{T}$, $\widetilde{S}$) the commutativity of

$$\begin{array}{ccc} HF^*(H^\alpha) & \xrightarrow{\widetilde{S}^{\alpha\beta}} & HF^*(H^\beta) \\ {\scriptstyle\tau_\alpha}\big\uparrow & & {\scriptstyle\tau_\beta}\big\uparrow \\ HM^*(f^\alpha) & \xrightarrow{\widetilde{T}^{\alpha\beta}} & HM^*(f^\beta) \end{array} \tag{29}$$

which means that τ defined in (14) induces the homomorphism $\mathcal{T}$ defined on $HM^*(M)$ with the values in $HF^*(T^*M)$. The inverse isomorphism ϕ defined in (12) induces the isomorphism in the cohomology, denote it by δ. More precisely, $\delta := \sigma_f \circ \phi \circ \sigma_H^{-1}$. In the same way as before, δ induces the homomorphism $\mathcal{S} : HF^*(T^*M) \to HM^*(M)$. It follows from $\phi \circ \psi = \mathrm{Id}$ that $\delta \circ \tau = \mathrm{Id}$, so we have

$$\mathcal{S} \circ \mathcal{T}([a]) = \mathcal{S}([\tau(a)]) = [\delta(\tau(a))] = [a],$$

and, in the same way, $\mathcal{T} \circ \mathcal{S} = \mathrm{Id}$. Thus $\mathcal{T}$ and $\mathcal{S}$ are indeed isomorphisms.

Now the commutativity of the diagram (13) proved in Theorem 1 and (23) imply

$$\begin{aligned} \mathcal{T}\,([a] \cup_M [b]) &= \mathcal{T}\,([a \cup_M b]) = [\tau(a \cup_M b)] \\ &= [\tau(a) \cup_F \tau(b)] = [\tau(a)] \cup_F [\tau(b)] \\ &= \mathcal{T}\,([a]) \cup_F \mathcal{T}\,([b])\,, \end{aligned}$$

so the proof of the Theorem follows. $\square$

Remark 1. One of the generalizations of the previous constructions are multiple products, i.e., products with more than one entrance. In the Morse case, we consider the tree T with $m+1$ edges, one interior and $m+1$ exterior vertices (see Figure 4) and identify the edges e_i, for $i = 1, \dots, m$ with $(-\infty, 0]$ (incoming edges) and the edge e_{m+1} with $[0, +\infty)$ (outgoing edge). Let f_i, for $i = 1, \dots, m+1$ be Morse functions on M, and let $g_i := -f_i$. Denote by $\vec{p} = (p_1, p_2, \dots, p_{m+1})$ and by $\vec{g} = (g_1, g_2, \dots, g_{m+1})$. Consider the set $\mathcal{M}(\vec{p}, \vec{g})$ of all mappings $I : T \to M$ such that, for $\gamma_i := I|_{e_i}$, $i = 1, \dots m+1$, it holds:

$$\begin{cases} \frac{d\gamma_i}{ds} = -\nabla(g_i)(\gamma_i) \\ \gamma_i(-\infty) = p_i, \;\; i = 1, \dots m \\ \gamma_{m+1}(+\infty) = p_{m+1} \end{cases}$$

(see Figure 4). Up to generic choices, $\mathcal{M}(\vec{p}, \vec{g})$ is a smooth manifold of dimension $d := \dim \mathcal{M}(\vec{p}, \vec{g}) = m_{f_{m+1}}(p_{m+1}) - m_{f_1}(p_1) - \dots - m_{f_m}(p_m)$ and, for $d = 0$ (i.e. for $p_i \in CM^{k_i}(f_i)$, $i = 1 \dots, m$, $p_{m+1} \in CM^{k_1 + \dots + k_m}(f_{m+1})$), denote by $n(\vec{p}, \vec{g})$ the cardinality of $\mathcal{M}(\vec{p}, \vec{g})$ mod 2. Define

$$\Psi_M(\vec{g}) := \sum_{\vec{p}} n(\vec{p}, \vec{g}) p_1 \otimes \dots \otimes p_m \otimes p_{m+1} \in$$
$$CM_{n-k_1}(g_1) \otimes \dots \otimes CM_{n-k_m}(g_2) \otimes CM_{n-(k_1+\dots+k_m)}(g_{m+1}).$$

For $a_i \in CM^k(f_i) \cong CM^{n-k}(g_i)$, $i = 1, \dots, m$, by using the contraction map,

$$\langle a_1 \otimes \dots \otimes a_m, \Psi_M(\vec{g}) \rangle \in CM_{n-(k_1+\dots+k_m)}(g_{m+1})$$

define

$$\begin{aligned} &\mathcal{O}_M(a_1 \otimes \dots \otimes a_m) \\ &\qquad := \left[\sigma_{f_{m+1}}(\langle a_1 \otimes \dots \otimes a_m, \Psi_M(\vec{g}) \rangle)\right] \in HM^{k_1+\dots+k_m}(f_{m+1}). \end{aligned}$$

As before, one can easily check that $\mathcal{O}_M$ does not depend on the choice of the cycles a_i in the same cohomology classes, hence that it is defined on the cohomology level.

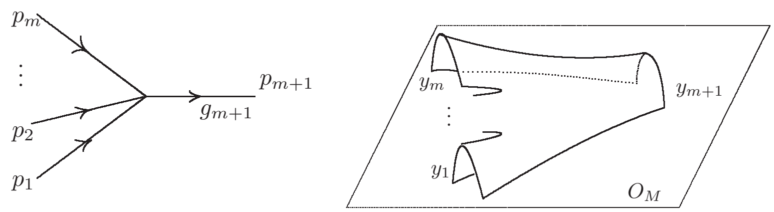

FIGURE 4. Multiple products in Morse and Floer theory.

The multiple products in Floer homology are defined similarly, using the numbers of solutions of

$$\begin{cases} \frac{\partial u_i}{\partial s} + J\left(\frac{\partial u_i}{\partial t} - X_{\tilde{\rho}(s)G_i}(u_i)\right) = 0,\ i = 1, \dots, m \\ u_i(-\infty, t) = y_i(t),\ i = 1, \dots, m \\ \frac{\partial u_{m+1}}{\partial s} + J\left(\frac{\partial u_{m+1}}{\partial t} - X_{\rho(s)G_{m+1}}(u_{m+1})\right) = 0 \\ u_{m+1}(+\infty, t) = y_{m+1}(t) \\ u(\partial\Sigma) \subset O_M \\ \overline{\partial}(u|_{\Sigma_0}) = 0 \end{cases}$$

where G_i, y_i are as before and Σ is a Riemannian surface of genus zero with $m+1$ semi-strips-ends (see Figure 4). By repeating the previous construction we obtain

$$\mathcal{O}_F : HF^{k_1}(H_1) \otimes \dots \otimes HF^{k_m}(H_m) \to HF^{k_1+\dots+k_m}(H_{m+1}).$$

As before, from the analysis of the boundary of a one-dimensional manifold of type (26) (but with $m+1$ instead of three exterior vertices, i.e., Hamiltonian paths) it follows that the operators $\mathcal{O}$ are well defined as the operators

$$\begin{aligned} \mathcal{O}_M &: HM^{k_1}(M) \otimes \cdots \otimes HM^{k_m}(M) \longrightarrow HM^{k_1+\dots+k_m}(M) \\ \mathcal{O}_F &: HF^{k_1}(T^*M) \otimes \cdots \otimes HF^{k_m}(T^*M) \longrightarrow HF^{k_1+\dots+k_m}(T^*M). \end{aligned}$$

So from the commutativity of (29) and of the diagram

$$\begin{tikzcd} HF^{k_1}(H_1) \otimes \cdots \otimes HF^{k_m}(H_m) \arrow[r, "\mathcal{O}_F"] & HF^{k_1+\dots+k_m}(H_{m+1}) \\ HM^{k_1}(f_1) \otimes \dots \otimes HM^{k_m}(f_m) \arrow[r, "\mathcal{O}_M"] \arrow[u, "\tau_1 \otimes \dots \otimes \tau_m"] & HM^{k_1+\dots+k_m}(f_{m+1}) \arrow[u, "\tau_{m+1}"'] \end{tikzcd}$$

(this commutativity follows from the same arguments as is Section 2) we derive the following

Theorem 3. *The* PSS *isomorphism* $\mathcal{T} : HM^*(M) \to HF^*(T^*M)$ *defined in Theorem* 2 *is an isomorphism that preserves the products* $\mathcal{O}$, *i.e.*:

$$\mathcal{T}(\mathcal{O}_M(a_1 \otimes \cdots \otimes a_m)) = \mathcal{O}_F(\mathcal{T}(a_1) \otimes \cdots \otimes \mathcal{T}(a_m))$$

for $a_i \in HM^*(M)$.

Remark 2. The result of Theorem 3 can be generalized to the case of Floer homology for Lagrangian submanifold L of the symplectic manifold (P, ω), when $\omega|_{\pi_2(P,L)} = 0$. Indeed, this assumption provides that no bubbles appear in the limit of a sequence of (perturbed) holomorphic maps, so the compactness arguments rest the same as in the case of cotangent bundle.

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